This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

R. P. STUDLEY & CO.
BOOK BINDERS, PRINTERS
AND
Lithographers
S. W. COR. MAIN & OLIVE STS
ST. LOUIS.

LIBRARY
UNIVERSITY OF
CALIFORNIA
SANTA CRUZ

Lick Observatory Library.

Purchased from the

Phebe Hearst Endowment Fund.

A.D. 1892

Ed. J. Holden

"1866"

St. Louis 28th Sept. 1865.

-711-

A TREATISE

ON

INFINITESIMAL CALCULUS;

CONTAINING

DIFFERENTIAL AND INTEGRAL CALCULUS, CALCULUS OF VARIATIONS, APPLICATIONS TO ALGEBRA AND GEOMETRY, AND ANALYTICAL MECHANICS.

BY

BARTHOLOMEW PRICE, M.A., F.R.S., F.R.A.S.,

FELLOW AND TUTOR OF PEMBROKE COLLEGE, AND
SEDLEIAN PROFESSOR OF NATURAL PHILOSOPHY, OXFORD.

VOL. I.

DIFFERENTIAL CALCULUS.

SECOND EDITION.

"Les progrès de la science ne sont vraiment fructueux, que quand ils amènent aussi le progrès des Traités élémentaires."—CH. DUPIN.

OXFORD:
AT THE UNIVERSITY PRESS.
M.DCCC.LVII.

1857.

QA
303
P94
1857
V. 1

PREFACE

TO

THE SECOND EDITION.

THE character of Infinitesimal Calculus as a progressive science has been so well sustained during the last few years, that in preparing for the press this second edition of my first volume I have found it necessary to make more alterations than might otherwise have been desirable. In Great Britain, no less than in other parts of the world, and especially in Germany and France, have the boundaries of our science been advanced: new methods and new processes have been devised, new theorems have been discovered, and to new forms of subject-matter the principles of the Calculus have been applied. Every one who has watched the progress of the science will hardly need to be told of Sir W. R. Hamilton, Cayley, Boole, Donkin, Spottiswoode, Sylvester, Salmon, Hargreave, any more than of Jacobi, Dirichlet, Joachimsthal, Hesse, Plücker, Steiner, Sturm, Liouville. To all these in their several degrees is due the credit of having brought the subject to a state of perfection never heretofore attained; and a treatise pretending to be didactic and educational would ill deserve those epithets if the discoveries of such eminent mathematicians were not noticed in it.

a 2

The character of my treatise is, as heretofore, elementary. I have endeavoured to explain as clearly as possible the language, the symbols, the first processes of Infinitesimal Calculus; and with the object of presenting the principles in a form less repulsive than is usual with English writers many illustrations are introduced which may at first sight appear foreign to the subject; yet they are not so; for the thoughtful reader will detect in these applications abstract conceptions which are fundamental in the Calculus. And he will also see that though he may have been ignorant of the nomenclature of the science, yet that neither the idea nor the apparatus of it is entirely new to him. Throughout the Treatise, and especially in the early part of it, examples are inserted to give the student an aptness in applying general rules to particular cases, as well as to aid him in the detection of universal principles lying under particular examples. The Treatise also includes the higher parts of the science, because it is intended for the use of the most advanced students in our universities; and thus in many parts theorems are introduced which are close upon the boundaries of our knowledge.

The matter was first delivered orally in lecture, and subsequently written; whence has arisen the colloquial style; and which it has been thought expedient to retain, under the conviction that it invests a book with a personal and living character more akin to the explanations of a speaking teacher, and thereby infuses life into what might be otherwise dry text; and in some few passages wherein disputed questions are discussed, objections are stated as if urged by an opponent.

I do not care to claim any originality as to the matter of the Treatise; it is enough for me if I have been able to state the principles of our science with perspicuity, and to arrange its several parts in such order as to remove difficulties which I venture to think are not necessarily connected with the subject. The sources whence the matter is derived are various, and for the most part foreign; all cannot be specified, and probably I am indebted to others for more than I am aware of; every book, and perhaps every sentence and every conversation, may leave an impress which, unconsciously to himself, modifies an author's opinions. Almost all which has been taken from other sources, and is not specially acknowledged, has been so long known and commented on, as to have become *publici juris;* Euler, Lagrange, Lacroix, Peacock, Monge, Dupin, it is quite unnecessary to mention; a few names occur in the foot- and other notes. I am however under especial obligation to M. Cauchy in his various treatises and memoirs, to his Redacteur M. l'Abbé Moigno; to M. Navier; to the late Mr. D. F. Gregory; to Professor De Morgan, the author of the treatise published by the Society for the Diffusion of Useful Knowledge; to Professor Donkin; to Mr. W. Spottiswoode, M. A.; both of the University of Oxford, and from whom I have received valuable assistance and advice in many parts of the Treatise: and generally too to the Journals of Liouville and Crelle, mines of precious materials to the mathematical student.

The process of assimilating a body of matter so large, and of such diverse origin, as that of Infinitesimal Calculus, is necessarily long; and in the course of it the question arises, under what principle is all to be harmonized? the parts are seemingly dissonant, what renders them consistent? "whence is the string obtained on which the pearls are to be hung?"

An inquiry of this kind is far too large to be answered within the limits of a preface, but some few remarks are necessary for a due understanding of our method.

Infinitesimal Calculus, both in its pure and applied forms, whether of Geometry or of Mechanics, is a branch of the science of Number; its symbols are of the same kind, and are operated on according to the same laws; they are applied subject to the same conditions, are interpreted on the same principle, and lead to analogous results. What then is its specific characteristic? In the parts of the science of Number which are supposed to have been previously studied, viz. in Arithmetic and Algebra (as it is called), numbers are finite and discontinuous; but Number also admits of being continuous; that is, is capable of gradual growth and of infinitesimal increase. Number under this aspect is what Infinitesimal Calculus contemplates: and investigates the new properties of it, the new symbols required to express them, and the new laws to which they are subject; it has thus to create its own materials, and these materials are infinitesimals.

The method therefore has been at first to unfold those properties of Number which are necessary to the construction of the science of infinitesimals; and then so to describe continuous Number and the infinitesimal elements of its growth, in their essential qualities, as to be able to enunciate certain axiomatic properties of them, from which the Calculus may be evolved. These are stated in seven Theorems in Art. 9; of which perhaps the most important is Theorem VI, presenting the character of infinitesimals in their broadest view, viz. that they are such that a finite number of them has no value at all when added to a finite quantity. These Theorems are the ultimate propositions of the science: from them the other truths are inferred, and on them does the correctness of the processes depend. In order to the subsequent use of the advantage in the way of inference possessed by algebraical symbols, we express them in mathematical language, and then proceed to deduce the consequences with which they are pregnant; and this deductive process is continued through the Treatise. In the course of it however, and especially in the applied parts, the subject-matter becomes enlarged, new relations are introduced, and new names are required; thus definitions become necessary, and these are so constructed as to contain in germ the substance of the Articles dependent on them.

A due understanding of these axioms and definitions is plainly of the utmost importance; because on it does it depend whether we work with mere symbols, or whether the symbols are *σημεῖα* of philosophical ideas which we comprehend. With the object of guarding against such superficial knowledge, which is useful neither in its results nor as an intellectual exercise, geometrical interpretation of infinitesimals has been often introduced, and magnified diagrams exhibit lines, areas, angles, &c. which are represented by symbols of infinitesimals: every process, nay every step in every process, admits of such geometrical translation; and it is most desirable that the student should exercise himself in it: by so doing he will have a most certain test whether his operations are according to the laws of correct inference, or whether he is merely applying mnemonic rules and groping his way in the dark by some obscure road, and drawing his conclusion as it were only by riddles.

The Calculus then is that of infinitesimals; but as the most convenient mode of forming them shews them in the light of differences between two states at a "very small" distance apart, they have obtained the name of Differentials; and hence the name of "Differential Calculus" has arisen; the latter term is retained, although it refers to the mode of generating the materials, and not to any pregnant property of them when generated. The notation also and language of Lagrange's Calculus of derived-functions has been employed, because it expresses in a most convenient form some of the earliest results of operations with infinitesimals, and their relations to finite quantities: but I would warn the reader especially against supposing that our Calculus is founded on the notion of derived-functions as the coefficients of the terms of a series; they are not considered in that relation at their first admission, and it is only by a course of reasoning that they afterwards become so. Expansion in a series has been admitted fundamentally but once, viz. in Art. 21, and whatever may be the final issue of the question as to Convergent and Divergent Series, it cannot affect that which I have employed, inasmuch as it is proved to be Convergent.

And here I must say a few words on the exactness of our processes. It is supposed that the results cannot be truly accurate, because certain quantities are neglected; thus, a finite number of infinitesimals is neglected if it is added to or subtracted from a finite quantity; and similarly a finite quantity must be and is neglected when it is added to or subtracted from an infinity. And it is supposed that an error is hereby introduced, which vitiates and destroys the accuracy of the whole work, and that the results are at last true only because a compensation is made. Now let the nature of our Calculus be clearly understood; it is of itself a part of the science of number; its subject matter is continuous number; and according as its results and theorems in number are exact or not, so is its exactness to be tested. It is capable of application to other sciences; those of Geometry, Motion, &c.; but its truth is not to be tried by the results of these subjects, because their matter may not be so conformable to that of our Calculus, but that discrepancy may be between them. In Infinitesimal Calculus properly so called, our symbols are symbols of number only: we make our own materials; our infinitesimals and infinities are created by us; and are subject to certain conditions which we choose to impose on them; we make them subject to certain laws; and so long as they are employed consistently with these conditions and these laws, and in accordance with the rules of correct inference, the conclusions which they lead to are strictly correct. There is no error: the neglecting of infinitesimals is a necessary step in our process, and therefore it is that I have used the language of necessity: I have said in Theorem VI that a finite sum of infinitesimals *must* be neglected, not *may* be neglected, when it is added to or subtracted from a finite quantity: were this not done, infinitesimals would not be what they are, and our rules for the discovery of them would be other than they are. When however we apply to other subject matter, say of space or of motion, the conception of infinitesimals, it may be that this particular subject-matter does not admit of the continuous infinitesimal change with that exactness which the conception of numerical infinitesimal growth is capable of; it may be that there is a discrepancy, and consequently an error; for which afterwards compensation has to be made. Thus, for instance, we may in our conception of the infinitesimal Calculus as applied to Geometry assume the line joining two consecutive points in a circle to be straight, and represent it by a symbol which denotes a straight line; whereas from the geometrical definition of a circle we know that the curvature of a circle is continuous, and that the line joining two points of it, however near together they are, cannot be straight; and thus our symbols, though representatives of such straight lines, only approximately represent them. In this case doubtless there may be an error; an error not in the work of the calculus; that is true and exact; but because the geometrical quantities are not adequately expressed by the symbols; but when by means of integration we pass from the infinitesimal element to the finite function, then the finite function becomes the exact and adequate representative of the geometrical quantity, and a compensation has taken place in the act of passing from the infinitesimal element to the finite function. On investigation it will, I venture to think, be found that the exactness of the Calculus has been impugned on these and similar grounds; and therefore that it has been unfairly impugned: let it be tried on its own principles; on them I venture to say it will stand the attack. It creates its own materials, and is subject to its own laws; let it not be condemned because other materials, which you try to bring within its grasp, refuse to submit to these laws.

The Volume consists of two Parts: in the former are investigated the Theorems of the Differential Calculus so called, and in the latter the applications to Geometry of two and three dimensions are discussed.

The Chapters mark the salient divisions of the matter: and these again are subdivided into Articles, which for the sake of reference are numbered continuously through the Volume, and have their numerals placed in the corners on the top of the pages. The bracketed numerals attached to the more important equations are separate for each Chapter; and the references are usually made to the numbers of the equation and of the Article. The Analytical Table of Contents exhibits a résumé of the matter of the Treatise.

Pembroke College, Oxford,
June 6, 1857.

ANALYTICAL TABLE OF CONTENTS.

PART I.

ANALYTICAL INVESTIGATIONS.

CHAPTER I.

PRINCIPLES AND EXPLANATION OF TERMS.

SECTION 1.—*Introductory; Number, its properties, affections, and science.*

Art. Page
1. *Number*, and some properties of it 7
2. The modes of forming number 8
3. The abstract character of number 9
4. Numbers are variable and constant 11
5. The words *definite* and *indefinite*, *infinite*, *finite*, *infinitesimal* .. 12
6. Number varies continuously and discontinuously 15
7. Infinitesimal Calculus considers continuously-varying number 16
8. Infinities and infinitesimals, their orders and their symbols .. 18
9. Fundamental theorems on infinities and infinitesimals 21
10. Examples on the above theorems 23
11. The relation of the finite to the infinite and the infinitesimal . 24
12. Functions; on dependent and independent variables 25
13–15. Functions are implicit and explicit, algebraical and transcendental, simple and compound, continuous and discontinuous 27
16. The generation of continuous quantity 29
17. The particular mode of generating continuous number, as it is considered in the Differential Calculus 31
18, 19. Derivation and derived-functions 32
20. Description of the Differential Calculus 34

SECTION 2.—*Fundamental Lemmas of the Infinitesimal Calculus.*

21. Evaluation of $(1+x)^{\frac{1}{x}}$, when x is infinitesimal 34
22. Tan x, x, and sin x are equal, when x is infinitesimal 37
23. 2 versin $x = x^2$, when x is infinitesimal 39

SECTION 3.—*Examples of the infinitesimal method.*

24. Illustrative examples on differentiation 40

CHAPTER II.

CONSTRUCTION OF RULES FOR DIFFERENTIATION OF FUNCTIONS.

SECTION 1.—*The differentiation of explicit functions of one variable.*

25–29. Rules for differentiating
$f(x) \pm c,\ cf(x),\ f(x) \pm \phi(x) \pm \ldots,\ f(x) \times \phi(x),\ \dfrac{f(x)}{\phi(x)}$. 51
30. Differentiation of x^n................................ 54
31. Differentiation of compound functions 56
32–36. Differentiation of a^x, e^x, $\log_a x$, $\log_e x$ 57
37. Differentiation of a product of many functions 60
38–45. Differentiation of circular functions 61

SECTION 2.—*The differentiation of functions of many variables.*

46. Functions of many variables 70
47. Differentiation of a function of two independent variables ... 73
48. Differentiation of an implicit function of two variables 75
49, 50. Differentiation of a function of many independent variables 77
51. Differentiation of functions of many variables, all of which are not independent 79
52, 53. Differentiation of functions of many variables, when the variables enter in particular combinations 81

CHAPTER III.

SUCCESSIVE DIFFERENTIATION.

SECTION 1.—*Successive differentiation of an explicit function of one variable.*

54. Explanation of the process, and corresponding nomenclature and symbols. Examples 89
55. Leibnitz's Theorem for successive differentiation 94
56. The nth differential of $f(x)$, in terms of successive values of $f(x)$ 97

SECTION 2.—*Expansion of an explicit function of one variable.*

57. Maclaurin's Theorem.............................. 97
58. Limits of Maclaurin's Theorem 99
59. Examples of Maclaurin's Theorem 100

60–63. Certain trigonometrical expressions and theorems 103
64–66. The roots of $+1$ and of -1 108
67. Imaginary logarithms 113

SECTION 3.—*Theory of the equicrescent variable, and Taylor's Theorem.*

68. The relation between y and its equivalent $f(x)$, of the equation $y = f(x)$, when x has been successively increased n times 115
69, 70. Modification of the preceding, when x is equicrescent .. 116
71. Taylor's Series 119
72. Limits of Taylor's Series 119
73. Examples of Taylor's Series 121
74. Expansion of $f(x)$ in ascending powers of $x-a$ 123

SECTION 4.—*Change of equicrescent variable, and transformation of differential expressions.*

75. Different forms of the problem 123
76. Requisite formulæ for a function of two variables 124
77. Transformations in terms of a new variable 125

SECTION 5.—*Successive differentiation of functions of many independent variables.*

78. Explanation of the symbols 128
79. The order of successive differentiations with respect to many variables is indifferent 131
80, 81. Application of the principles of the preceding Articles to functions of two and more variables 133
82. Euler's Theorems of homogeneous functions 137
83. Transformation of partial derived functions 139

SECTION 6.—*Successive differentiation of implicit functions.*

84. Calculation of derived-functions when the relation between the variables is given in an implicit form 141
85. Examples of preceding formulæ 142
86. Expansion of one of the variables of an implicit function in terms of the other by means of Maclaurin's Theorem .. 143
87. Calculation and properties of Bernoulli's numbers 145
88. Examples of Bernoulli's numbers 146
89. The sums of the powers of the reciprocals of the natural numbers 148
90. Lagrange's Theorem 149
91. Examples of Lagrange's Theorem 153

92. Expansion of $f(x)$ in ascending powers of $\phi(x)$ 155
93. Laplace's Theorem 156
94. Extension of Maclaurin's Theorem 157
95. Arbogast's process of derivation 158
96. Extension to implicit functions 162
97. Extension to functions of many series 163

SECTION 7.—*The formation of differential equations by means of elimination.*

98. Elimination of constants from an explicit function 165
99. Elimination of constants from an implicit function 167
100. Elimination of given functions 169
101. Trigonometrical relations expressed by differential equations 171
102. Formation of differential equations in terms of three and more variables 172
103–105. Formation of differential equations by the elimination of arbitrary functions 174

SECTION 8.—*Transformation of partial differential expressions.*

106. Transformation of expressions involving partial derived-functions into their equivalents in terms of other variables 180
107, 108. Examples illustrative of the preceding principles 182

CHAPTER IV.

CERTAIN RELATIONS BETWEEN FUNCTIONS AND DERIVED-FUNCTIONS.

109. Extension of preceding principles to the consideration of $f(x)$, when x receives a finite increment 187
110. According as $f'(x)$ is positive or negative, so does $f(x)$ increase and diminish with x, or as x increases, $f(x)$ decreases, and *vice versâ* 188
111. The proof that

$$\mathrm{F}(x_n) - \mathrm{F}(x_0) = (x_n - x_0)\,\mathrm{F}'\{x_0 + \theta(x_n - x_0)\},$$

$\mathrm{F}(x)$ being finite and continuous for all values of x between x_n and x_0 190
112. The proof that under stated conditions

$$\frac{\mathrm{F}(x_0 + h) - \mathrm{F}(x_0)}{f(x_0 + h) - f(x_0)} = \frac{\mathrm{F}'(x_0 + \theta h)}{f'(x_0 + \theta h)}$$ 191

113. Hence also under certain conditions

$$\frac{\mathrm{F}(x_0 + h) - \mathrm{F}(x_0)}{f(x_0 + h) - f(x_0)} = \frac{\mathrm{F}^n(x_0 + \theta h)}{f^n(x_0 + \theta h)};$$

and corollaries are deduced therefrom 193

114. If in the theorem of the last Article $f(x) = (x-x_0)^n$, then $F(x_0+h) - F(x_0) = \frac{h^n}{1.2.3\ldots n} F^n(x_0+\theta h)$ 195
115. Certain corollaries of the theorem of Art. 114 195
116. Particular cases of the theorem of Art. 114 196

CHAPTER V.

THE ORDERS OF INFINITESIMALS, AND EVALUATION OF INDETERMINATE QUANTITIES.

117. *Unity* accurately defined 198

Section 1.—*Orders of infinitesimals.*

118. Definition of order of infinitesimals 199
119. Mode of determining the order of an infinitesimal 200
120. Examples wherein orders of infinitesimals are determined 201

Section 2.—*Evaluation of quantities of the forms* $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$.

121. The cause why quantities assume the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ .. 202
122–124. Modes of evaluating quantities of the form $\frac{0}{0}$ 202
125. Evaluation of quantities of the form $\frac{\infty}{\infty}$ 206
126. Evaluation of quantities of the form $0 \times \infty$ 209
127. Evaluation of quantities of the form $\infty - \infty$ 209

Section 3.—*Evaluation of quantities of the forms* 0^0, ∞^0, 1^∞, 0^∞.

128. Mode of evaluating, and examples of, such quantities 210

CHAPTER VI.

EXPANSION OF FUNCTIONS.

Section 1.—*Functions of one variable.*

129. An accurate proof of Taylor's Series.................. 212
130. The imperfect form of it given in Art. 71 214
131. Deduction of Maclaurin's Theorem from Taylor's 214
132. The imperfect form of Maclaurin's Theorem given in Art. 57 215
133. Another form of Taylor's Theorem 215
134, 135. The limits of Taylor's and Maclaurin's Series 215
136. The failure of Taylor's and Maclaurin's Series 217

SECTION 2.—*Functions of many variables.*

137. The theorem for functions of two variables analogous to that of Art. 111 222
138. A particular form of the preceding 224
139. Evaluation of indeterminate quantities which are functions of two independent variables 225
140. Expansion of $F(x+h, y+k)$ 228
141. Examples of the preceding 229
142. Expansion of $F(x+h, y+k, z+l, \ldots)$ 230
143. Extension of the process of derivation to functions of two and more variables 233

CHAPTER VII.

MAXIMA AND MINIMA.

144. Definition of maximum and minimum 235

SECTION 1.—*Maxima and minima of explicit functions of one variable.*

145. Method of determining such singular values of $f(x)$ 236
146. Easy detection of the singular values when $f'(x)$ is algebraical 237
147. Geometrical representation of the criteria 238
148. Examples illustrative of the preceding methods 238
149, 150. The forms of the derived-functions of $f(x)$, when it has such singular values; and a method of thereby determining maxima and minima of $f(x)$ 240
151. Examples of maxima and minima 243
152. The determination of the number of maxima and minima of a given function 247
153. The absolute maximum and minimum 249

SECTION 2.—*Maxima and minima of implicit functions of two independent variables.*

154. Method of determining such values, and examples 250

SECTION 3.—*Maxima and minima of an explicit function of two independent variables.*

155. Definition of maxima and minima of functions of two variables 252
156, 157. Method of determining criteria of such singular values 253
158. Another method of determining such singular values 255

159. Application of the method to total minima 258
160. The sufficiency of the process 259
161. Examples of the process 260
162. A consideration of a case wherein the requisite conditions are not fulfilled 262

Section 4.—*Maxima and minima of functions of three and more independent variables.*

163. Conditions of such singular values of a function of three independent variables 263
164. The requisite conditions in the most general case 264
165. The method of least squares 266
166. Examples of the method of least squares 270

Section 5.—*Maxima and minima of functions when all the variables are not independent.*

167. Investigation of the most general case of many variables .. 271
168. Discussion of the case of two variables which are connected by a given equation 273
169. Examples illustrative of the preceding methods 274

CHAPTER VIII.

THE THEORY OF ALGEBRAICAL EXPRESSIONS.

170. Object of the Chapter is the discussion of an algebraical expression in its synthetical form 279
171, 172. The continuity of algebraical expressions 280
173. Proof that every equation has a root.................. 282
174. If a is a root of $f(x)$, $f(x)$ is divisible by $x-a$ 284
175. Every equation has as many roots as it has dimensions .. 284
176. The roots of $f'(x)$ are intermediate to those of $f(x)$ 285
177. If $f(x)$ has m equal roots, $f'(x)$ has $m-1$ roots equal to them 287
178. Sturm's Theorem 287
179. Examples in which Sturm's Theorem is applied 291
180. The criteria of the number of impossible roots of an equation 292
181. Fourier's Theorem 293
182. Des Cartes' rule of signs 295
183. Newton's method for finding a number greater than the greatest root of an equation 295
184. Method of approximating to the root of an equation 295

PART II.

GEOMETRICAL APPLICATIONS.

CHAPTER IX.

ON GEOMETRY.

SECTION 1.—*The adjustment of the principles of Pure Geometry and Infinitesimal Calculus.*

185. The application to space of the law of continuity 297

186. The definitions of some geometrical terms founded on the said law 298

187. Corroboration of the preceding modes of interpretation from some known examples 301

SECTION 2.—*On an extension of the symbols of direction.*

188. Necessity of symbols of direction 303

189. Interpretation of $+^n$ and of $+^{\frac{1}{2}}$ 305

190. Interpretation of $(-)^{\frac{1}{2}}$ or of $+^{\frac{1}{4}}$ 307

191. Other modes of indicating direction 307

192. Examples in illustration of the mode.................. 308

SECTION 3.—*On the generation of some plane curves of higher orders.*

193–206. On the Cissoid, the Witch, the Conchoid, the Lemniscata, the Logarithmic Curve, the Catenary, the Tractory, the Cycloid, the Companion to the Cycloid, the Epitrochoid and its varieties 311

SECTION 4.—*General properties of algebraical curves of the* n*th degree.*

207. Various forms, and the number of terms of an algebraical equation of the nth degree 320

208. The homogeneous forms, in terms of three variables 322

209. The order of the resultant of many algebraical equations .. 324

210. The number of the points of intersection of two curves of the mth and nth degree; the theorems of Newton and Cotes 325

211. The number of given points through which a curve of the nth degree may pass 327
212. The number of these points which may be on a curve of the mth degree 330

CHAPTER X.

PLANE CURVES REFERRED TO RECTANGULAR COORDINATES.

SECTION 1.—*Tangents and normals.*

213. The principle of application of infinitesimal calculus to the properties of curves 331
214. Definition of, and equation to, a tangent 332
215. Modification of the equation in case the function is (α) implicit, (β) homogeneous 333
216. Equation to the tangent, when the equation to the curve is homogeneous and of three variables 334
217. Equation to the normal 334
218. Values of ds, and of $\sin \tau$, $\cos \tau$, $\sin \psi$, $\cos \psi$ 336
219. Discussion of the equations to the tangent and the normal, and of lines and quantities connected with them 338
220. Explanation of the course of a curve at points where $\frac{dy}{dx} = 0$, and $= \infty$ 340
221. Examples illustrative of the preceding articles 341
222. General properties of the tangent of a curve of the nth degree 344
223. The number of tangents which can be drawn to a curve from a given point. The class of a curve 346
224. Joachimsthal's method of tangents 348
225. The equation of the pencil of tangents 350
226. Base-curve, Pole, and Polars 351
227. The first polar envelope 353
228. Successive polars 354
229. The number of normals which can be drawn to a curve from a given point. Evolutes 355
230. The normal is the longest or shortest line that can be drawn to a curve through a given point 356

SECTION 2.—*On asymptotes to plane curves referred to rectangular coordinates.*

231. On rectilinear asymptotes 357

232. Method of determining asymptotes by means of expansion in descending powers of x 358
233. Examples of the method 359
234. Asymptotes are also tangents to a curve at an infinite distance 360
235. Examples of the method 361
236. On asymptotes out of the plane of reference 363
237. Curvilinear asymptotes 363
238, 239. The equations of asymptotes are factors of the general equation of the curve 364

Section 3.—*Direction of curvature and points of inflexion.*

240. Direct proof that a curve is convex or concave downwards according as $\frac{d^2y}{dx^2}$ is positive or negative 367
241. Another proof of the same theorem by means of an expansion 368
242. Examples in illustration 370
243. Interpretation of the preceding results by the infinitesimal method 371
244. Criterion of points of inflexion when the equation of the curve is an implicit function 373
245. If the curve is of the nth degree the number of points of inflexion within a finite distance of the origin cannot exceed $3n(n-2)$ 374
246. The Hessian of a curve 375
247. Multiple tangents 376

Section 4.—*Multiple points.*

248. At a multiple point $\frac{dy}{dx} = \frac{0}{0}$; double points and their varieties 378
249. Examples of double points 381
250. An explicit function is explained which well exhibits some of the peculiarities of cusps 384
251. The number of double points of a curve of the nth degree . 385
252. The number of cusps of a curve of the nth degree 387
253. The relation of a curve to its Hessian 388
254. Double points and the Hessian when the equation of the curve is homogeneous and of three variables 390
255. Triple points; and an example 391

256. Quadruple points 392
257. Algebraical property of equations giving rise to multiple points 392

Section 5.—*On tracing curves by means of their equations.*

258. General hints as to the analysis of the equation 393
259. Rules for tracing curves, and method to be followed 398
260. Examples of equations analysed and of curves traced 398

CHAPTER XI.

PLANE CURVES REFERRED TO POLAR COORDINATES.

Section 1.—*Extension of the modes of interpretation, and on the equations to some polar curves.*

251. Interpretation of r and θ when affected with negative signs 411
262–265. Equations to the Spiral of Archimedes, the Reciprocal Spiral, the Lituus, and the Logarithmic Spiral 412
266–268. Equations in terms of r and p of the involute of the circle, the circle, and the epicycloid 414

Section 2.—*Tangents and normals to polar curves.*

269. The values of ds, and of lines and quantities connected with the tangent and the normal 415
270. Other values of p 417
271. The same quantities in terms of p and r 417
272. Examples in illustration of the preceding 418

Section 3.—*Asymptotes to polar curves.*

273. Means of determining rectilinear asymptotes 420
274. Examples in illustration 421
275. Asymptotic circles 422

Section 4.—*Direction of curvature and points of inflexion.*

276. The convexity or concavity of a curve towards the pole .. 423

Section 5.—*Tracing polar curves.*

277 General hints for analysing the equations and tracing the curves 424
278. Rules for tracing polar curves 425
279. Examples of equations analysed and of curves traced 426

CHAPTER XII.

CURVATURE OF PLANE CURVES.

SECTION 1.—*Curves referred to rectangular coordinates.*

280. Explanation of curvature; definition of curvature of a circle 432
281. Definition of radius of curvature and of circle of curvature . 433
282. Mathematical values of radius of curvature 434
283. Illustrative examples 436
284. The angle of contingence, and its relation to the radius of curvature.................................. 437
285. Other values of ρ and of $d\tau$ 437
286. Value of the radius of curvature, when the equation is an implicit function 439
287. The number of points in a curve at which the curvature is the same................................... 440

SECTION 2.—*On evolutes referred to rectangular coordinates.*

288. Mode of generating an evolute, and formulæ for determining its equation 441
289. Examples of evolutes............................ 442
290. On the circle of curvature.......................... 447
291. The normal to the curve passes through two consecutive points of the evolute 448
292. Mutual relation of the evolute and involute, and origin of their respective names 448
293. All evolutes are rectifiable. Examples................ 449
294. The normal to the involute touches the evolute. The relation between the radii of curvature of the involute and the evolute 451
295, 296. Explanation of the preceding theorems by the infinitesimal method 452
297. The order and the class of the evolute; singular properties of the involute produce singular properties of the evolute. 454

SECTION 3.—*Curvature of polar curves.*

298–300. Length of radius of curvature 456
301. On the chord of curvature, its definition and value 458
302. Examples in illustration 458

SECTION 4.—*Evolutes of polar curves.*

303. Method of finding the equation to the evolute; and examples.................................. 459

CHAPTER XIII.

CONTACT OF CURVES AND ENVELOPES.

SECTION 1.—*The theory of contact.*

304. Contact depends on the number of consecutive points which are common to two curves 461
305. And therefore on the number of derived-functions which are equal at the point common to the two curves 462
306. Theorems deduced from the latter mode of considering contact 462
307. Two curves which have a common tangent intersect or not at the common point, according as the contact is of an even or odd order 464
308. Order of contact dependent on the number of undetermined constants in the equation to a curve 464
309, 310. Cases considered of curves involving two and three arbitrary constants. Examples 465
311. Conditions under which a circle can have contact of the third order 468
312. Contact of curves with given curves 469

SECTION 2.—*The theory of envelopes.*

313. Explanation of the subject of envelopes, families of curves, variable parameters.............................. 471
314. Form of the problem when only one parameter is involved 472
315. Examples of envelopes 473
316. General case of n parameters and $(n-1)$ conditions 475
317. Examples in illustration.......................... 476

SECTION 3.—*The theory of reciprocation.*

318. First polar envelope 480
319. The degree of the first polar envelope 481
320. Reciprocal curves 483
321. Properties of reciprocal polars 484

SECTION 4.—*On caustics.*

322. On the formation of caustics 486
323, 324. Caustic by reflexion of systems of parallel rays 487
325. General properties of such caustics 491

326. Particular case of the caustics by reflexion at a spherical surface 493
327. Caustic by reflexion on a logarithmic spiral 494
328. General properties of caustics by refraction 495
329. All caustics are rectifiable 496
330. Caustic by refraction at a plane surface 497

CHAPTER XIV.

APPLICATIONS OF THE CALCULUS TO CURVED SURFACES.

331. The equations to a straight line and to a plane 498
332. The equation to a tangent plane to a curved surface 500
333. The direction-cosines of the tangent-plane 501
334, 335. Modified forms of the equation to the tangent plane, when the equation to the surface is (α) explicit, (β) homogeneous and algebraical 502
336. The equations to a normal of a curved surface 503
337. The equations to a perpendicular through the origin on a tangent plane 504
338. Examples in illustration of the preceding 504
339. Singular forms of tangent planes Cones of the second and third orders 506

CHAPTER XV.

APPLICATION OF THE CALCULUS TO CURVES IN SPACE.

340. The equations of curves in space 509
341. The equations to the tangent line 509
342. The equation to the normal plane 511
343, 344. The equation to the osculating plane 511
345. The equations to the binormal 512
346. Equations to the principal normal 513
347. Examples of the preceding formulæ 514
348. The distinguishing criterion of plane and non-plane curves 516
349. Geodesic lines, and lines of greatest slope 516

CHAPTER XVI.

THE GENERAL AND PARTIAL-DIFFERENTIAL EQUATIONS OF SURFACES.

350. The mode of formation of surfaces by means of generators and directors 518

SECTION 1.—*Surfaces generated by straight lines.*

351. Ruled surfaces 520

352. Developable surfaces 521
353–355. Conical surfaces 521
356–358. Cylindrical surfaces 526
359, 360. The equations to developable surfaces 531
361, 362. The edge of regression 533
363. Examples of developable surfaces 534
364. Skew surfaces 535
365. The partial-differential equation to skew surfaces 536
366, 367. Conoidal surfaces 537
368. Case of two directors and a generator parallel to a given plane 538

SECTION 2.—*On surfaces generated by circles.*

369–371. Surfaces of revolution 539
372–374. Tubular surfaces 542

CHAPTER XVII.

CURVATURE OF CURVES IN SPACE.

375. Mode of measuring absolute curvature; angle of contingence 547
376. Mode of measuring torsion; radius of torsion 548
377, 378. Radius of absolute curvature 549
379. Angle of curvature 551
380, 381. Geometrical illustrations 552
382. Torsion 553
383. Singular values of curvature and torsion 555
384. The polar surface 555
385. Equation to the polar surface 556
386, 387. The polar line and locus of polar lines 557
388. The osculating sphere 557
389. Evolutes of non-plane curves 559
390, 391. Geometrical illustrations 561
392. Complex flexure and its measure 564
393. The osculating surface 565
394, 395. The rectifying surface and the rectifying line 566
396. Application to the helix 567

CHAPTER XVIII.

CURVATURE OF CURVED SURFACES.

397. Normal sections 570
398, 399. Curvature of principal normal sections 571

400. Perpendicularity of normal sections 574
401. Curvature of any normal section 575
402. Normal sections of maximum and minimum curvature.... 576
403, 404. Euler's theorem of the curvature of normal sections.. 576
405. Application to the ellipsoid 578
406. Singular values of radii of curvature................ 579
407. Umbilics 581
408. Examples of umbilics 582
409. Lines of curvature................................. 584
410. The Theorem of Dupin 586
411. Three confocal surfaces of the second order 588
412. Locus-surface of centres of principal curvature.......... 589
413. Modification of the conditions when the equation is explicit 591
414. Meunier's theorem of oblique sections 593
415. Explanation of properties by means of the indicatrix 594
416. Osculating surfaces 597
417. Measure of curvature 598

CHAPTER XIX.

ON THE CALCULUS OF OPERATIONS.

418. The distinction between symbols of operation and symbols of quantities 600
419. The laws of commutation, distribution, and iteration 601
420. The extension of the same to algebraical functions 603
421. The law of total differentiation 604
422. Three fundamental theorems......................... 605
423. Another form of the preceding 606
424. Illustrative examples 607
425. Leibnitz's Theorem, and particular forms.............. 608
426. Another form of Leibnitz's Theorem 609
427. The symbolical form of Taylor's Series................ 609
428. Extension of Euler's Theorem 610

INFINITESIMAL CALCULUS.

PRELIMINARY THEOREMS.

As the following Algebraical Theorems will be frequently applied in the course of the treatise, it is convenient to prove them once for all, and, for the sake of reference, to place them at the beginning of the work.

I.

If there be any number of *equal* fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots \frac{a_n}{b_n}$, each of them is equal to

$$\frac{a_1 + a_2 + a_3 + \dots\dots + a_n}{b_1 + b_2 + b_3 + \dots\dots + b_n};$$

and, if $m_1, m_2, m_3, \dots\dots m_n$ be any multipliers, to

$$\frac{m_1 a_1 + m_2 a_2 + m_3 a_3 + \dots\dots + m_n a_n}{m_1 b_1 + m_2 b_2 + m_3 b_3 + \dots\dots + m_n b_n};$$

and to

$$\frac{\{a_1^2 + a_2^2 + a_3^2 + \dots\dots + a_n^2\}^{\frac{1}{2}}}{\{b_1^2 + b_2^2 + b_3^2 + \dots\dots + b_n^2\}^{\frac{1}{2}}}.$$

Let each of the fractions $= r$, that is, let

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots\dots = \frac{a_n}{b_n} = r;$$

$$\begin{array}{lll} \therefore \quad a_1 = b_1 r & m_1 a_1 = m_1 b_1 r & a_1^2 = b_1^2 r^2 \\ \quad\;\; a_2 = b_2 r & m_2 a_2 = m_2 b_2 r & a_2^2 = b_2^2 r^2 \\ \quad\;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot & \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot & \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\ \quad\;\; a_n = b_n r & m_n a_n = m_n b_n r & a_n^2 = b_n^2 r^2. \end{array}$$

Whence, by addition and division,

$$\begin{aligned} r &= \frac{a_1 + a_2 + a_3 + \dots\dots + a_n}{b_1 + b_2 + b_3 + \dots\dots + b_n}; \\ &= \frac{m_1 a_1 + m_2 a_2 + m_3 a_3 + \dots\dots + m_n a_n}{m_1 b_1 + m_2 b_2 + m_3 b_3 + \dots\dots + m_n b_n}; \\ &= \frac{\{a_1^2 + a_2^2 + a_3^2 + \dots\dots + a_n^2\}^{\frac{1}{2}}}{\{b_1^2 + b_2^2 + b_3^2 + \dots\dots + b_n^2\}^{\frac{1}{2}}} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots\dots = \frac{a_n}{b_n}. \end{aligned}$$

Q. E. D.

The geometrical form of the first of these theorems is the twelfth proposition of the fifth book of Euclid.

Similarly,

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots\ldots = \frac{a_n}{b_n} = \frac{\{a_1{}^r + a_2{}^r + \ldots\ldots + a_n{}^r\}^{\frac{1}{r}}}{\{b_1{}^r + b_2{}^r + \ldots\ldots + b_n{}^r\}^{\frac{1}{r}}}.$$

II.

When three unknown quantities are involved in three (so called) *linear* equations of the forms

$$a_1x + b_1y + c_1z = d_1 \qquad (1)$$

$$a_2x + b_2y + c_2z = d_2 \qquad (2)$$

$$a_3x + b_3y + c_3z = d_3 \qquad (3)$$

the following is the most convenient method of determining each of the unknown quantities in terms of the constants.

Multiply (1) by λ_1, (2) by λ_2, (3) by λ_3, and add; then

$$(a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3)\,x + (b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3)\,y$$
$$+ (c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3)\,z = d_1\lambda_1 + d_2\lambda_2 + d_3\lambda_3.$$

As three undetermined quantities, viz. $\lambda_1\ \lambda_2\ \lambda_3$, have been introduced, we may make three suppositions respecting them: leaving one to be determined hereafter, let two be, that the coefficients of y and z be equal to zero; so that

$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = 0, \qquad c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 = 0;$$

whence by elimination,

$$\frac{\lambda_1}{b_2c_3 - c_2b_3} = \frac{\lambda_2}{b_3c_1 - c_3b_1} = \frac{\lambda_3}{b_1c_2 - c_1b_2}.$$

But thus the ratio only of the multipliers has been determined, and therefore *any* numbers bearing to one another the above ratio would satisfy the requisite conditions; to take however the most simple numbers, let the third condition be introduced, and be, that each of the above fractions be equal to unity; and therefore

$$\lambda_1 = b_2c_3 - c_2b_3, \qquad \lambda_2 = b_3c_1 - c_3b_1, \qquad \lambda_3 = b_1c_2 - c_1b_2;$$

and therefore

$$x = \frac{d_1(b_2c_3 - c_2b_3) + d_2(b_3c_1 - c_3b_1) + d_3(b_1c_2 - c_1b_2)}{a_1(b_2c_3 - c_2b_3) + a_2(b_3c_1 - c_3b_1) + a_3(b_1c_2 - c_1b_2)}.$$

By similar processes, if the coefficients of z and x had been equated to zero, the result would have been

$$y = \frac{d_1(c_2a_3-a_2c_3)+d_2(c_3a_1-a_3c_1)+d_3(c_1a_2-a_1c_2)}{b_1(c_2a_3-a_2c_3)+b_2(c_3a_1-a_3c_1)+b_3(c_1a_2-a_1c_2)},$$

and if the coefficients of x and y had been equated to zero,

$$z = \frac{d_1(a_2b_3-b_2a_3)+d_2(a_3b_1-b_3a_1)+d_3(a_1b_2-b_1a_2)}{c_1(a_2b_3-b_2a_3)+c_2(a_3b_1-b_3a_1)+c_3(a_1b_2-b_1a_2)}.$$

This method of elimination is generally known by the name of Lagrange's Rule of Cross-Multiplication, the origin of which term is sufficiently obvious from the *form* of the multipliers: two examples are subjoined for the sake of practice, but the student is recommended to exercise himself in many others.

Ex. 1.

$$\begin{array}{r|r|r|r|}
 & x & y & z \\
2x+4y+5z = 49 & 2 & 12 & -11 \\
3x+5y+6z = 64 & -1 & -12 & 10 \\
4x+3y+4z = 55 & -1 & 3 & -2
\end{array}$$

the values of the multipliers corresponding to the several variables being arranged in the annexed vertical rows: whence

$$\begin{array}{rl}
x(\ \ 4-\ 3-4) = & 98-\ 64-\ 55 \\
y(\ 48-60+9) = & 588-768+165 \\
z(-55+60-8) = & -539+640-110
\end{array}
\quad \therefore \quad \left.\begin{array}{l} x = 7 \\ y = 5 \\ z = 3 \end{array}\right\}$$

Ex. 2.

$$\begin{array}{r|r|r|r|}
 & x & y & z \\
3x-7y+4z = 1 & 7 & 4 & 1 \\
-5x+9y-\ z = 22 & -1 & -1 & -1 \\
x-2y+\ z = 0 & -29 & -17 & -8
\end{array}$$

the values of the several multipliers being arranged as before: whence

$$\begin{array}{rl}
x(\ \ 21+5-29) = & 7-22 \\
y(-28-9+34) = & 4-22 \\
z(\ \ 4+1-\ 8) = & 1-22
\end{array}
\quad \therefore \quad \left.\begin{array}{l} x = 5 \\ y = 6 \\ z = 7 \end{array}\right\}$$

The following is a particular case to which the preceding process of cross-multiplication is applicable.

Let there be three equations of the forms

$$\left.\begin{array}{r} a_1x+b_1y+c_1z = 0 \\ a_2x+b_2y+c_2z = 0 \\ a_3x+b_3y+c_3z = 0 \end{array}\right\},$$

which involve only two unknown quantities, say $\frac{x}{z}$ and $\frac{y}{z}$; because each equation may be divided through by z; hence it is evident that a relation exists between the coefficients: this relation is to be determined; and may be found in the following manner. Since $d_1 = d_2 = d_3 = 0$, if we multiply the first, second, and third severally by the multipliers determined as above, we shall have, after addition,

$$a_1 (b_2 c_3 - c_2 b_3) + a_2 (b_3 c_1 - c_1 b_3) + a_3 (b_1 c_2 - c_1 b_2) = 0,$$

which is the relation among the coefficients of the three preceding equations, when they coexist.

This theorem is one of the simplest in the method of Determinants: the general principles of which are explained in Mr. Spottiswoode's "Elementary Theorems relating to Determinants," Crelle's Mathematical Journal, Vol. LI, Berlin, 1856; and in the Appendix to "Salmon's Higher Plane Curves," Dublin, 1852.

III.

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ are quantities of the same sign, and $a_1, a_2, a_3, \ldots\ldots a_n$ are any other homogeneous quantities capable of addition and subtraction, then $a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 + \ldots + a_n \alpha_n$ is equal to $(\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n)$ multiplied into some quantity greater than the least, and less than the greatest, of the quantities $a_1, a_2, a_3, \ldots a_n$.

The proof of this proposition depends on the fact, if both terms of an inequality are multiplied or divided by a positive number, the sign of inequality remains the same; that is, the quantity which was greater before the multiplication is the greater after it; but if the terms are multiplied by a negative number, the sign of the inequality is reversed: that is, a $>$ is changed into a $<$, and a $<$ into a $>$. This is easily shewn by an example; as, for instance, 5 is greater than 2; let each side be multiplied by $+4$, then $20 > 8$; again, let each side be multiplied by a negative number, as -2: the $>$ is changed into a $<$, $-10 < -4$, because -10 is less than -4.

Let L be the least and G the greatest of the quantities $a_1, a_2, a_3, \ldots\ldots a_n$; then, with the exception of the two cases of the greatest and the least of the quantities, whereby however the final result is not vitiated, we have the following inequalities,

$$a_1 \text{ is } > \text{L}, < \text{G}$$
$$a_2 \text{ is } > \text{L}, < \text{G}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_n \text{ is } > \text{L}, < \text{G};$$

First, let the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ be positive, so that the signs of the above inequalities will not be changed when they are multiplied as follows:

$$\alpha_1 a_1 \text{ is } > \text{L}\alpha_1, < \text{G}\alpha_1$$
$$\alpha_2 a_2 \text{ is } > \text{L}\alpha_2, < \text{G}\alpha_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\alpha_n a_n \text{ is } > \text{L}\alpha_n, < \text{G}\alpha_n;$$

and therefore by addition

$$\alpha_1 a_1 + \alpha_2 a_2 + \ldots\ldots + \alpha_n a_n \text{ is } > \text{L}(\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_n)$$
$$- \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; < \text{G}(\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_n)$$
$$\therefore \quad \alpha_1 a_1 + \alpha_2 a_2 + \ldots\ldots + \alpha_n a_n = (\alpha_1 + \alpha_2 + \alpha_3 + \ldots\ldots + \alpha_n)$$
$$\times \text{ some } \textit{mean} \text{ value of the } a\text{'s},$$

signifying by *mean* value, a quantity greater than the least, and less than the greatest.

Secondly, if $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ be negative, the signs of the inequalities would have been changed in the first multiplication, and would have been again changed in the final result, because $\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_n$ would be a negative quantity, and thus the same result would follow: and therefore the proposition is proved.

IV.

If $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots\ldots \frac{a_n}{b_n}$ is a series of fractions, the numerators of which are of either sign, and *the denominators all of the same sign*, then $\frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n}$ is equal to some quantity greater than the least and less than the greatest of the given fractions.

First, let all the denominators be positive, and let L be the least and G the greatest of the fractions; then

$$\frac{a_1}{b_1} \text{ is } > \text{L}, < \text{G}$$
$$\frac{a_2}{b_2} \text{ is } > \text{L}, < \text{G}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\frac{a_n}{b_n} \text{ is } > \text{L}, < \text{G}.$$

Let the inequalities be severally multiplied by the positive quantities $b_1, b_2, b_3, \ldots b_n$, by which process the signs are not changed; then

$$\begin{aligned} a_1 &\text{ is } > \mathrm{L}b_1, < \mathrm{G}b_1 \\ a_2 &\text{ is } > \mathrm{L}b_2, < \mathrm{G}b_2 \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_n &\text{ is } > \mathrm{L}b_n, < \mathrm{G}b_n; \end{aligned}$$

and therefore by addition

$$\begin{aligned} a_1+a_2+a_3+\ldots+a_n &\text{ is } > \mathrm{L}(b_1+b_2+b_3+\ldots+b_n) \\ \text{- - - - - - -} &\text{ is } \overset{<}{\text{[illegible]}}\ \mathrm{G}(b_1+b_2+b_3+\ldots+b_n), \end{aligned}$$

and therefore

$\dfrac{a_1+a_2+a_3+\ldots+a_n}{b_1+b_2+b_3+\ldots+b_n}$ is $> \mathrm{L}, < \mathrm{G}$, and therefore is equal to some *mean* value of the fractions. Q. E. D.

Secondly, let $b_1, b_2, b_3, \ldots b_n$ be negative; then, as before;

$$\begin{aligned} \frac{a_1}{b_1} &\text{ is } > \mathrm{L}, < \mathrm{G} \\ \frac{a_2}{b_2} &\text{ is } > \mathrm{L}, < \mathrm{G} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{a_n}{b_n} &\text{ is } > \mathrm{L}, < \mathrm{G}; \end{aligned}$$

let these inequalities be severally multiplied by the negative quantities $b_1, b_2, b_3, \ldots\ldots b_n$, so that the signs of them are changed; then

$$\begin{aligned} a_1 &\text{ is } < \mathrm{L}b_1, > \mathrm{G}b_1 \\ a_2 &\text{ is } < \mathrm{L}b_2, > \mathrm{G}b_2 \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_n &\text{ is } < \mathrm{L}b_n, > \mathrm{G}b_n; \end{aligned}$$

$$\begin{aligned} \therefore\quad a_1+a_2+a_3+\ldots+a_n &\text{ is } < \mathrm{L}(b_1+b_2+\ldots+b_n) \\ \text{- - - - - - - -} &\text{ is } > \mathrm{G}(b_1+b_2+\ldots+b_n); \end{aligned}$$

whence, bearing in mind that the sign of an inequality is changed when it is divided by a negative quantity,

$$\frac{a_1+a_2+a_3+\ldots+a_n}{b_1+b_2+b_3+\ldots+b_n} \text{ is } > \mathrm{L}, < \mathrm{G}.$$

Q. E. D.

DIFFERENTIAL CALCULUS.

PART I.

ANALYTICAL INVESTIGATIONS.

CHAPTER I.

GENERAL PRINCIPLES, AND EXPLANATION OF TERMS.

SECTION 1.—*Introductory; on Number, its properties, affections, and science.*

ARTICLE 1.] Infinitesimal Calculus is a branch of that science, the aggregate of the rules and operations of which French writers call "Le Calcul," but for which we have no more specific name than the Science of Number*. A metaphysical inquiry into the origin and nature of *number* would be out of place in a didactic treatise such as the present, and would also be superfluous; because it will be sufficient for the student to have that notion of it which an ordinary knowledge of arithmetic and algebra implies; but it is well to recall his attention to certain axiomatic properties of it, and to bring into greater prominence those from which the Infinitesimal Calculus is deduced.

Number is *the ratio or relation which two quantities of the same kind bear to one another in respect of quantuplicity.* By quantity I mean whatever is capable of measurement; whether it be geometrical space, or weight, or time, or heat, or light, or velocity, or any thing else; that, viz., of which we can predicate muchness in reply to the question "how much?" or number of times in answer to "how many times?" The above phænomena may be severally the substrata of mixed sciences, as they are called, but they can only be treated of in accordance with the rules of the science of number, because they are capable of addition and division and measurement, at least in conception, if not in act; and it is only when a problem can be reduced to a question of pure number that it can be brought within the domain of the Calculus; and when this condition is satisfied, the science of number supplies, as it were, the skeleton or framework on which such mixed science is treated; and considers the subject-matter not in its concrete and physical and phænomenal, but in its abstract state, as measured, and in the measure, as the correct representative of it. It does not take cognizance of this or that weight or colour, but of the *number of times* such a weight contains another weight, and so on; and this is what I mean by the terms "in respect of quantuplicity" in the above definition of number; and thus it is that the science is so general, almost universal, because all the subject-matter of the Physical Sciences conforms to its requirements in respect of admitting of measurement.

---

* M. Comte writes, "Le Calcul a pour objet propre de résoudre toutes les questions de *nombres*."—*Philosophie Positive*, vol. i. p. 143.

2.] There are two modes of measuring quantities, and thus of arriving at abstract numbers from the concrete magnitudes. Firstly, a certain amount of the given quantity is taken, which for the sake of convenience is called an unit of that particular quantity, and with it any other amount is compared, the principle of comparison being assigned by the particular science whose subject-matter the quantity is; and if the latter amount be divisible into two or three or more parts, severally equal to each other and to the unit, we say that the latter amount is twice or three times or more times the unit, and thus arrive at the abstract number; and the problem of determining other relations arising out of this one belongs to the science of number. The unit, it is to be observed, is arbitrary; thus, for instance, if a certain volume of matter of given density, say a cubic inch of distilled water, be considered the volume-unit, and its density be called the density-unit, then if two cubic inches of distilled water could be compressed into one inch, its density would be *two*. Hereby we arrive at abstract number by directly comparing any given concrete quantity with the unit of that quantity. Secondly, we may estimate quantity without *formally* introducing the unit; for suppose of two unequal quantities, one to be divisible into two equal parts, and the other into three parts, equal to each other and to each of the divided parts of the former quantity; then, although if one of the equal parts were taken as the unit, one quantity would contain two and the other three units: yet we may omit the unit, and say that the ratio of one quantity to the other is that of the numbers two to three. Thus, if two lines admit of being resolved into 3 and 7 parts respectively, which are equal to each other, the ratio of the lines will be represented by the numerical ratio $3 : 7$; and this mode of measuring quantities is independent of the amount of the chosen unit; for if the above lines had been divided into 6 and 14 equal parts respectively, or into $3n$ and $7n$ equal parts respectively, the unit would have been only one-half or one-nth part of what it was in the previous resolution; and yet the ratio of the lines, or of the numbers which represent them, viz. $6 : 14$, or $3n : 7n$, would have been the same as before. And this mode of representing lines by numbers is equally applicable to areas, hours, weights, &c. all of which may be so related to each other as to admit of resolution into 3 and 7 equal parts respectively, and thus be represented by the ratio of the numbers $3 : 7$. It is manifest that only quantities of the same kind can be measured in either of the above methods, and hence it is only from comparing *homogeneous* quantities that numbers can be formed. The principle and mode of comparison however must be assigned by the particular science whose subject-matter the quantity is; thereby are its concrete materials abstracted and brought within the range of the Science of Number, which has thus to deal with only abstract quantuplicities; and thus (which is a point of the utmost importance, and deserves the most careful attention) the symbols $2, 3, 4, \ldots\ldots a, b, c, \ldots\ldots x, y, x$, do not represent concrete quantities, such as 2 *ounces*, or a *hours*, or x *feet*, but *abstract numbers;* and it is the properties of these that the Science of Number has to discuss*.

3.] The distinguishing characteristic of such numbers is, that they remain after any operation the same in *kind* which they were before. When two numbers are added or subtracted, the sum or difference is a number; when two numbers are multiplied or divided, the resultant is the same in kind as each of the components: as they are numbers, so is it number. Thus $2 \times 3 = 6$, and 6 is an abstract number of the same kind as 2 and 3; that is, twice thrice is equivalent to six times; $8 \div 4 = 2$, that is, one-fourth part of eight times is twice. The same is also true of the operations of Involution and Evolution. These remarks are important, because if the symbols represent concrete quantities the results would be otherwise; thus in the *analogous* operation of geometrical multiplication, if two linear inches be "multiplied" by three linear inches, the result is six square inches: and therefore in the process we have changed from linear to superficial quantity. And so, if the six superficial inches be multiplied by four linear inches, the result is 24 cubic inches, or inches of volume, and thus by the last process we have passed from superficial magnitude to solid content. By the operations then the kind has changed; and as all the dimensions which space admits of have been exhausted, it is impossible to multiply together more than three geometrical lines; thus two linear inches multiplied into themselves four times is an impossibility; or, in other words, a geometrical line cannot be raised to any power above the third. And the possibility of extracting roots is confined within narrower limits; thus the square root of a superficial area is a possible quantity, and so is the cube root of a solid content, being in each case a line. The case of geometrical multiplication is the most favourable one, for in other subject-matter, which is only uni-dimensional, we cannot at all multiply the concrete quantities, and can only divide them when they are homogeneous. It is absurd to speak of a pound multiplied into a pound, or of the product of an hour by an hour: such operations are impossible, and have no meaning; but we can divide two pounds by one pound, and thereby arrive at the abstract number 2, because the inverse process is possible, and may therefore be undone; that is, because we can multiply any concrete unit by an abstract number; but we can no more divide pounds weight by pounds sterling than we can multiply hours by degrees of heat. These remarks on the abstract character of Number are relevant to the present subject, inasmuch as they prove the correctness of an expression such as

---

* The two modes of measuring quantities and of thereby forming numbers, correspond to the two aspects in which arithmetical fractions are viewed; one in which the denominator assigns the unit, and the numerator the number of times it is to be taken, and the other in which they are considered as the expressions of arithmetical ratios. See Peacock's Algebra, 2nd edition, vol. i. pp. 54, 155.

$$ay^2 + bxy + cx^2 + dy + ex + f;$$

because each symbol by itself expressing a number, each term, whether consisting of one or two or three factors, is a number also: and thus they may be added, and the whole expression is homogeneous and correct. Whereas, did such symbols represent concrete quantities, as, for example, geometrical lines, the first three terms would represent solid content, the next two superficial area, and the last lineal length: and thus they would be heterogeneous, and could not be added. Also from such a point of view, a term ay^3, consisting of more than three dimensions, would be uninterpretable and impossible.

Hence we conclude, that

(1) The symbols, whose laws and combinations are considered in the Science of Number, express the number of times any thing is taken; and the science, disregarding the concrete thing, discusses the properties of the abstract number.

(2) As Infinitesimal Calculus is a branch of the Science of Number, the symbols which will be employed in the following work represent number only; and the properties of number will be considered only as they are represented by symbols.

This latter conclusion is important, as it restricts the subject-matter to symbols, and our discussion to their laws and properties.

4.] The numbers or quantities which are employed in the following treatise are of two kinds, *constants* and *variables*.

Constant numbers are those which have the same determinate value throughout a given operation or problem: though in another operation, or considered in another relation, they may vary. Such are the symbols 2, 3, 4, and for the most part those in algebra, which are represented by the early letters of the alphabet: constant numbers are specific in form and value.

A *variable* number is that which is capable of receiving values different from each other, and generally admits of any value, though it may by the conditions of a problem be restricted to values of a particular kind, or within certain limits; variable quantities are *general* in form, though they admit of specific values: they are generally represented by the later letters of the alphabet.

5.] That we may avoid misconception as to the following terms, which will be frequently employed, it is necessary to give detailed explanations of them; viz. *Definite*, *Indefinite*; *Infinite*, *Finite*, *Infinitesimal*.

Definite means *determinate* and *assigned;* thus constants are definite. Indefinite numbers are those whose values are not assigned; they are represented by general symbols, and may therefore, so far, have any value.

To form an accurate and due conception of the latter three terms, and of the means of symbolizing and estimating them, requires a knowledge of the whole Calculus: we must therefore have recourse to analogous illustration, in hope that the student may glean from it such notions, imperfect though they will be, as will enable him to understand the technical language of the science.

By *finite* we generally mean that which is within reach, or may be brought within reach, of our senses. Thus a ton, or an ounce, may be taken as the unit of weight, and any number of tons or ounces which the senses perceive would be considered a finite weight; and many animalculæ, which, on account of their minuteness, are beyond the power of unassisted vision, would nevertheless be considered finite, because they may be brought within it by means of the microscope, and may be measured. Or again, inasmuch as the senses are the media by which impressions of external objects are conveyed to the mind, and as the mind conceives them when so conveyed, we apply the term *finite* to those magnitudes, the relation of which to other magnitudes of the same kind the mind is capable of conceiving*. The powers therefore of our senses and mind place the limit to the finite; but those magnitudes which severally transcend these limits by reason of their being too great or too small, we call *infinite* and *infinitesimal* (or infinitely small). Thus, when our senses fail to perform their office of transmitting to the mind what it would think about, by reason of the object being too large or too small, or when for a similar reason the mind fails to be capable of considering the relation of such objects, and when the most delicate subsidiary instruments for assisting the senses are employed to bring within their reach what was beyond them, and yet in vain, then we are on the boundary of the infinite or infinitesimal: of the infinite if the object be too vast, of the infinitesimal if it be too minute. Physical Science affords instances of both these cases. The distances of those fixed stars of which the parallax has not been discovered must be so great, that 400 millions of miles are not appreciable in comparison of them; considering then these millions of miles to be a finite quantity, yet no *sensible* change is made in the distance of a star by the addition or subtraction of them. Nay, more than this; we can employ our millions of miles to greater advantage; we can make them the base of a triangle whose vertex is the star, and yet, great as is the delicacy of our astronomical instruments, the sides of the triangle are to all appearance parallel. This then is a case where we cannot compare two geometrical distances, on account of the immensity of one of them. Considering then the 400 millions of miles to be a *finite* quantity, the distance of the star is *infinite*. Again: if one grain weight of aloetic acid be added to five pounds of pure water, the whole will after a short time assume a fine crimson colour, which could not happen unless the grain of aloetic acid had been divided and equally diffused throughout the whole volume. Now it is possible to see a quantity of water as small as a thousandth part of a grain, and such a portion of the solution would contain a thirty-five millionth part of a grain of aloetic acid. We have therefore actually divided this substance into thirty-five millions of parts, and the most delicate microscope does not so far magnify the atoms of the acid that they should be separately visible in the water; yet there they are, and are so small as to be beyond the limit of our vision, even though it be increased many thousand times: they are *infinitesimal*, though the sum of them is finite; and as they are so small, there must be an infinity of them. Hence also we have a new aspect of such quantities. In reference to a finite quantity, infinity and infinitesimal are reciprocal terms, each implying the other; the finite quantity may be the infinitesimal infinitely-quantupled, and the infinitesimal an element of the finite quantity, when it is resolved into an infinity of parts. Again: an infinite quantity may be so large, as not only to surpass the compass of our senses, but also to surpass quantities which are from their magnitude beyond them; that is, there may be infinite quantities beyond infinite quantities, and others again beyond these: and thus there may be quantities infinitely greater than infinities, and there may be *orders of infinities*. Astronomy supplies instances of such quantities, to an extent within the reach of sight, by means of the telescope, but beyond the range of the micrometer as a means of measuring distance; assuming as the unit of length the mean radius of the earth's orbit, which is about 95 millions of miles, we can compare with it the mean radii of the orbits of the other planets, and thus determine the *relative* sizes of their orbits; but when we extend our observations to other bodies in the celestial space, we find stars situated at such a distance from the sun, that, taking the star to be the vertex of an isosceles triangle, and the base to be a line through the sun's centre and of 190 millions of miles in length, the vertical angle of the triangle is less than $1''$, and in the case of Capella is computed to be $0''.046$; were the vertical angle $1''$, it can easily be shewn that the distance would be 20 billions of miles; and as it is determinable, we may say that it is comparable with finite quantities, though on the verge of the infinite. "In such numbers the imagination is lost; the mode we have of conceiving such intervals at all is by the time which it will take light to traverse them. Light, we know, travels at the rate of 192 thousand miles per second; it will occupy therefore three years and eighty-three days to traverse the distance in question. Now as this is an inferior limit, which it is already ascertained that even the brightest and therefore (in the absence of all other indications) the nearest stars exceed, what are we to allow for those innumerable stars of the smaller magnitudes which the telescope discloses to us? What for the dimensions of the galaxy in whose remoter regions the united lustre of myriads of stars is perceptible in powerful telescopes as a feeble nebulous gleam*?" Here then we have not only finite but also infinite distances, and spaces infinitely greater than these infinite distances; that is, we have successive orders of infinities, and in an ascending scale from finite distances, of which our senses are cognizant, to those infinite spaces which surpass our powers of measurement.

---

* Peacock's Algebra, vol. ii. p. 294.

So again may any one of the small particles of the aloetic acid which has been dissolved in the water be conceived to be analysed into other parts infinite in number, each one of which will therefore be infinitesimally small in comparison of its original particle; that is, one small particle may be conceived to be distributed through a finite volume, and thus to be resolved into other particles infinitely less than itself: and thus we may arrive at orders of infinitesimals, each one being infinitesimally less than that of which it is an element. It is also to be observed, that in the resolution of a finite quantity into infinitesimals of successive orders, different orders of infinities arise which severally correspond to the orders of infinitesimals, to which they are so related that the product of the infinity and infinitesimal of the same order is equal to the original finite quantity.

---

* Sir John Herschel's Outlines of Astronomy, Art. 800 and following.

Here perhaps it may be asked, *when* does a quantity pass from the finite to the infinite and to the infinitesimal? How many finite quantities must be added to make an infinity, and into how many parts must a finite quantity be resolved so that each should be infinitesimal? An answer to such questions may be beyond our power; and it may be a matter of words only: but it is also beside the object and requirements of the Calculus. We have nothing to do with concrete quantities; the instances above cited are for the sake of *illustration* only: to give the reader a rough notion of the principles; such as may serve their purpose until other and more accurate ones take their place; our subject-matter is *number*, and number as represented by symbols: and the form of the symbols, and the subsidiary symbols which will be derived from them, as will be shewn in the sequel, enable us to overcome the apparent difficulty. For we shall create our numbers, and our subsidiary numbers, subject to certain laws: and therefore, so long as they are applied within the conditions prescribed by these laws, all results correctly inferred will also be correct.

6.] To resume then the course of the exposition from the end of the 4th Article: variable number may change value in two ways, either *continuously* or *discontinuously*.

A *quantity* or *number* varies discontinuously when it passes abruptly from one value to another, as by the addition of a finite quantity. Thus the passage from 1 to 2, and from 2 to 3, and so on, is made discontinuously, viz. by the abrupt addition of the number 1; similarly, if x is finite, we pass from $2x$ to $4x$, and from $4x$ to $8x$, by the successive addition of $2x$ and $4x$; and the changes are made "per saltus."

But continuous increase is when number *grows*, that is, passes from one value to another only by going through all the intermediate numbers, whereby the successive increments or augments which the numbers receive are infinitesimal; thus, if we pass from 3 to 4 not only by going through 3.1, 3.2, 3.3, and if we pass from 3 to 3.1 not only by going through 3.01, 3.02, 3.03, ... and if we pass from 3 to 3.01 not only by going through 3.001, 3.002, ... and so on to any *finite* number of divisions, the increase is discontinuous; but if the number of divisions be infinite, and if the lesser number pass into the greater number by receiving at each successive step an infinitesimal increase, the mode of increase is continuous. For the sake of illustration let us consider the case of motion. Consider the gliding motion of a worm, and suppose it to pass uniformly over an inch in a minute; if the space through which the worm has passed be estimated at the end of each minute only, the space will apparently be discontinuously increased by an inch: but if the space be measured at the end of each infinitesimal or very short lapse of time, the increase during that instant will be very small, and if the instants be infinitesimal, the space, we say, will have increased by infinitesimal increments. The earth's motion in its orbit, the running out of water, the gradual radiation of heat, the growth of a tree, are all instances of a similar continuous increase; and it is worth observing, that it was from such cases that the Calculus had its origin. But if we count horses or men, we count discontinuously: we pass "per saltum" from one man to two men; we cannot divide a man into infinitesimal elements; each man is an unit whose personal existence does not admit of such infinitesimal subdivision. Hence it appears, that numerical continuity requires infinite numerical divisibility, and expresses the property of quantity considered under the aspect of generation by growth: thus the difference of the two modes of increase is one of degree and not of kind. Hence also we have a criterion of them; the difference between two successive numbers is finite or infinitesimal, according as the mode of increase is discontinuous or continuous.

7.] The subject-matter of arithmetic and of algebra (commonly so called) is discontinuous number. The numbers 8, 9, 10, ... $a, b, c, \dots x, y, z, \dots$ as they are commonly employed, are discontinuous; we pass from one to another "per saltus," and do not contemplate the mode of continuous increase. The distinction between the two sciences appears to be the following: In arithmetic are discussed the properties of numbers which have certain determinate values, and can have none other; in algebra we treat of symbols which are general in form, and either have specific values, as the constants a, b, c; or admit of having one or more such values, as the variables x, y, z.

Infinitesimal Calculus, on the contrary, considers number in its aspect of continuous growth. In this lies its distinctive character: for whereas Arithmetic and Algebra treat of finite and discontinuous number, it treats of continuous, and especially of infinite and infinitesimal number.

These terms however are at present too vague to be the foundations of philosophical investigations, and therefore I proceed in this and the four following Articles to enuntiate certain axioms concerning infinitesimals and infinities, their orders and their relations, which flow from an adequate conception of continuous number, and which do not admit of proof by deduction from more general principles. Thus our method of inquiry is similar to that of pure geometry; and we shall shortly find our notions of the *materies* of infinitesimal calculus sufficiently definite to enable us to deduce from them results which will not be wanting in precision.

The value towards which an expression converges nearer than by any assignable difference, while the symbol on which it depends approaches to any assigned value, is called a *limit* or *limiting value.* If the assigned value of the symbol be zero, the limit is called the *inferior limit;* and if the value be infinity, it is called the *superior limit.*

Thus the inferior limit of $\frac{1}{1+x}$ is 1; although for every value of x greater than 0 the quantity is less than 1, yet the nearer x approaches to 0, the less becomes the difference between $\frac{1}{1+x}$ and 1; and the superior limit is 0; for as x increases, the quantity becomes less and less, and ultimately, when x is greater than any assignable quantity, the difference between $\frac{1}{1+x}$ and 0 is less than any quantity, and thus the limit is zero. So again, as the difference between x and -1 becomes less than any assignable quantity, $\frac{1}{1+x}$ approaches to infinity; similarly the inferior limit of $\tan x$ is 0, and, as x becomes $\frac{\pi}{2}$, the difference between $\tan\frac{\pi}{2}$ and infinity vanishes, and infinity is the limit of $\tan\frac{\pi}{2}$.

Again: suppose that we have a series of the form

$$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots\ldots \frac{n+1}{n},$$

the number of terms of which is infinite; although the terms become less and less as we proceed from left to right, yet the value of the last term does not approach to 0 but to unity, as is manifest from the form of it; unity then is the limit of the last term of the series.

In geometry a circle is the limit towards which the perimeter of an inscribed polygon converges, as the number of sides is infinitely increased, and as thereby the lengths of the sides become infinitesimally small. Some examples of finding limits will be given in the sequel. Hence absolute *zero* is the inferior limit of an infinitesimal, and absolute infinity is the superior limit of a quantity which is greater than any assignable quantity.

8.] The symbols by which we shall represent an infinity and an infinitesimal are ∞ and 0: the relation of which is, that if a represent a finite quantity, $\infty = \frac{a}{0}$, and $0 = \frac{a}{\infty}$. We shall attach a more definite meaning to these relations, if we consider a dividend to be the product of the divisor and quotient; thus there is no *finite* quantity which, when multiplied into an infinitesimal, will produce a finite product: nothing short of an infinity can do it; and from the illustrations of Article 5 it appears that it must be an infinity of a particular kind. It will be observed, that 0 does not represent absolute zero, and that ∞ does not express absolute infinity.

If then any finite numerical quantity be divided into any number of equal or unequal parts, as the case may be, the larger the number of parts is, the smaller is each part; and if the number of parts be infinitely great, each part is an infinitesimal: and the less the difference is between the number of parts and absolute infinity, the less is also the difference between each part and absolute zero. Suppose a to be a finite determinate quantity, and to be divided into x equal parts; then each part $= \frac{a}{x}$; and, if x is infinitely great, $\frac{a}{x}$ is an infinitesimal, x and $\frac{a}{x}$ being thus symbols of an infinity and an infinitesimal, which mutually imply and are reciprocal to each other.

Suppose again $\frac{a}{x}$ to be divided into x equal parts, then each part is equal to $\frac{a}{x^2}$, and x^2 is the number of parts into which a has been divided: thus x^2 and $\frac{a}{x^2}$ severally represent an infinity and an infinitesimal, which are reciprocal to and mutually imply each other. By similar and subsequent divisions we may find x^3, $\frac{a}{x^3}$, ... x^n, $\frac{a}{x^n}$, and therefore other infinities and infinitesimals which are relative to each other.

Or again, suppose i to be an infinitesimal element of a, so that a is divided into $\frac{a}{i}$ equal parts, then $\frac{a}{i}$ is an infinity, and is relative to the infinitesimal i. And again, suppose a to be resolved into elements each of which is equal to i^2, then $\frac{a}{i^2}$ is the number of equal parts, and $\frac{a}{i^2}$ is an infinity which is relative to the infinitesimal i^2; similarly by subsequent resolutions may other infinitesimals and infinities i^3, $\frac{a}{i^3}$, ... i^n, $\frac{a}{i^n}$, be formed, which mutually involve each other.

Now although *generally* infinities and infinitesimals are symbolized respectively by ∞ and 0, yet it is manifest that all of each kind are not equal; not only do infinitesimals differ from absolute zero, but they may also differ from each other: and so may infinities differ from each other, and from a quantity which transcends every assignable quantity, that is, from absolute infinity. Hence the need of classifying such quantities.

Assuming then the order to depend on the exponent, it is plain that such *orders* must exist relatively to a certain determinate quantity, which is the subject of the exponent, and which we call the *base*. Taking therefore x to be the *base* of infinities, let x^2, x^3, ... x^n be infinities of the second, third, ... nth orders; and taking i to be the base of infinitesimals, let i^2, i^3, ... i^n be infinitesimals of the second, third, ... nth orders respectively. Similarly $\frac{a}{x}$, $\frac{a}{x^2}$, ... $\frac{a}{x^n}$ are infinitesimals of the first, second, ... nth orders, if x be the infinity-base; and $\frac{a}{i}$, $\frac{a}{i^2}$, ... $\frac{a}{i^n}$ are infinities of the first, second, ... nth orders, if i be the infinitesimal-base: such properties evidently involve each other, if i and x are so related that $x = \frac{1}{i}$, or $i = \frac{1}{x}$. These, it is to be observed, are the *definitions* of the orders of infinities and infinitesimals. Similarly, if $x^{\frac{1}{2}}$ be the infinity-base, x, $x^{\frac{3}{2}}$, x^2 ...... would be infinities of the second, third, fourth orders; and if $i^{\frac{1}{3}}$ be the infinitesimal-base, $i^{\frac{2}{3}}$, i, $i^{\frac{4}{3}}$ ...... would be infinitesimals of the second, third, fourth orders respectively. *Order* then, it is to be remembered, is relative to a certain base.

Hence then it appears, that there will be a scale of infinities and of infinitesimals in regular sequence: such that an infinity of the nth order must be infinitely subdivided to produce an infinity of the $(n-1)$th order, and infinitely quantupled to produce one of the $(n+1)$th order: infinitesimals also bear such relations to those, on either side of them in the scale, that they are infinitesimal parts of the one, and the aggregate of an infinity of the other. Thus, if x be the symbol of infinity as above, x^0 will be the symbol of the finite quantity, and the scale will be

$$x^n, \dots x^2, \; x^1, \; x^0, \; x^{-1}, \; x^{-2}, \dots x^{-n};$$

and if i be the symbol of an infinitesimal, i^0 will represent the finite quantity, and the scale will be

$$i^{-n}, \dots i^{-2}, \; i^{-1}, \; i^0, \; i^1, \; i^2, \dots i^n,$$

the order in each scale being a descending one. Hence also, using the *general* symbols of such infinitesimals and infinities, viz. 0 and ∞, the scales become

$$0^{-n}, \dots 0^{-2}, \; 0^{-1}, \; 0^0, \; 0^1, \; 0^2, \dots 0^n,$$

$$\infty^n, \dots \infty^2, \; \infty^1, \; \infty^0, \; \infty^{-1}, \; \infty^{-2}, \dots \infty^{-n}.$$

Thus then, although the mind is incapable of forming adequate notions of infinities and infinitesimals as they were described in rough outline in Article 5, yet they may be brought within its grasp when they are symbolized as above*. It is true that they do not always present themselves under the simple forms herein investigated; but in a subsequent chapter methods will be discussed for determining the orders of the more complex forms: and it will then appear that we can always determine the order of an infinity or of an infinitesimal relatively to a given base; and we can thus arrange and collect into groups all those which are of the same order. Primarily, the chief work of the Calculus will be the formation of subsidiary infinitesimals and infinities by means of, and from, finite quantities: but ere we enter on that work, we must plainly state the laws to which such infinitesimals and infinities are to be subject: these laws are of the utmost importance; they are indeed the conditions under which our subsidiary infinitesimals and infinities exist at all; to us, at present, it is immaterial whether they are geometrical or other representatives, or only analogues: but it is necessary that they should be of a nature which is not inconsistent with the nature of number. If any one hesitates to accept the following propositions, which I have called Theorems, but which are in fact axiomatic statements of infinitesimals and infinities, let him bear in mind, that our quantities are what they are only by virtue of these conditions. In all the following Theorems the base is assumed to be the same.

---

* See Poisson, Traité de Mécanique, Tome I, pp. 14, 16, 2^{de} ed. Paris, 1833.

9.] Theorem I.—Infinities and infinitesimals, like finite quantities, admit of being multiplied and divided by finite numbers, and their order is not thereby changed: but multiplication or division by the base or any power of it changes the order of the infinity and of the infinitesimal.

Thus x^2 and $2x^2$ are infinities of the same order, and i and $\frac{i}{4}$ are infinitesimals of the same order; $i^n \times i^m = i^{n+m}$, that is, by multiplication of the base raised to a power the order of the infinitesimal is changed.

Theorem II.—The product of an infinity and of an infinitesimal of the same order is a finite number.

Thus $x^2 \times \frac{a}{x^2} = a$; $i \times \frac{a}{i} = a$.

Theorem III.—The product of an infinity and of an infinitesimal of different orders is an infinity or an infinitesimal, according as the order of the infinity is higher or lower than that of the infinitesimal; and the order of the product depends on the difference of the orders of the component factors.

Thus $x^3 \times \frac{a}{x} = ax^2$; $x^n \times \frac{a}{x^m} = ax^{n-m} = \frac{a}{x^{m-n}}$, the former or latter form being taken according as n is greater or less than m, and therefore the result accordingly is an infinity or an infinitesimal. Similarly, $i^2 \times \frac{a}{i^4} = \frac{a}{i^2}$.

THEOREM IV.—The ratio to each other of two infinities or infinitesimals of the same order is finite.

Let a and b be two finite numbers, and thus let $\frac{a}{i^n}$, $\frac{b}{i^n}$ be two infinities of the same (viz. the nth) order; then their ratio is $a : b$; similarly, if ai^n, bi^n be two infinitesimals of the nth order, their ratio is $a : b$, that is, the same as before. This is also manifest geometrically. Let there be two concentric circles, the radius of one of which is double that of the other, and in them let two regular polygons of the same number of sides be described; each side of the larger is always double each side of the smaller; and as this is true whatever is the number of the sides, it is true when the number is infinitely great; in which case each side becomes infinitesimally small: and if the number in both polygons is the same, the sides are infinitesimals of the same order, and bear to each other the finite ratio of 2 : 1.

Hence also it follows that quantities, whose symbolical form is $\frac{0}{0}$, are indeterminate by virtue of that form, and may be either infinite, finite, or infinitesimal, and that such determinate values depend on the relation of the order of infinitesimal in the numerator to that in the denominator; that is, if the infinitesimal in the denominator be of a higher order than that in the numerator, the determinate value is infinite; if the orders are the same, the value is finite; and if that in the numerator is higher than that in the denominator, the value is infinitesimal.

Thus $\frac{(a^2-x^2)^{\frac{1}{2}}}{a-x} = \frac{0}{0}$ when $x = a$; but dividing out $(a-x)^{\frac{1}{2}}$, the result is $\frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}} = \frac{(2a)^{\frac{1}{2}}}{0} = \infty$, when $x = a$.

Similarly $\frac{a(1-x^2)}{b(1-x)} = \frac{a \times 0}{b \times 0}$, when $x = 1$, $= \frac{a(1+x)}{b} = \frac{2a}{b}$; and $\frac{(a-x)^2}{a^2-x^2} = \frac{0}{0}$, when $x = a$, $= \frac{a-x}{a+x} = \frac{0}{2a}$ when $x = a$.

Similar results are also manifestly true of infinities and their different orders.

THEOREM V.—The sum of two infinities or infinitesimals of the same order is the product of the infinity or infinitesimal by the sum of their coefficients; and the difference is an infinity or infinitesimal of the same order, except when the coefficients are equal, in which case it is absolutely zero.

Thus $ai^n + bi^n = (a+b)i^n$, $ai^n - bi^n = (a-b)i^n$, $ai^n - ai^n = 0$.

THEOREM VI.—Since an infinitesimal is derived from a FINITE number by the resolution of the finite number into an infinity of parts, the ratio of a FINITE number of such infinitesimals to the original number, is that of 0 to 1; a finite number therefore of such infinitesimal parts can have no value at all when added to a finite quantity: it must be neglected.

Thus if a and b are finite numbers, and i be an infinitesimal, of such an expression as $a+bi$, the latter part must be neglected; bi has no value at all when added to a.

THEOREM VII.—For a similar reason a finite quantity can have no value when added to an infinity, and must therefore be neglected.

Thus of $ax+b$, if a is a finite number and x is an infinity, the finite quantity b must be neglected, and the expression is equal to ax.

Similarly, in expressions involving the sum or difference of infinitesimals of different orders which have finite coefficients, all the higher infinitesimals must be neglected, and the lower ones alone retained. Thus let a and b be two finite quantities, and i^n and i^{n+r} two infinitesimals; then

$$ai^n + bi^{n+r} = i^n\{a + bi^r\},$$

the latter part of which is equal to a by Theorem VI; and therefore

$$ai^n + bi^{n+r} = ai^n.$$

Similarly, $a + bi + ci^2 + \dots + ki^n = a.$

And similarly, if an expression involves the algebraical sum of infinities of various orders, whose coefficients are finite, the expression is equal to the infinity of the highest order, and all the others, and the finite quantities, can have no value, when added to it, and must be neglected.

10.] To enable the student to appreciate the importance of the above theorems, some examples are subjoined:

Ex. 1. To find the inferior and superior limits of

$$\frac{ax^3+bx^2+cx+e}{mx^3+nx^2+px+q}.$$

By Theorem VI the inferior limits of the numerator and denominator are severally e and q, and by Theorem VII the superior limits are severally ax^3 and mx^3; hence the superior and inferior limits of the fraction are severally $\frac{a}{m}$ and $\frac{e}{q}$.

Ex. 2. To find the inferior and superior limits of

$$\frac{b+a^x}{a+b^x}.$$

If $x=0$, the inferior limit becomes $\frac{b+1}{a+1}$; and if $x=\infty$, the limit is $\left(\frac{a}{b}\right)^{\infty}$, which is ∞ or 0, according as a is greater or less than b.

11.] If any one idea or conception is pregnant with the whole Calculus, it is that contained in Theorems VI and VII; they enuntiate the essential properties of infinitesimals and their reciprocal infinities: such as flow immediately from, inasmuch as they are involved in, any adequate notion of such a mode of resolution as the Calculus contemplates; were not the properties of infinitesimals such as the theorems import, the Calculus would not be what it is: from them it takes its rise, and whatever its genius be, such have they imparted to it.

On inspecting the scales of infinities and infinitesimals which are given above, it will be observed that the finite quantity is represented by the symbol which has 0 for its exponent: the reason of which by the common law of indices is manifest from the examples given in illustration of Theorem IV of Article 9; and on the correctness of thus representing it more will be said hereafter. And it will also be observed, that all the symbols on one side of it represent infinities, and all on the other infinitesimals; but it is quite arbitrary which grade shall be considered finite, or the one intermediate to the infinite and the infinitesimal. Borrowing an analogy from the senses, as explained above, we make them the test of finiteness, but such is not necessary; and doubtless, were our senses much more delicate than they are, we should start from some order lower than we do, and call that finite which we now call infinitesimal; and if we were living amongst bodies and distances which were comparable with the distances of the fixed stars from the sun, they would doubtless be our finite quantities, and what are now finite would become infinitesimal.

A pertinent illustration of the preceding Theorems is found in the modern treatises on algebraical geometry. The general equation of the first degree is assumed to be of the form

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C} = 0,$$

where A, B and C are finite constants; and in the course of the discussion of this equation we meet with the paradoxical result $\mathrm{C} = 0$, where C is not and does not admit of being zero. Thus, for instance, we may in the course of an investigation arrive at the equation $5 = 0$; and this equation is, whether for satisfactory reasons or otherwise I will not now inquire, said to represent a straight line at an infinite distance from the origin. Now we have here the reason for this seeming impossibility: if $\mathrm{C} = 0$, it is implied that C is one term of an equation, the other terms of which are infinite; and that in addition or subtraction with them, the term C must be omitted; in other words, the other terms of the equation satisfy the equation, and the constant term C must be omitted: in order however that the nature of the equation may not be forgotten or overlooked, it is left in the seemingly paradoxical form

$$\mathrm{C} = 0.$$

Thus if the preceding equation takes the form $\mathrm{C} = 0$, where C is not zero, it is implied that x and y are infinite, and have different signs, and that the equation is satisfied by them; and thus the equation represents a line at an infinite distance. For many cases of this curious result I may refer the reader to Salmon's Conic Sections, Dublin, 1855, 3rd edit. Art. 64.

12.] Thus far we have spoken of single symbols, and of their properties; it is of *continuous* variables that we shall treat, and we shall not introduce discontinuous ones without special statement. Now it is plain that two or more such variables may be combined with constants in an equation, and may be such that a change of value of one may involve a corresponding change of value of one or more of the others; when this is the case, such variables are said to depend on, and to be *functions* of, each other: and the equation which expresses the mode of dependence is said to be a *function* of such variables.

If one variable is involved in such an expression, it is said to be a function of one variable; if two variables are involved, to be a function of two variables; and so on. Thus $\sin x$, e^{ax}, $\log x$, $\sqrt{(a^2-x^2)}$, are functions of one variable, viz. x; e^{ax+by}, $\tan(ax+by)$, x^y are functions of two variables, x and y; xyz, $x^2+y^2+z^2$ are functions of three variables: similarly we may have functions of more variables. Functions are designated by the symbols $\mathrm{F}, f, \phi, \psi$, &c. Thus $\mathrm{F}(x)$ means a function of one variable x, combined or not with constants as the case may be; $\mathrm{F}(x^2)$ means a function of x^2; $\phi(x, y)$ symbolizes a function of two variables; $\psi(x, y, z)$ a function of three variables: thus these functional symbols are *general*, and the specific forms of them are the particular functions which arise from operations in algebra, trigonometry, &c. Thus if $\mathrm{F}(x) = \cos x$, F is the general symbol of an operation of which cos is the specific instance; similarly would $\tan x$, $\log x$, e^{ax}, $\sqrt{(a^2-x^2)}$, be all represented by $\mathrm{F}(x)$; and $\log(x+y)$ would be represented by $f(x, y)$.

Now as such equations represent the mode of mutual interdependence of two or more variables in their symbolized state, so in their unsymbolized state they express the relation between, and the law of, certain causes and effects. Suppose a mass of metal to have been heated to a certain temperature, and that we have to find the temperature at any subsequent time; this latter quantity will depend on (say) three circumstances, viz. the original temperature, the law of radiation of heat, and the length of intervening time. Moreover suppose the law of relation of these four circumstances to be known, (which it is,) and it to be possible to express that law in a symbolical form; then the equation of dependence will involve four variables, viz. the original temperature, the time elapsed, the law of radiation, and the present temperature, and thus will be a function of four variables; but we shall also say, that the present temperature is a function of three other variables, and write it as follows:

Present temperature = F (original temperature, time, law of radiation).

Now it is possible that any one of the last three variables may vary without involving any change of the other two, in which case however the "present temperature" must vary also;

and as a similar variation of any other of the three may take place, there may be three separate variations of it due to the separate variations of each of the three variables on which it depends. On this account it is called a *dependent* variable, and each of the others is called an *independent* variable.

13.] Functions are said to be *implicit* and *explicit*, according as they assume the form of one or the other of those of the last Article. When by any artifice or operation, as, for instance, by the algebraical solution of an equation, one variable is expressed in terms of all the others, then it is said to be an explicit function of them; but when it is not solved, and all the variables remain involved in one expression, then the function is said to be implicit. Thus the illustrating case of the last Article will be an implicit function of four variables, if the quantities are combined in the form,

$$\text{F (original temperature, time, law of radiation, present temperature)} = 0;$$

and the present temperature becomes an explicit function of three variables, if it is written in the form,

$$\text{Present temperature} = \text{F (original temperature, time, law of radiation).}$$

Thus $x^2+y^2-a^2=0$ is an implicit function of two variables, but $y=(a^2-x^2)^{\frac{1}{2}}$ is an explicit function of one variable, of which y is the dependent and x the independent variable; and $y=f(x)$ is the general form of such explicit functions, and $\text{F}(x,y)=c$ (c being a constant) is the general form of an implicit function of two variables: and in these forms x and y are called the *subjects* of the functional symbols f and F. So again: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is an implicit function of three variables of the form $\text{F}(x,y,z)=c$; whereas $z=c\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{\frac{1}{2}}$, which is of the form $z=f(x,y)$, is an explicit function of two variables. Implicit functions are often written in the form,

$$u=\text{F}(x,y,z,\ldots)=c, \text{ or } =0,$$

as the case may be.

The terms *dependent* and *independent* variables have reference to explicit functions. When functions are implicit, there are no general marks whereby to determine the variable, which may first most conveniently change value.

14.] Functions have again been divided into two classes, *algebraical* and *transcendental*: the former being those functions which involve the operations of addition, subtraction, multiplication, division, involution, and evolution, or the algebraical sum of many such functions; the latter those wherein the operations symbolized are such as e^x, $\log_e x$, $\sin x$, $\sec^{-1} x$; that is, where they are either exponential, logarithmic, or circular. This however is a division not necessary to our present purpose.

Functions again may be *simple* or *compound*; that is, according as one or many operations, the results of which are the functions in question, are involved. Thus $y = \sin x$, $y = \log_a x$, are simple functions of x; but $y = \log \sin x$, $y = e^{\tan ax}$ are compound functions; compound functions are thus functions of functions.

It is necessary to observe, that, if two functions are represented by the same functional symbol, they are formed in the same manner by means of the variables which they involve. Thus if $f(x) = \sin x$, $f(y) = \sin y$; if $f(x) = e^{bx}$, $f(y) = e^{by}$.

15.] Functions may be either *continuous* or *discontinuous*. A continuous function is subject to the two following conditions:

1st. As the variable gradually changes, the function must gradually change.

2nd. The law symbolized by the functional character must not abruptly change.

When these two conditions are not satisfied, the function is discontinuous.

Thus, for instance, both conditions are fulfilled in the functions

$$y = ax + b, \quad y = \sin x;$$

in which, as the variable x changes, the value of the function also changes, but changes gradually, and there is no abrupt passage from one value to another; and the law symbolized by the functional character does not change, but always remains the same: but if the function were such as to express a line of the form in fig. 1, so that BA should be a continuous curve drawn after some determinate law, but at A the law suddenly should change, and the curve, from being, say, a circle, become a straight line, then the second of the above conditions is not satisfied, and the function is discontinuous. A is called a point of discontinuity. As an instance of a function of this description

Price Sup

$\sin x = \frac{1}{2\sqrt{-1}}\{$

$\cos x = \frac{1}{2}\ \{$

also $\cos(x+$

whence &c

which is the r

of the assump

the complete

$S = \frac{\varepsilon^{x}}{2}$

$= -$

$= \frac{\cos}{}$

In this form making x

which is indetermin

Price) $= -\frac{0}{2\sin x}$

have $\cos x + \cos 3x +$

which is easily veri

is continuous as

being an integer

That is the form of .

Comes indetermina

determinate, & wh

iple of π we have

(Note

$= \frac{\beta}{2} \quad [S'] = \frac{\cos n\alpha \sin n\alpha}{\sin\alpha} = \frac{\sin 2n\alpha}{2\sin\alpha}$

ate for $n = \infty$ and not (as given by

. For a finite value of n, we

$\cos 5\alpha + \ldots \cos(2n-1)\alpha = \frac{\sin 2n\alpha}{2\sin\alpha}$

ied for $n = 1\ 2$ etc. This quantity

long as α is so; if $\alpha = m\pi$, m

it becomes $(-1)^m n = \frac{0}{0}$.

the sum in the 2d member be-

te, although the first member is

ows that when $n = \infty$ & α a mult

$\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots = \pm\infty$

by Prof. Wm Chauvenet.)

the following may be mentioned. Replacing the circular quantities by their exponential values, it may easily be proved that

$$\cos a+\cos(a+\beta)+\cos(a+2\beta)+\ldots \text{ ad infin.} = -\frac{\sin\left(a-\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}.$$

Suppose that $a=\frac{\beta}{2}$, then the series becomes

$$\cos a+\cos 3a+\cos 5a+\ldots \text{ ad infin.} = -\frac{0}{2\sin a}:$$

but if $a =$ any multiple of π, the sum of the series assumes the indeterminate form $\frac{0}{0}$; hence we have this remarkable result, each term of the series varies continuously with a, but the sum of the series varies discontinuously, being always zero, except when a passes through some multiple of π, when the sum of the series suddenly and abruptly becomes $\frac{0}{0}$; that is, some indeterminate quantity; thus we have a series of points of discontinuity.

It is of continuous functions of continuous variables generally that we shall treat; and if discontinuous functions are introduced, they will be considered only for those values of the variables for which they are continuous.

16.] There are two different modes of viewing such continuous functions and variables, both of which will be convenient for the future purposes of the treatise. Firstly, suppose x_1 and x_2 to be two definite values of a variable number x, of which x_2 is the larger; and suppose the difference x_2-x_1 to be finite, and to be resolved into an infinite number of equal parts, each of which is therefore an infinitesimal; then the passage from x_1 to x_2 may be made by the successive addition of such infinitesimal elements, the whole sum of which is of course the finite quantity. Secondly, the idea of *motion* or continuous *growth* may be introduced, and we may conceive number to be in a gradually increasing state; and thus, if the *rate* of increase be finite, the increment due to a finite time will be finite, and that due to an infinitesimal interval of time will be an infinitesimal. The former mode we have hitherto invariably considered, except in the illustrations of Art. 6, but as we have now to deduce infinitesimals from finite quantities, and the latter method is more convenient for that purpose, we shall apply it. Both manifestly lead to identical quantities and results; for whereas in the one we consider a quantity during the process of generation, and the infinitesimal elements as they are successively produced; so in the other, we resolve the finite quantity when generated into its infinitesimal elements: in the latter then we arrive at the finite quantity from the elements, in the former we derive the elements from the finite quantity. The latter idea is the more complex, inasmuch as it involves motion and perhaps time, but adapts itself more readily to mechanical questions; and the former is undoubtedly best suited to geometry. The former of these two ideas is that on which Leibnitz conducted his investigations; the latter is for the most part that which sir Isaac Newton has embodied in the Principia, although some of the lemmas involve the former.

Thus suppose we consider the arc of a quadrant of a circle of radius a; its length is $\frac{\pi a}{2}$, which if we resolve into infinitesimal elements, each element will be the distance between two consecutive points: and the two points will be taken so near together, that the line joining them *must* be considered straight; and thus must the circle be conceived to be made up of an infinity of infinitesimal *straight* lines, and the tangent at any point is the line which coincides with the straight line joining the point and its consecutive point; that is, the tangent is the element produced. In elementary treatises on conics, the tangent to a conic is defined to be the straight line passing through *two* points on a curve infinitesimally near to each other, and its equation is derived from this definition. Similarly must all continuous curves be considered as composed of infinitesimal straight lines, and all surfaces of infinitesimal plane areas, and all solids of infinitesimal elements, and all concrete bodies as made up of infinitesimal corpuscles. Or if we consider the quadrant to be generated by a point moving according to a given law, and the motion to be carried on during a finite time, and the time to be resolved into very short instants, then the space passed over in one of these instants is the infinitesimal increment of the curve; and the direction in which the point is moving at the time of generating the element is that of the tangent of the circle at the point. In this view points generate lines, lines generate surfaces, and surfaces generate solids.

17.] Let us now investigate these properties with greater precision, and with reference to symbolized number; and let the first subjects of our consideration be continuous functions of continuous variables; and let us take the simple case of an explicit function of one variable, of the form

$$y = f(x); \tag{1}$$

let us consider it in two successive states, and first at a finite interval apart.

Let Δ be used as an abbreviation of *difference*, and represent a finite increment of a function or of a variable; so that Δx* and Δy represent the finite increments which x and y receive, and $\Delta f(x)$ represents the finite change in $f(x)$ due to the finite augment of the independent variable x; whence we have,

$$\Delta f(x) = \Delta y = f(x + \Delta x) - f(x). \tag{2}$$

Thus $\Delta f(x)$ is the quantity by which $f(x)$ is increased, as the variable on which it depends is increased, and is therefore called the difference of $f(x)$.

Now suppose these increments to become infinitesimal, in which case we shall use d, the abbreviation of *differential* or small difference, to symbolize them: so that dx and dy represent the infinitesimal increments which x and y receive, and $df(x)$ the infinitesimal increment which $f(x)$ receives, owing to the infinitesimal increment of its independent variable; so that (2) becomes

$$df(x) = dy = f(x + dx) - f(x). \tag{3}$$

Thus $df(x)$ is the infinitesimal quantity by which $f(x)$ is increased, by reason of the infinitesimal increase of the variable on which it depends, and is therefore called *the differential* of $f(x)$.

As the symbols d and Δ will be employed throughout the treatise in the meanings here assigned to them, let the difference between them be noticed. Also let it be observed, that when they are prefixed to x or y or $f(x)$, they have not the effect of multiplication: that is, dx is not d times x; but their power is that of an operation performed on the subject to which they are prefixed: thus dx is the differential of x, the infinitesimal increment of x, or the infinitesimal quantity by which x grows: so $df(x)$ is the infinitesimal increment of $f(x)$ when x has been increased by the infinitesimal quantity dx.

---

* Δx may be negative; in which case it might perhaps be more properly called a decrement; but in the following treatise we shall use the words augments and increments to express the variations in the values of the variables, whether such variations cause them to increase or decrease.

From the form (3), which the most convenient mode of determining such infinitesimal increments takes, has the name "Differential Calculus" arisen.

The operation of determining the values of the differentials of the functions due to the differential increase of the variable is called Differentiation, and is the first work of the Calculus; and we are said to differentiate the function *with respect to* that variable, owing to the change of which the function changes. Hereby the materials will be formed, which are subject to the axioms previously enuntiated, and the other laws of which will be subsequently developed. We shall generally differentiate directly and without the intervention of any other symbols than those introduced above, but sometimes it is convenient to put the result in the following form.

18.] Since the left-hand member of equation (3) is an infinitesimal of, say, a first order, the right-hand member must be an infinitesimal of the same order; an infinitesimal must therefore be a factor of it. But the only infinitesimal that it involves is dx, therefore we may reasonably presume that dx will be the factor; the presumption however must be verified by subsequent investigations. Dividing both sides by dx, we have

$$\frac{dy}{dx} = \frac{d.f(x)}{dx} = \frac{f(x+dx)-f(x)}{dx},$$

of which the last member is of the form $\frac{0}{0}$, but will be a finite quantity, if our presumption is correct. Let us represent it by $f'(x)$, so that

$$\frac{dy}{dx} = \frac{d.f(x)}{dx} = f'(x); \tag{4}$$

$$\therefore \quad dy = d.f(x) = f'(x)\,dx: \tag{5}$$

and therefore from (3),

$$f'(x)\,dx = f(x+dx)-f(x). \tag{6}$$

$f'(x)$ is called *the derived function* of $f(x)$, and represents the ratio of the differential of the function to the differential of the variable; and therefore if it is known, the absolute change of the function due to the change of the variable is also known. Also $f'(x)$ is called *the differential coefficient*, because it is the coefficient of dx in the equation $d.f(x) = f'(x)\,dx$; and $f'(x)\,dx$ is by (6) the excess of the function when its subject variable is infinitesimally increased over its value when the subject is not increased.

It is also manifest, that generally

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) + \mathrm{R}, \tag{7}$$

when R is some residual quantity, which must be neglected when Δx becomes dx.

The process by which derived functions are determined is called *Derivation*: and evidently by it a function becomes the parent of some other function. On this principle Lagrange, one of the most eminent mathematicians of the eighteenth century, has constructed his works on the Infinitesimal Calculus, viz. "Théorie des Fonctions Analytiques," and its sequel "Leçons sur le Calcul des Fonctions." It is however beside my present purpose to unfold the process, because it is so different to that which I have made fundamental; and it is impossible adequately to discuss the reasons why the preceding principles are preferred, because fully to do so requires a knowledge of all the Calculus. On the latter subject however a few words are said in the Preface.

19.] Similarly, in considering a function of many variables, such as

$$u = \mathrm{F}(x, y, z, \ldots),$$

we shall use Δ and d to symbolize respectively finite and infinitesimal changes: so that Δu, Δx, Δy, ... represent finite, and du, dx, dy, dz ... infinitesimal quantities; wherefore

$$\Delta u = \mathrm{F}(x+\Delta x, y+\Delta y, \ldots) - \mathrm{F}(x, y, \ldots) \tag{8}$$

$$du = \mathrm{F}(x+dx, y+dy, z+dz, \ldots) - \mathrm{F}(x, y, z, \ldots). \tag{9}$$

It is, however, more convenient to reserve the complete inquiry into the variations of functions of many variables for a later part of our treatise.

20.] Hence we may describe the Differential Calculus as follows:

The Differential Calculus is a general method and system of rules by which are determined the corresponding changes of functions and variables, when the variations of the variables are infinitesimal; and the code of laws to which they are subject, and conformably to which they may be applied to questions of Geometry and Physics.

Before however we proceed to give either general rules for the differentiation of functions, or to illustrate the process by particular examples, it is necessary to determine the values of two functions for certain values of the variables, which will be frequently applied hereafter.

SECTION 2.—*Fundamental Lemmas of the Infinitesimal Calculus.*

21.] LEMMA I. To evaluate $(1+x)^{\frac{1}{x}}$, when x is an infinitesimal.

By the Binomial Theorem

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots;$$

let $n=\frac{1}{x}$,

$$\therefore\ (1+x)^{\frac{1}{x}} = 1+\frac{1}{x}x+\frac{1}{x}\Big(\frac{1}{x}-1\Big)\frac{x^2}{1.2}+\frac{1}{x}\Big(\frac{1}{x}-1\Big)\Big(\frac{1}{x}-2\Big)\frac{x^3}{1.2.3}+\ldots$$

$$= 1+1+\frac{1-x}{1.2}+\frac{(1-x)(1-2x)}{1.2.3}+\ldots\quad.$$

Let x be some small positive fractional number; then it is plain that each factor in the numerators of the several terms of the series is less than 1; and therefore, no term being negative, the whole series is greater than its first two terms, that is, is greater than 2. Also since

$$3 = 1+\Big(1-\frac{1}{2}\Big)^{-1} = 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots,$$

each term of this after the second is greater than the corresponding term in the series above, and therefore the whole series is greater; and therefore, when x is a small positive fractional number, $(1+x)^{\frac{1}{x}}$ is equal to some number greater then 2 and less than 3.

Let x be an infinitesimal, and thus be symbolized by 0, in which case by virtue of Theorem VI, Art. 9, we have

$$(1+0)^{\frac{1}{0}} = 1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\frac{1}{1.2.3.4.5}+\dots$$

which must be summed arithmetically as follows:

$$\begin{array}{ccl}
1 & = & 1\cdot \\
1 & = & 1\cdot 0000000 \\
\frac{1}{1.2} & = & \cdot 5000000 \\
\frac{1}{1.2.3} & = & \cdot 1666666 \\
\frac{1}{1.2.3.4} & = & \cdot 0416666 \\
\frac{1}{1.2\dots 4.5} & = & \cdot 0083333 \\
\frac{1}{1.2\dots 5.6} & = & \cdot 0013888 \\
\frac{1}{1.2\dots 6.7} & = & \cdot 0001984 \\
\frac{1}{1.2\dots 7.8} & = & \cdot 0000248 \\
\frac{1}{1.2\dots 8.9} & = & \cdot 0000027 \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot
\end{array}$$

whence by addition we have

$$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\dots \quad = \quad 2\cdot 7182818\dots\dots$$

which is the correct sum of the series to seven places of decimals. This arithmetical number, which is incommensurable* with any digit of the decimal scale of notation, and which is identical with the base of the Napierian logarithms, is symbolized by e; so that we have

$$\begin{aligned}(1+x)^{\frac{1}{x}} &= 2.7182818 \\ &= e, \qquad (10)\end{aligned}$$

when x is an infinitesimal; and wherever e is met with in the course of this work, it is used as the symbol for this arithmetical quantity.

Again, if x were a small negative fractional quantity, then

---

* Peacock's Algebra, Vol. I, Art. 203, Cambridge, 1842.

$1+x$ is a positive quantity less than 1; let $1+x=\frac{1}{1+z}$, where z is a small positive quantity less than 1, and becomes infinitesimal simultaneously with x; then

$$(1+x)^{\frac{1}{x}} = \left(\frac{1}{1+z}\right)^{-\frac{1+z}{z}} = (1+z)^{\frac{1+z}{z}} = \left\{(1+z)^{\frac{1}{z}}\right\}^{1+z},$$

which, when z is infinitesimal, becomes e by the former part of the Lemma.

Hence we conclude that when x is a small positive or negative quantity, approximating to 0, $(1+x)^{\frac{1}{x}}$ approximates to the value e; that is, differs from e by a quantity less than any assignable quantity, when x diminishes without limit; that is, when x is an infinitesimal, $(1+x)^{\frac{1}{x}}=e$.

Corollary I. Hence, taking Napierian logarithms of both sides of the last equation,

$$\frac{1}{x}\log_e(1+x) = \log_e e = 1;$$

$$\therefore\quad \log_e(1+x) = x, \text{ when } x \text{ is an infinitesimal.} \qquad (11)$$

Cor. II. And taking logarithms to the base a, we have

$$\begin{aligned}\frac{1}{x}\log_a(1+x) &= \log_a e,\\ &= \frac{1}{\log_e a}, \text{ by the theory of logarithms;}\end{aligned}$$

$$\therefore\quad \log_a(1+x) = \frac{x}{\log_e a}, \text{ when } x \text{ is infinitesimal.} \qquad (12)$$

Cor. III. As the above determination of the value of e is independent of any previously-expanded logarithmic series, it is well to shew how other results follow from it.

$$\begin{aligned}\left\{(1+x)^{\frac{1}{x}}\right\}^y &= (1+x)^{\frac{y}{x}}\\ &= 1+y+\frac{y(y-x)}{1.2}+\frac{y(y-x)(y-2x)}{1.2.3}+\dots\end{aligned}$$

Let x be infinitesimal, whence

$$e^y = 1+y+\frac{y^2}{1.2}+\frac{y^3}{1.2.3}+\dots$$

Again, since

$$(1+x)^{\frac{y}{x}} = e^{\frac{y}{x}\log_e(1+x)},$$

expanding both sides we have

$$1 + \frac{y}{1} + \frac{y(y-x)}{1.2} + \frac{y(y-x)(y-2x)}{1.2.3} + \dots$$

$$= 1 + \frac{y}{x}\log_e(1+x) + \frac{1}{1.2}\frac{y^2}{x^2}\{\log_e(1+x)\}^2 + \dots;$$

cancelling the units, dividing through by y, making y infinitesimal, and equating the finite terms, we have

$$\frac{\log_e(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots$$

$$\therefore \quad \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

22.] **Lemma II.** To prove that $\tan x$, x, and $\sin x$ may be used indifferently for each other, when x is an infinitesimal arc, and the radius is finite.

Let AP (see fig. 2) be the arc of a circle whose radius $= 1$, and let the arc AP $= x$, x being the circular measure of the angle PCA; let AT be the tangent and PM the sine of the arc. At P draw a tangent to the circle, viz. PT′, and draw the other lines as in the figure; then, since TPT′ is a right angle, TT′ is greater than PT′,

$$\therefore \quad \text{AT} > \text{AT}' + \text{T}'\text{P};$$

and because two sides of a triangle are greater than the third,

$$\text{RT}' + \text{T}'\text{S} > \text{RS};$$

$$\therefore \quad \textit{à fortiori} \ \ \text{AT} > \text{AR} + \text{RS} + \text{SP}:$$

and similarly, if tangents are drawn to the arc at points between A and Q and Q and P, it may be shewn that the sum of all the lines similar to SR is less than AT; but the limit of all such lines is the circular arc; therefore AT is greater than the arc.

Again, the chord AP is greater than PM, which is the sine of x, and PQ + QA is greater than AP; therefore

$$\text{PQ} + \text{QA} > \text{PM}:$$

and, drawing other chords from A and P to intermediate points on the arcs, it may be shewn that the sum of such chords is greater than the chord AP, and therefore, *à fortiori*, than PM; and as the arc AP is the limit of all such chords, it follows that it is greater than the sine PM. Therefore the arc is less than its tangent and greater than its sine.

Again, bearing in mind that $\cos x$ by its definition $= 1$, when x is an infinitesimal, we have

$$\frac{\sin x}{\tan x} = \cos x = 1, \text{ when } x \text{ is an infinitesimal;}$$

whence it follows that $\sin x = \tan x$, when x is an infinitesimal; and since x is, as above shewn, always intermediate to these, it is clear that all three are equal to, and therefore may be used indifferently for, each other.

Hence, when x is an infinitesimal,

$$\sin x = x = \tan x; \tag{13}$$

which proposition is frequently expressed in the form,

The limiting ratio of the sine, the arc, and the tangent is that of equality.

This result is also involved in the former Lemma. The exponential value of the sine gives us,

$$\begin{aligned}\sin x &= \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}, \\ &= \frac{e^{-x\sqrt{-1}}}{2\sqrt{-1}}\{e^{2x\sqrt{-1}} - 1\}.\end{aligned}$$

To evaluate $e^{2x\sqrt{-1}} - 1$, when x is an infinitesimal;

$$\text{let} \quad e^{2x\sqrt{-1}} - 1 = z,$$

so that x and z are simultaneously infinitesimal;

$$\begin{aligned}\therefore \quad e^{2x\sqrt{-1}} &= 1 + z, \\ \text{and} \quad 2x\sqrt{-1} &= \log_e(1+z) \\ &= z,\end{aligned}$$

when z is an infinitesimal, by Cor. I, Lemma I.

Hence $e^{2x\sqrt{-1}} - 1 = 2x\sqrt{-1}$;

$$\begin{aligned}\therefore \quad \sin x &= \frac{2x\sqrt{-1}\, e^{-x\sqrt{-1}}}{2\sqrt{-1}} \\ &= x e^{-x\sqrt{-1}} \\ &= x,\end{aligned}$$

when x is an infinitesimal.

Hence also it follows that

$$\tan x = \frac{1}{\sqrt{-1}} \cdot \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = x, \text{ when } x \text{ is an infinitesimal.}$$

Cor. I. When x is an infinitesimal, chord $x = x$.

For since $\text{ch}\, x = 2\sin\frac{x}{2}$, and by Lemma II, $\sin\frac{x}{2} = \frac{x}{2}$, when x is an infinitesimal,

$$\therefore \quad \text{ch}\, x = 2\frac{x}{2} = x, \text{ when } x \text{ is an infinitesimal:}$$

that is, the chord of an infinitesimal arc is equal to the arc; which is the VIIth Lemma of the first Section of Book I of Newton's Principia.

COR. II. Hence also, if x is infinitesimal,

$$\sin^{-1}x = x = \tan^{-1}x: \tag{14}$$

that is, the sine and the tangent may be used indifferently for the arc, when the arc is infinitesimal.

23.] Although it may be beside our defined path, yet it is worth while to shew, that the arc and the sine are equal, when the arc is infinitesimal, only by omitting terms of a higher order, and which *must* be neglected in accordance with Theorem VI, Art. 9.

Assuming the validity of the trigonometrical proof of the series,

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \dots$$

if x is an infinitesimal, x^3, x^5, ... must be neglected as they are algebraically added to x, and we have

$$\sin x = x.$$

Also since

$$\begin{aligned} \text{versed-sine of } x &= 1 - \cos x \\ &= 1 - \left\{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots\right\} \\ &= \frac{x^2}{1.2} - \frac{x^4}{1.2.3.4} + \dots \end{aligned}$$

it follows by reason of Theorem VI, Article 9, that, if x is infinitesimal,

$$\text{versin}\, x = \frac{x^2}{2};$$

and therefore, if the arc is an infinitesimal of the first order, the versed-sine of it is an infinitesimal of the second order. The geometrical proof of this truth is so manifest, that it is unnecessary to do more than to suggest it to the student.

SECTION 3.—*Examples of the Infinitesimal method.*

24.] The principles above explained are sufficient for a variety of problems; examples of which are subjoined, to give the student an insight into the *kind* of processes which he has to perform.

Ex. 1. To differentiate x^2, that is, to determine the change of x^2 due to an infinitesimal change of value of x.

$$\begin{aligned} \text{Let} \qquad y &= x^2, \\ \therefore \quad y+\Delta y &= (x+\Delta x)^2, \\ \therefore \quad \Delta y &= (x+\Delta x)^2-x^2 \\ &= 2x\,\Delta x+(\Delta x)^2; \end{aligned}$$

and therefore, taking differentials instead of differences,

$$dy = d.x^2 = 2x\,dx,$$

omitting the term $(dx)^2$, which can have no value, because it is an infinitesimal of the second order, and added to one of the first order.

Hence $\frac{dy}{dx} = 2x$, and therefore if, in accordance with the notation of derived functions,

$$f(x) = x^2,$$

$$\frac{d.f(x)}{dx} = f'(x) = 2x;$$

which result, and others of a similar kind, as will be shewn hereafter, justifies the presumption of Article 18, that the ratio of the infinitesimal variation of $f(x)$ to that of x is a finite quantity.

The above process may thus be explained geometrically:

Let x represent the straight line AP, fig. 3; and suppose AP to be increased by PQ, which is represented by Δx; then from the figure it is plain that the square is increased by the rectangles DB, BQ, and the square BR, the values of which are $x \times \Delta x$, $x \times \Delta x$, $(\Delta x)^2$; whence

$$\Delta y = \Delta.x^2 = 2x\,\Delta x+(\Delta x)^2.$$

Now let PQ be infinitesimal, that is, let Δx become dx, whereby also Δy becomes dy, and we have

$$dy = d.x^2 = 2x\,dx+(dx)^2;$$

but $(dx)^2$ must be omitted for the following reason: $x\,dx$ symbolizes approximately a straight line, of which the length is x, and the breadth, if one may so speak, is dx; but $(dx)^2$ represents a square whose side is dx; and as dx is an infinitesimal, its square is *a point*, and as it will require an infinity of such points to make a straight line, and as the coefficient of $(dx)^2$ is not infinity, we must neglect it; that is, in calculating the enlargement of the square due to the enlargement of a side, we take account of the infinitely narrow rectangles which adjoin the sides, but must neglect the small point which is required to complete the square, and which is situated at one of the angles, as at B, and no error is committed by our so doing. Or, if we introduce the idea of motion, the enlargement of the square is due to the moving forwards of the two sides PB and CB, and the rectangles by which the square is increased are the several spaces passed over by the sides, which are the spaces contained between the lines before and after the motion; and as the spaces through which the lines have passed are very small, the lines being considered to be in two immediately *successive* positions, the small element at B becomes a point, and, as we have not an infinity of such points, the accuracy of our result is not destroyed by neglecting this small quantity; and therefore, again, the increase of the square due to the infinitesimal increase of the side is $2x\,dx$.

Ex. 2. To differentiate $\dfrac{x}{\{a^2+x^2\}^{\frac{1}{2}}}$.

$$\text{Let} \qquad y = \frac{x}{\{a^2+x^2\}^{\frac{1}{2}}},$$

$$\therefore \quad y+\Delta y = \frac{x+\Delta x}{\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}},$$

$$\Delta y = \frac{x+\Delta x}{\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}} - \frac{x}{\{a^2+x^2\}^{\frac{1}{2}}}$$

$$= \frac{(x+\Delta x)\,\{a^2+x^2\}^{\frac{1}{2}} - x\,\{a^2+x^2+2x\,\Delta x+(\Delta x)^2\}^{\frac{1}{2}}}{\{a^2+x^2\}^{\frac{1}{2}}\,\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}};$$

and expanding the second member of the numerator by the Binomial Theorem, and neglecting the terms involving the square and higher powers of Δx, which will become infinitesimals of an order to be omitted, we have

$$\Delta y = \frac{(x+\Delta x)\,\{a^2+x^2\}^{\frac{1}{2}} - x\,\{(a^2+x^2)^{\frac{1}{2}} + x\,\Delta x\,(a^2+x^2)^{-\frac{1}{2}}\}}{\{a^2+x^2\}^{\frac{1}{2}}\,\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}}$$

$$= \frac{\Delta x\,\{a^2+x^2\}^{\frac{1}{2}} - x^2\,\Delta x\,\{a^2+x^2\}^{-\frac{1}{2}}}{\{a^2+x^2\}^{\frac{1}{2}}\,\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}}$$

$$= \frac{a^2\,\Delta x}{(a^2+x^2)\,\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}};$$

whence, taking differentials, and omitting dx, where it is added to the finite quantity x, we have

$$dy = \frac{a^2\,dx}{\{a^2+x^2\}^{\frac{3}{2}}}; \qquad \therefore \quad \frac{dy}{dx} = \frac{a^2}{\{a^2+x^2\}^{\frac{3}{2}}};$$

and therefore, if $f(x) = \dfrac{x}{\{a^2+x^2\}^{\frac{1}{2}}}$, $f'(x) = \dfrac{a^2}{\{a^2+x^2\}^{\frac{3}{2}}}$.

Ex. 3. To differentiate $\dfrac{e^x}{e^x+1}$.

Let $\quad y = \dfrac{e^x}{e^x+1}, \qquad \therefore \quad y+\Delta y = \dfrac{e^{x+\Delta x}}{e^{x+\Delta x}+1};$

$$\therefore \quad \Delta y = \frac{e^{x+\Delta x}}{e^{x+\Delta x}+1} - \frac{e^x}{e^x+1}$$

$$= \frac{e^{x+\Delta x}\,(e^{x}+1) - e^x\,(e^{x+\Delta x}+1)}{(e^{x+\Delta x}+1)\,(e^x+1)}$$

$$= \frac{e^x\,\{e^{\Delta x}-1\}}{(e^{x+\Delta x}+1)\,(e^x+1)};$$

to evaluate $e^{\Delta x}-1$, when Δx becomes an infinitesimal dx;

$$\text{let} \qquad e^{\Delta x}-1 = z,$$

so that Δx and z are simultaneously infinitesimal;

$$\therefore \quad e^{\Delta x} = 1+z, \qquad \Delta x = \log_e(1+z),$$

$$\therefore \quad dx = z, \text{ by Cor. I, Lemma I};$$

$$\therefore \quad e^{dx}-1 = dx;$$

replacing therefore $e^{\Delta x}-1$ by its equivalent in the above equation, and omitting dx when added to the finite quantity x, we have

$$dy = \frac{e^x\,dx}{\{e^x+1\}^2}.$$

If therefore

$$f(x) = \frac{e^x}{e^x+1}, \qquad f'(x) = \frac{e^x}{\{e^x+1\}^2}.$$

Ex. 4. To differentiate $\cos x \sin 2x$.

$$\text{Let} \qquad y = \cos x \sin 2x,$$

$$\therefore \quad y + \Delta y = \cos(x + \Delta x) \sin 2(x + \Delta x);$$

$$\begin{aligned} \Delta y &= \cos(x + \Delta x) \sin 2(x + \Delta x) - \cos x \sin 2x \\ &= (\cos x \cos \Delta x - \sin x \sin \Delta x)(\sin 2x \cos 2\Delta x + \cos 2x \sin 2\Delta x) \\ &\qquad - \cos x \sin 2x; \end{aligned}$$

and taking differentials instead of differences, and therefore by reason of Lemma II replacing the sine of an infinitesimal arc by the arc itself, and the cosine by unity, we have

$$dy = (\cos x - dx \sin x)(\sin 2x + 2\,dx \cos 2x) - \cos x \sin 2x,$$

$$dy = 2\,dx \cos x \cos 2x - dx \sin x \sin 2x,$$

omitting, as is necessary, the term involving $(dx)^2$. If therefore

$$f(x) = \cos x \sin 2x, \qquad f'(x) = 2 \cos x \cos 2x - \sin x \sin 2x.$$

The above examples are instances of differentiating from first principles.

Ex. 5. To determine the perimeter and the area of a circle of given radius.

As in this and the two following examples, and indeed throughout the whole work, it is necessary to have a clear notion of the relation between circular and grădual (that is, by degrees) measures of angles, I propose once for all to say a few words on the subject.

In Euclid, Book VI, Prop. XXXIII, it is proved that in the same circle, or in equal circles, angles at the centre have the same ratio which the arcs on which they stand have to each other: hence we conclude that in the same circle, or in equal circles, the arcs vary as the angles which they subtend at the centre.

Also let us suppose in two circles of unequal radii, which we will assume to be concentric, although this assumption is unnecessary, two regular polygons of the same number of sides to be inscribed; then the perimeters of these polygons evidently vary as the radii of the circles. Let the number of the sides be infinitely increased, so that each side becomes a straight line of infinitesimal length: the lengths of these sides evidently still vary as the radii; for the infinite increase in the number of these sides, and thereby the infinitesimal diminution of the lengths of the sides, does not change this ratio. But when the number of the sides is infinite, the polygon becomes a circle, and its perimeter becomes the circumference of the circle; so that by this method we infer that the perimeters of circles vary as their radii. The same relation is also true of any parts of the perimeters; and therefore of two concentric circles the arcs subtending the same angle at the common centre vary as the radii of the circles. If therefore we combine this result with the preceding, we have the following proposition:

The circular arcs subtending different angles with different radii vary conjointly as the angles and the radii of the arcs.

Let s, r, θ be the symbols representing an arc, a radius, and the subtended angle; these symbols expressing numbers; the length-unit of the radius and arc being the same, and the angle-unit being at present undetermined. Let k be a coefficient of variation. So that the preceding proposition expressed mathematically is

$$s = kr\theta.$$

But k is undetermined; to determine it, *let us assume the unit-angle to be that, the subtending arc of which is equal to the radius;* let the importance of this assumption be noted. Then in the preceding equation $\theta = 1$, when $s = r$; and therefore $k = 1$; and

$$s = r\theta, \tag{15}$$

and $$\theta = \frac{s}{r}. \tag{16}$$

Now a right-angle is a quantity independent of any arbitrary assumption. In ordinary trigonometry it has been found convenient to divide it into 90 equal parts or angles, each of which is called a *degree:* this indeed is the definition of a degree.

Here then are two different unit-angles. The latter is much more arbitrary than the former; we might divide a right angle into 100 or into any other number of equal parts, if such a division were more convenient than that into 90 equal parts. But the principle of the former is founded on certain geometrical properties of the circle. Thus then we have, what we will call technically, *the unit-angle*, and a degree. A question arises, what relation exists between these angles? can the relation be expressed by a fraction? or by any other number? We reply, no; no number commensurable with our scale can express it. We may roughly get a notion of the relation in the following way:

The unit-angle is that of which the subtending arc is equal to the radius: now the *chord* of 60 degrees is equal to the radius; and the chord is less than the arc: so that the unit-angle is less than 60°; it ultimately results that the unit-angle is 57.29578... degrees. Let us however assume the number which gives the ratio of two right angles to the unit-angle to be represented by π: so that π is the number of times the unit-angle is contained in two right angles. Thus if Θ represents the unit-angle,

$$\pi \times \Theta = 180°. \tag{17}$$

The numerical value of π (to five places of decimals) is 3.14159... and the fraction which most conveniently, although of course not correctly, represents it is $\frac{355}{113}$; the number is however incommensurable, and has actually been calculated within the last four years by Mr. Shanks of Houghton le Spring in the county of Durham, to the astonishing number of 607 places of decimals. Let the reader therefore be careful as to the meaning of the symbol π: it is a *number*, and a *number* only. By some authors it has been used as equivalent to two right angles or 180°: such an use is incorrect as to form; and although commonly used can be justified only by understanding it to be the coefficient of Θ, which is the unit-angle; and thus makes the quantity $\pi\Theta$ to be a concrete angle.

In the same way I would observe, that when the symbol x or θ is used for an angle the symbol is a number, and implies that x times or θ times the unit-angle is taken.

One other remark must be made; the sine, cosine, tangent, &c. are affections of angles, and not of circular arcs: but since by the preceding equation $s = \theta$, if $r = 1$; the trigonometrical functions become affections of arcs of a circle, if the radius is unity: and thus if x is an arc, $\sin x$, $\tan x$, &c. are intelligible.

Now to return to our problem: let ACD, fig. 4, be the circle, of which let the radius be a; take O the centre, and let the angle AOC be the nth part of four right angles, and AC be the side of a regular polygon of n sides inscribed in the circle; and make a construction such as is represented in the figure.

Then if 2π is the symbol for four right angles,

$$\text{AOC} = \frac{2\pi}{n}, \quad \text{and} \quad \text{AOB} = \frac{\pi}{n};$$

$$\therefore \quad \text{AB} = a \sin\frac{\pi}{n}, \quad \text{OB} = a \cos\frac{\pi}{n};$$

$$\therefore \quad \text{the perimeter of the polygon} = n.\text{AC} = 2n.\text{AB} = 2na \sin\frac{\pi}{n};$$

the area of the polygon $= n \times$ area of triangle OAC $= n.\text{AB}.\text{OB}$

$$= na^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n}.$$

Now when the number of the sides of the polygon is infinitely increased, the perimeter and the area coincide respectively with the perimeter and the area of a circle; in which case $\frac{\pi}{n}$ becomes an infinitesimal angle, and its sine must be replaced by the arc to the radius unity, and its cosine by unity. Hence we have

$$\text{Perimeter of circle} = 2na\frac{\pi}{n} = 2\pi a;$$

$$\text{Area of circle} = na^2\frac{\pi}{n} = \pi a^2.$$

Ex. 6. To determine the convex surface and content of a right circular cone.

Let A, fig. 5, be the vertex of the cone, and C be the centre of its circular base; let $\text{AC} = a$, $\text{CB} = b$, and let PQ be a side of a regular polygon of n sides circumscribing the circular base, so that $\text{PCQ} = \frac{2\pi}{n}$; whence it is plain that

$$\text{PQ} = 2\,\text{PB} = 2b\tan\frac{\pi}{n}.$$

Therefore the area of the triangle $\text{APQ} = \text{AB} \times \text{BP}$

$$= \{a^2+b^2\}^{\frac{1}{2}}\, b\tan\frac{\pi}{n};$$

therefore the whole convex area of the circumscribed pyramid of n sides, each of which is similar to APQ, $= nb\{a^2+b^2\}^{\frac{1}{2}}\tan\frac{\pi}{n}$.

But when n is infinitely increased, the surface of the pyramid coincides with the convex surface of the cone, and replacing $\tan\frac{\pi}{n}$ by $\frac{\pi}{n}$ in accordance with Lemma II, we have

$$\text{Convex surface of cone} = nb\,\{a^2+b^2\}^{\frac{1}{2}}\frac{\pi}{n}$$

$$= \pi b\,\{a^2+b^2\}^{\frac{1}{2}};$$

that is, the convex surface of a cone is equal to the product of the slant side by the semi-circumference of the base.

This proposition may also be arrived at by the following process. If the convex surface is developed into a plane surface, and it is clearly capable of such developement, it becomes a sector of a circle, of which the centre is the vertex of the cone, the length of the arc is the circumference of the base of the cone, and the radius is the slant side of the cone. And as the sector may be resolved into infinitesimal triangles with a common vertex, and all of the same altitude; so the area of it is equal to half the rectangle of which two adjacent sides are the radius of the circle and the length of the containing arc: and thus as the containing arc $= 2\pi b$, and the radius $= (a^2+b^2)^{\frac{1}{2}}$,

$$\text{the convex surface of the cone } = \pi b (a^2 + b^2)^{\frac{1}{2}}.$$

As the area of the base of the cone $= \pi b^2$, the whole surface $= \pi b^2 + \pi b \{a^2+b^2\}^{\frac{1}{2}}$.

Again, by Euclid XII, 7, the content of the pyramid APQC is one-third of that of the prism of the same altitude and the same base; and therefore

$$\begin{aligned} \text{Content of circumscribed pyramid } &= \frac{n}{3}.\text{AC}.\text{CB}.\text{BP} \\ &= \frac{n}{3} ab^2 \tan \frac{\pi}{n}. \end{aligned}$$

Let the number of the sides of the polygon circumscribing the base be infinitely increased, in which case the pyramid assumes the limiting form of the cone, and $\frac{\pi}{n}$ becomes an infinitesimal angle, so that it must replace $\tan \frac{\pi}{n}$, and we have

$$\begin{aligned} \text{Content of cone } &= \frac{n}{3} ab^2 \frac{\pi}{n} \\ &= \frac{\pi a b^2}{3}. \end{aligned}$$

Now it is plain that $\pi a b^2$ is the content of a cylinder of altitude a described on the circular base whose area is πb^2; hence it follows, that the content of a cone is one-third of that of its circumscribed cylinder.

Ex. 7. To determine the surface and content of a sphere of given radius.

Suppose the circle ACBD, fig. 6, to be a plane central section of the sphere, or to be that by the revolution of which, about its diameter BOA, the sphere is generated. Let the radius $= a$, and be divided into n equal parts, each of which is therefore equal to $\frac{a}{n}$, and suppose MN to be one of these parts, then

$$\text{MN} = \frac{a}{n}.$$

Draw the ordinates MP, NQ, including between them the arc PQ of the circle. Then, if PQ is infinitesimal, and which therefore must be considered straight, the *zone* of the sphere generated by the revolution of PQ about AO is equal to a rectangle, of which one side is PQ and the other is the circular path described by P, as it revolves; that is,

$$\text{the surface of the zone} = \text{PQ} \times 2\pi \times \text{MP}.$$

$$\text{Let POA} = \theta; \quad \therefore \quad \text{PQ} = \text{MN} \operatorname{cosec}\theta = \frac{a}{n \sin\theta},$$
$$\text{MP} = a \sin\theta;$$
$$\therefore \quad \text{surface of zone} = \frac{2\pi a^2}{n}.$$

And since the surface of the whole sphere is equal to $2n$ times the surface of such a zone, we have

$$\text{the surface of the sphere} = 4\pi a^2,$$

and is equal therefore to four times the area of a great circle of the sphere.

Now suppose a cylinder to be described about the sphere, and to be of the same altitude as the sphere, and to be generated by the revolution of the rectangle AEFB about the diameter BA; and suppose two planes drawn through M and N, perpendicular to BOA, so as to intercept a zone of the cylinder whose height is P′Q′ or MN: then, since a is the radius of the cylinder, the surface of the intercepted zone $= 2\pi a \times$ MN; but from above,

$$\text{corresponding surface of zone of sphere} = 2\pi a \frac{a}{n} = 2\pi a \times \text{MN}.$$

Hence, if a cylinder be described about a given sphere, and if the surface of the cylinder be divided into zones by planes perpendicular to its axis, the surfaces of the intercepted zones of the cylinder and the sphere are equal; and therefore,

The whole surface of the sphere is equal to the convex surface of its circumscribed cylinder.

Again, to determine the content of a sphere.

Let a polyhedron be described about the sphere; let $s =$ the area of a face, and $a =$ the radius of the sphere; then the content of the pyramid, whose vertex is at the centre of the sphere and base is s, is $\frac{as}{3}$: and a similar expression is true of each face; let the number of faces be infinitely increased, and let each face become infinitesimal; then the polyhedron ultimately coincides with the sphere, and the sum of the s's becomes the surface of the sphere, and is therefore equal to $4\pi a^2$; therefore the content of the sphere $= \dfrac{4\pi a^3}{3}$.

Ex. 8. To determine the area of a parabola.

Let OPQBA, fig. 7, be an area bounded by the parabola OPB, the axis OA, and the ordinate AB which is parallel to the tangent at the vertex O.

$$\text{Let}\qquad \left.\begin{array}{l}\text{OM} = x\\ \text{MP} = y\end{array}\right\},\qquad \left.\begin{array}{l}\text{OA} = a\\ \text{AB} = b\end{array}\right\};$$

$$\therefore\ \text{the equation to the curve is } y^2 = \frac{b^2}{a}x.$$

Let AB be divided into n equal parts, and through the several points of division let lines be drawn parallel to OA; let K and L be respectively the rth and the $(r+1)$th points of division, and therefore $\text{KL} = \dfrac{b}{n}$; then, if n is taken very large, the area of PQLK = KL × KP.

$$\begin{aligned}\text{But}\qquad \text{KP} &= \text{OA}-\text{OM}\\ &= a - \frac{a}{b^2}\text{MP}^2\\ &= a - \frac{a}{b^2}\left(r\frac{b}{n}\right)^2\\ &= a - \frac{r^2}{n^2}a\end{aligned}$$

$$\begin{aligned}\therefore\quad \text{area PQKL} &= \frac{b}{n}\left\{a - \frac{r^2}{n^2}a\right\}\\ &= \frac{ab}{n}\left\{1 - \frac{r^2}{n^2}\right\};\end{aligned}$$

and as this expression is true of every slice similar to PQKL, the whole area is the sum of all such, which are obtained by giving to r successively the values $1, 2, 3, \ldots (n-1), n$; therefore

$$\begin{aligned}\text{the area of the parabola} &= \frac{ab}{n}\left\{1 - \frac{1^2}{n^2} + 1 - \frac{2^2}{n^2} + \ldots + 1 - \frac{n^2}{n^2}\right\}\\ &= \frac{ab}{n}\left\{n - \frac{1^2+2^2+\ldots+n^2}{n^2}\right\}\\ &= \frac{ab}{n}\left\{n - \frac{(2n+1)\,n(n+1)}{6n^2}\right\}\end{aligned}$$

$$\text{the area of the parabola} = ab\left\{1 - \frac{(2n+1)(n+1)}{6n^2}\right\}$$

$$= \frac{2}{3}ab,$$

when n becomes infinity.

Hence the area of such a parabola is equal to two-thirds of that of the circumscribed rectangle.

This last example, however, more properly belongs to the Integral Calculus.

The student will find no difficulty in applying the infinitesimal method to the proof of the following problems:

(1) The tangent of an ellipse makes equal angles with the focal distances.

(2) The tangent of a parabola makes equal angles with the focal distance and a line through the point of contact and parallel to the axis.

(3) The curve which cuts a series of confocal ellipses at right angles is a confocal hyperbola.

(4) The area of an ellipse is to that of a circle whose radius is the semi-axis major of the ellipse as the axis minor is to the axis major.

Many other problems will be introduced hereafter.

CHAPTER II.

CONSTRUCTION OF RULES FOR THE DIFFERENTIATION OF FUNCTIONS.

SECTION 1.—*The differentiation of explicit functions of one variable.*

25.] THE nature of infinitesimals, their mode of discovery, the means of expressing them, and the kind of problems to which they are applicable, have been explained in the preceding Chapter. I propose therefore to investigate in the first place certain properties which are generally true of all functions, and in the second place to consider certain special forms; because we shall thereby construct rules which will be useful subordinately, and we shall be spared the labour of continually having recourse to first principles. The convenience of this method is manifest.

If a constant is connected with a function of a variable by the symbol of addition or subtraction, it disappears in the process of differentiation.

$$\begin{aligned}
\text{Let} \qquad y &= f(x) \pm c, \\
y + \Delta y &= f(x + \Delta x) \pm c; \\
\therefore \quad \Delta y &= f(x + \Delta x) - f(x), \\
dy &= f(x + dx) - f(x) \\
&= d.f(x) \\
&= f'(x)\,dx, \text{ by Art. 18.} \qquad (1)
\end{aligned}$$

This result is manifest from the very nature of constants, which do not admit of increase, and therefore have the same value in both states in which the function is considered, and therefore disappear in the subtraction; thus we may say, that the differential of a constant is zero.

Ex. 1. $y = a + x, \qquad dy = dx.$

Ex. 2. $y = x - a, \qquad dy = dx.$

26.] A constant connected with a function of a variable by the processes of multiplication or division is not changed by differentiation.

Let
$$\begin{aligned} y &= cf(x), \\ y+\Delta y &= cf(x+\Delta x), \\ \Delta y &= c\{f(x+\Delta x)-f(x)\}, \\ dy &= c\{f(x+dx)-f(x)\} \\ &= cd.f(x) \\ &= cf'(x)\,dx. \end{aligned} \tag{2}$$

Or again:

Let
$$\begin{aligned} y &= \frac{1}{c}f(x), \\ \Delta y &= \frac{1}{c}\{f(x+\Delta x)-f(x)\}, \\ dy &= \frac{1}{c}\{f(x+dx)-f(x)\} \\ &= \frac{1}{c}d.f(x) \\ &= \frac{1}{c}f'(x)\,dx. \end{aligned}$$

Suppose $c = -1$, then
$$\begin{aligned} y &= -f(x), \\ dy &= -d.f(x) \\ &= -f'(x)\,dx. \end{aligned}$$

Ex. 1. $y = ax+b$, $\quad dy = a\,dx$.

Ex. 2. $y = \frac{x}{a}-c$, $\quad dy = \frac{dx}{a}$.

27.] The differentiation of an algebraical sum of functions.

Let
$$\begin{aligned} y &= f(x) \pm \text{F}(x) \pm \phi(x) + \ldots\ldots \\ y+\Delta y &= f(x+\Delta x) \pm \text{F}(x+\Delta x) \pm \phi(x+\Delta x) + \ldots\ldots \\ \Delta y &= f(x+\Delta x)-f(x) \pm \{\text{F}(x+\Delta x)-\text{F}(x)\} \\ &\qquad \pm \{\phi(x+\Delta x)-\phi(x)\} + \ldots\ldots \\ \therefore\; dy &= f(x+dx)-f(x) \pm \{\text{F}(x+dx)-\text{F}(x)\} \\ &\qquad \pm \{\phi(x+dx)-\phi(x)\} \pm \ldots\ldots \\ &= d.f(x) \pm d.\text{F}(x) \pm d.\phi(x) \pm \ldots\ldots \\ &= f'(x)\,dx \pm \text{F}'(x)\,dx \pm \phi'(x)\,dx \pm \ldots\ldots \end{aligned} \tag{3}$$

Hence the differential of the algebraical sum of functions is equal to the sum of the differentials of the functions.

Ex. 1.
$$y = ax - bf(x) + c,$$
$$\therefore \quad dy = a\,dx - bf'(x)\,dx.$$

From the last two Articles it is plain, that if the function to be differentiated is of the form
$$y = f(x) + \sqrt{-1}\,\phi(x),$$
one of the functions being what is commonly called imaginary or impossible,
$$\begin{aligned} dy &= d.f(x) + \sqrt{-1}\,d.\phi(x) \\ &= f'(x)\,dx + \sqrt{-1}\,\phi'(x)\,dx. \end{aligned}$$
Whence it appears that, in the differentiation of impossible quantities, we may treat the symbol of impossibility in the same way as we treat an ordinary constant or symbol of affection.

28.] Differentiation of a product of two functions of a variable.

Let
$$\begin{aligned} y &= f(x) \times \phi(x), \\ \therefore \quad y + \Delta y &= f(x+\Delta x) \times \phi(x+\Delta x), \\ \therefore \quad \Delta y &= f(x+\Delta x) \times \phi(x+\Delta x) - f(x) \times \phi(x); \end{aligned}$$
whence, adding and subtracting the required quantities,
$$\begin{aligned} \Delta y = {} & \{f(x+\Delta x) - f(x)\}\,\phi(x) + \{\phi(x+\Delta x) - \phi(x)\}\,f(x) \\ & + \{f(x+\Delta x) - f(x)\}\,\{\phi(x+\Delta x) - \phi(x)\}, \end{aligned}$$
$$\therefore \quad dy = d.f(x) \times \phi(x) + d.\phi(x) \times f(x) + d.f(x) \times d.\phi(x);$$
and omitting the last term, which is an infinitesimal of the second order, we have
$$\begin{aligned} dy &= d.f(x) \times \phi(x) + d.\phi(x) \times f(x), \\ &= f'(x)\,\phi(x)\,dx + \phi'(x)\,f(x)\,dx. \end{aligned} \qquad (4)$$
Hence it follows that, the differential of the product of two functions is the sum of the products of each function and the differential of the other.

Ex. 1.
$$\begin{aligned} y &= (b+cx)\,(e-ax), \\ \therefore \quad dy &= c\,dx\,(e-ax) - a\,dx\,(b+cx), \\ &= (ec - ab - 2acx)\,dx. \end{aligned}$$

29.] Differentiation of the quotient of two functions.

Let
$$y = \frac{f(x)}{\phi(x)}, \qquad y + \Delta y = \frac{f(x+\Delta x)}{\phi(x+\Delta x)};$$

$$\therefore \quad \Delta y = \frac{f(x+\Delta x)}{\phi(x+\Delta x)} - \frac{f(x)}{\phi(x)}$$

$$= \frac{f(x+\Delta x)\,\phi(x) - f(x)\,\phi(x+\Delta x)}{\phi(x)\,\phi(x+\Delta x)}$$

$$= \frac{\{f(x+\Delta x) - f(x)\}\,\phi(x) - \{\phi(x+\Delta x) - \phi(x)\}\,f(x)}{\phi(x)\,\phi(x+\Delta x)};$$

$$\therefore \quad dy = \frac{d.f(x) \times \phi(x) - d.\phi(x) \times f(x)}{\{\phi(x)\}^2},$$

omitting the infinitesimal dx in the denominator, because it is added to the finite quantity x; and therefore

$$dy = \frac{f'(x)\,\phi(x)\,dx - \phi'(x)\,f(x)\,dx}{\{\phi(x)\}^2}. \tag{5}$$

Ex. 1. $$y = \frac{a+bx}{b+ax},$$

$$dy = \frac{b\,dx\,(b+ax) - a\,dx\,(a+bx)}{(b+ax)^2}$$

$$= \frac{b^2-a^2}{(b+ax)^2}\,dx.$$

30.] Differentiation of x^n.

Let $$y = x^n, \qquad y+\Delta y = (x+\Delta x)^n,$$

$$\therefore \quad \Delta y = (x+\Delta x)^n - x^n$$

$$= x^n\left\{\left(1+\frac{\Delta x}{x}\right)^n - 1\right\}.$$

Let $$\left(1+\frac{\Delta x}{x}\right)^n - 1 = z,$$

so that Δx and z are simultaneously infinitesimal;

$$\therefore \quad \left(1+\frac{\Delta x}{x}\right)^n = 1+z, \qquad n\log_e\left(1+\frac{\Delta x}{x}\right) = \log_e(1+z).$$

Let Δx and z become infinitesimal; then by Cor. I, Lemma I, Art. 21,

$$n\frac{dx}{x} = z;$$

therefore when Δx is infinitesimal, $\left(1+\frac{\Delta x}{x}\right)^n - 1$ is to be replaced by $n\frac{dx}{x}$:

$$\therefore \quad dy = x^n n\frac{dx}{x},$$

$$= nx^{n-1}\,dx;$$

that is, $$d.x^n = nx^{n-1}\,dx. \tag{6}$$

Therefore to differentiate x^n we have the following rule: Multiply by the exponent, diminish the exponent by unity, and multiply by dx.

Suppose the exponent to be negative, then

$$y = x^{-n}, \qquad dy = -nx^{-(n+1)}\,dx.$$

And suppose $n = -1$,

$$y = \frac{1}{x} = x^{-1}, \qquad dy = -x^{-2}\,dx = -\frac{dx}{x^2}.$$

Suppose the index to be fractional, $n = \frac{p}{q}$;

$$y = x^{\frac{p}{q}}, \qquad dy = \frac{p}{q}\,x^{\frac{p}{q}-1}\,dx.$$

Also, since $\frac{dy}{dx} = nx^{n-1}$, it is manifest that if $f(x) = x^n$, $f'(x) = nx^{n-1}$; that is, the derived function of x^n is nx^{n-1}.

The above rule for differentiating x^n might also have been determined as follows:

$$y = x^n, \qquad y + \Delta y = (x + \Delta x)^n;$$

$$\therefore \quad \Delta y = x^n + \frac{n}{1}x^{n-1}\Delta x + \frac{n(n-1)}{1.2}x^{n-2}(\Delta x)^2 + \ldots - x^n$$

$$= \frac{n}{1}x^{n-1}\Delta x + \frac{n(n-1)}{1.2}x^{n-2}(\Delta x)^2 + \ldots;$$

let the increments be infinitesimal, and we have

$$dy = nx^{n-1}\,dx.$$

Ex. 1. $y = a + bx^3, \qquad \therefore \quad dy = 3bx^2\,dx.$

Ex. 2. $y = a\sqrt{x}, \qquad dy = \dfrac{a\,dx}{2\sqrt{x}}.$

Ex. 3. $y = c + \dfrac{b}{x^{\frac{1}{2}}}, \qquad dy = -\dfrac{b\,dx}{2x^{\frac{3}{2}}}.$

Ex. 4. $y = \dfrac{a}{x^2}, \qquad dy = -\dfrac{2a\,dx}{x^3}.$

Ex. 5. $y = (a + bx^n)(c - ex^m),$

$$dy = nbx^{n-1}\,dx\,(c - ex^m) - mex^{m-1}\,dx\,(a + bx^n).$$

Ex. 6. $y = \dfrac{x^2}{a^2 + x^2},$

$$dy = \frac{2x\,dx\,(a^2 + x^2) - 2x\,dx\,x^2}{(a^2 + x^2)^2}$$

$$= \frac{2a^2x\,dx}{(a^2 + x^2)^2}.$$

31.] Differentiation of a compound function, or of a function of a function.

Thus far we have calculated the change of value of a function of x, due to a small variation of x; but suppose the function to be compound, and thus to depend on x, not immediately, but by means of some function; what modification must the principles and results of the preceding Articles undergo? The only condition to which x is subject is, that it is a *continuous* and finite variable for the values assigned to it; and this condition will be fulfilled, if x is replaced by any *continuous* and finite function of x, and if its variation due to a small variation of x is infinitesimal: in this case therefore the original function will vary in consequence of the variation of this latter function. *Pari ratione* this latter function may be a compound function, and therefore not vary immediately by reason of the variation of its variable, but only through some other function; and so on to any number of functions. Taking then our former notation, in this case y is a function of a function of x, or a function of many functions of x, and its differential must be calculated as follows: and let us first take the case in which only two functions are involved: viz. where $y = f\{\phi(x)\}$. Since if $y = f(x)$, $dy = f'(x)\,dx$; therefore, if x is replaced by $\phi(x)$, dx must be replaced by the differential of $\phi(x)$, that is, by $\phi'(x)\,dx$; and therefore if

$$y = f\{\phi(x)\}, \qquad dy = f'\{\phi(x)\}\,\phi'(x)\,dx.$$

And so again, if the x involved in $\phi(x)$ is a function of x, say $\psi(x)$, $\phi(x)$ will be replaced by $\phi\{\psi(x)\}$, and $\phi'(x)\,dx$ by $\phi'\{\psi(x)\}\,\psi'(x)\,dx$; and a similar notation will adapt itself to a function of a function ...... (to n functions) of x. Thus if

$$y = f\{\phi[\psi(x)]\}, \qquad dy = f'\{\phi[\psi(x)]\}\,\phi'[\psi(x)]\,\psi'(x)\,dx.$$

Or we may substitute as follows:

$$\begin{aligned} \text{Let} \quad \psi(x) &= z, & \therefore\ dz &= \psi'(x)\,dx, \\ \phi[\psi(x)] &= \phi(z) = u, & \therefore\ du &= \phi'(z)\,dz, \\ f\{\phi[\psi(x)]\} &= f(u) = y, & \therefore\ dy &= f'(u)\,du; \\ \therefore\ dy &= f'(u)\,du \\ &= f'(u)\,\phi'(z)\,dz \\ &= f'(u)\,\phi'(z)\,\psi'(x)\,dx \\ &= \frac{dy}{du}\frac{du}{dz}\frac{dz}{dx}\,dx: \end{aligned} \tag{7}$$

But as generally it is convenient to make as few substitutions as possible, the former value of dy is preferable to the latter; the latter also, it is to be observed, is an identity.

Hence, by reason of the last Article, if

$$y = \{f(x)\}^n, \qquad dy = n\{f(x)\}^{n-1}f'(x)\,dx.$$

Ex. 1. $\quad y = f(x \pm a), \qquad dy = f'(x \pm a)\,dx.$

Ex. 2. $\quad y = f(cx), \qquad dy = cf'(cx)\,dx.$

Ex. 3. $\quad y = f(ax^n \pm b), \qquad dy = anf'(ax^n \pm b)\,x^{n-1}\,dx.$

Ex. 4. $\quad y = (ax^n + c)^m, \qquad dy = m(ax^n + c)^{m-1}\,nax^{n-1}\,dx.$

Ex. 5. $\quad y = (a + bx^3)^2\,(a - x)^{\frac{3}{2}},$

$$dy = (a + bx^3)\,(a - x)^{\frac{1}{2}}\,\frac{12abx^2 - 15bx^3 - 3a}{2}\,dx.$$

Ex. 6. $\quad y = \dfrac{(x+3)^3}{(x^2-2)^2},$

$$dy = \frac{3(x+3)^2\,(x^2-2)^2\,dx - 4x\,(x^2-2)\,(x+3)^3\,dx}{(x^2-2)^4}$$

$$= \frac{(x+3)^2\,(-x^2 - 12x - 6)}{(x^2-2)^3}\,dx.$$

32.] Differentiation of a^x.

Let $\quad y = a^x, \qquad y + \Delta y = a^{x+\Delta x};$

$$\therefore \quad \Delta y = a^{x+\Delta x} - a^x,$$

$$= a^x\{a^{\Delta x} - 1\}.$$

Let $a^{\Delta x} - 1 = z$, so that Δx and z are simultaneously infinitesimal;

$$\therefore \quad a^{\Delta x} = z + 1,$$

$$\Delta x \log_e a = \log_e (1 + z),$$

and by Cor. I, Lemma I, $\log_e (1 + z) = z$, when z is infinitesimal;

$$\therefore \quad dx \log_e a = z;$$

and therefore $a^{dx} - 1$ must be replaced by $dx \log_e a$;

$$\therefore \quad dy = a^x \log_e a . dx,$$

$$d.a^x = \log_e a . a^x\,dx: \tag{8}$$

that is, the differential of a^x is the product of a^x, of the Napierian logarithm of a, and of the differential of the exponent.

And hence, if $a^x = f(x)$, $f'(x) = \log_e a . a^x$.

33.] Differentiation of e^x.

In the last Article let the base a be replaced by the Napierian base e; in which case $\log_e a$ becomes $\log_e e$, which is equal to 1; and therefore

$$d.e^x = e^x\,dx: \qquad (9)$$

that is, the differential of e^x is the product of the quantity and the differential of the exponent.

Hence, if $f(x) = e^x$, $f'(x) = e^x$; and e^x is a quantity which is equal to its own derived function.

And similar results are true by reason of Article 31, when x is replaced by any function of x; hence we have

$$d.a^{f(x)} = \log_e a.a^{f(x)}f'(x)\,dx; \qquad (10)$$

$$d.e^{f(x)} = e^{f(x)}f'(x)\,dx. \qquad (11)$$

Ex. 1. $y = a^{cx}$, $dy = \log_e a.a^{cx}\,c\,dx$.

Ex. 2. $y = e^{(b+cx^2)}$, $dy = e^{(b+cx^2)}\,2cx\,dx$.

Ex. 3. $y = e^{-x^n}$, $dy = -e^{-x^n}nx^{n-1}dx$.

Ex. 4. $y = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$,

$$dy = \sqrt{-1}\,\{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\}\,dx.$$

Ex. 5. $y = e^x\,(x^2 - 2x + 2)$,

$$\begin{aligned} dy &= e^x\,\{x^2-2x+2\}\,dx + e^x\,\{2x-2\}\,dx, \\ &= e^x x^2\,dx. \end{aligned}$$

Ex. 6. $y = e^{e^x}$, $dy = e^{e^x}.e^x.dx$.

34.] Differentiation of $\log_a x$.

Let $y = \log_a x$, $y + \Delta y = \log_a(x + \Delta x)$;

$$\begin{aligned} \Delta y &= \log_a(x+\Delta x) - \log_a x, \\ &= \log_a\left(\frac{x+\Delta x}{x}\right) \\ &= \log_a\left(1+\frac{\Delta x}{x}\right); \end{aligned}$$

but when $\frac{\Delta x}{x}$ becomes an infinitesimal, viz. $\frac{dx}{x}$, $\log_a\left(1+\frac{\Delta x}{x}\right) = \frac{dx}{x\log_e a}$ by Cor. II, Lemma I, Art. 21; hence

$$dy = \frac{dx}{x\log_e a}; \qquad \therefore\ d.\log_a x = \frac{dx}{x\log_e a}. \qquad (12)$$

Therefore if

$$f(x) = \log_a x, \qquad f'(x) = \frac{1}{x\log_e a}.$$

Hence also

$$d.\log_a f(x) = \frac{f'(x)\,dx}{\log_e a.f(x)}. \qquad (13)$$

35.] Differentiation of $\log_e x$.

Suppose the base a of the last Article to be e, then since $\log_e e = 1$,

$$d.\log_e x = \frac{dx}{x}.$$

And therefore if

$$f(x) = \log_e x,$$

$$f'(x) = \frac{1}{x}.$$

Hence also

$$d.\log_e f(x) = \frac{f'(x)\,dx}{f(x)}. \tag{14}$$

Hence the differential of the Napierian logarithm of any function of x is equal to the differential of the function divided by the function itself.

36.] The general result of Article 34 might also have been found from that of Article 32, as follows:

$$y = \log_a x, \qquad \therefore \quad a^y = x;$$

$$\therefore \quad \log_e a . a^y\, dy = dx, \qquad \therefore \quad dy = \frac{dx}{x \log_e a}.$$

Ex. 1. $y = \log x^n$, $\qquad dy = \dfrac{n x^{n-1}\,dx}{x^n} = n\dfrac{dx}{x}.$

Ex. 2. $y = \log\{x + (1+x^2)^{\frac{1}{2}}\}$, $\qquad dy = \dfrac{dx}{\{1+x^2\}^{\frac{1}{2}}}.$

Ex. 3. $y = \log\log x$, $\qquad dy = \dfrac{d.\log x}{\log x} = \dfrac{dx}{x \log x}.$

Ex. 4. $y = x^2 \log x$,

$$dy = 2x\,dx \log_e x + x\,dx$$
$$= \{2\log_e x + 1\}\, x\,dx.$$

Ex. 5. $y = f(x) \times \phi(x)$,

$$\therefore \quad \log_e y = \log_e f(x) + \log_e \phi(x),$$

$$\frac{dy}{y} = \frac{f'(x)\,dx}{f(x)} + \frac{\phi'(x)\,dx}{\phi(x)},$$

$$\therefore \quad dy = f'(x) \times \phi(x)\,dx + \phi'(x) f(x)\,dx.$$

Ex. 6. $y = \{f(x)\}^{\phi(x)}$,

$$\log_e y = \phi(x) \log_e f(x),$$

$$\frac{dy}{y} = \phi'(x)\,dx . \log_e f(x) + \phi(x)\frac{f'(x)\,dx}{f(x)},$$

$$\therefore \quad dy = \{f(x)\}^{\phi(x)} \left\{ \phi'(x) \log_e f(x) + \phi(x)\frac{f'(x)}{f(x)} \right\} dx.$$

Ex. 7. $y = x^x$,

$$\log_e y = x\log_e x, \qquad \frac{dy}{y} = dx\log_e x + dx,$$

$$\therefore \quad dy = x^x(1+\log_e x)\,dx.$$

Ex. 8. $y = x^n$,

$$\log_e y = n\log_e x; \qquad \frac{dy}{y} = n\frac{dx}{x},$$

$$\therefore \quad dy = nx^{n-1}dx;$$

whence we have an independent proof of the result of Art. 30.

37.] The differentiation of functions of x connected with one another by multiplication or division.

$$y = f(x)\times \text{F}(x)\times\phi(x)\times \ldots\ldots$$

If all the functions are positive, we have

$$\log y = \log f(x) + \log \text{F}(x) + \log\phi(x) + \ldots\ldots;$$

$$\therefore \quad \frac{dy}{y} = \frac{d.f(x)}{f(x)} + \frac{d.\text{F}(x)}{\text{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \ldots\ldots$$

$$dy = d\{f(x)\,\text{F}(x)\,\phi(x)\ldots\}$$

$$= f(x)\,\text{F}(x)\,\phi(x)\ldots\left\{\frac{d.f(x)}{f(x)} + \frac{d.\text{F}(x)}{\text{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \ldots\right\}.$$

Or, if some of the functions are negative,

$$y^2 = [f(x)]^2\,[\text{F}(x)]^2\,[\phi(x)]^2\ldots\ldots;$$

$$\log(y^2) = \log[f(x)]^2 + \log[\text{F}(x)]^2 + \log[\phi(x)]^2 + \ldots\ldots$$

$$\frac{dy}{y} = \frac{d.f(x)}{f(x)} + \frac{d.\text{F}(x)}{\text{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \ldots\ldots$$

$$\therefore \quad dy = f(x)\,\text{F}(x)\,\phi(x)\ldots\left\{\frac{d.f(x)}{f(x)} + \frac{d.\text{F}(x)}{\text{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \ldots\right\}.$$

Therefore, in either case,

$$d\{f(x)\,\text{F}(x)\,\phi(x)\ldots\} = d.f(x)\,\text{F}(x)\,\phi(x)\ldots + d.\text{F}(x)f(x)\phi(x)\ldots$$
$$+\,d.\phi(x)f(x)\,\text{F}(x)\ldots + \ldots \quad (15)$$

And, if $\qquad y = \dfrac{f(x)}{\phi(x)}$,

$$\log(y^2) = \log[f(x)]^2 - \log[\phi(x)]^2;$$

$$\therefore \quad \frac{dy}{y} = \frac{d.f(x)}{f(x)} - \frac{d.\phi(x)}{\phi(x)},$$

$$\therefore \quad dy = d.\left\{\frac{\phi(x)}{f(x)}\right\} = \frac{f(x)}{\phi(x)}\left\{\frac{d.f(x)}{f(x)} - \frac{d.\phi(x)}{\phi(x)}\right\}$$

$$= \frac{d.f(x)\,\phi(x) - d.\phi(x)\,f(x)}{\{\phi(x)\}^2}$$

$$= \frac{f'(x)\,\phi(x)\,dx - \phi'(x)\,f(x)\,dx}{\{\phi(x)\}^2}; \qquad (16)$$

which result is the same as that obtained in Art. 29.

In the latter cases we have raised both sides to the square power, in order to avoid the apparent difficulty of taking the logarithms of negative quantities; but had this not been done, we should have had $\log_e(-1)$, which will be shewn in the next Chapter to be equal to $(2k+1)\,\pi\sqrt{-1}$, the differential of which is zero, because it is a constant quantity.

Ex. 1. $$y = (a+x)\,(b+2x)\,(c+3x),$$

$$dy = dx\,(b+2x)\,(c+3x) + 2\,dx\,(c+3x)\,(a+x) + 3\,dx\,(a+x)\,(b+2x).$$

Ex. 2. $$y = e^x\left\{\frac{x+1}{x-1}\right\}^{\frac{1}{2}},$$

$$\therefore \quad \log_e y = x + \frac{1}{2}\log(x+1) - \frac{1}{2}\log(x-1),$$

$$\frac{dy}{y} = dx + \frac{1}{2}\,\frac{dx}{x+1} - \frac{1}{2}\,\frac{dx}{x-1};$$

$$\therefore \quad dy = y\left\{dx - \frac{dx}{x^2-1}\right\}$$

$$= y\left\{\frac{x^2-2}{x^2-1}\right\}dx$$

$$= e^x\,\frac{(x^2-2)\,dx}{(x+1)^{\frac{1}{2}}\,(x-1)^{\frac{3}{2}}}.$$

And a complicated product or quotient may often be most easily differentiated by first taking the logarithms, and then differentiating and reducing.

38.] Differentiation of (α) $\sin x$, (β) $\cos x$, (γ) $\tan x$, (δ) $\sec x$, (ϵ) $\operatorname{versin} x$, (ζ) $\cot x$, (η) $\operatorname{cosec} x$.

(α) $$y = \sin x,$$

$$y + \Delta y = \sin(x + \Delta x);$$

$$\therefore \quad \Delta y = \sin(x+\Delta x) - \sin x;$$

and since $$\sin A - \sin B = 2\cos\frac{A+B}{2}\,\sin\frac{A-B}{2},$$

$$\therefore \quad \Delta y = 2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}.$$

Let the increments be infinitesimal; then replacing $\sin\frac{dx}{2}$ by $\frac{dx}{2}$ in accordance with Lemma II, Article 22, and omitting $\frac{dx}{2}$ which is added to x, we have

$$dy = 2\cos x\frac{dx}{2};$$

$$\therefore \quad d.\sin x = \cos x\,dx. \tag{17}$$

If therefore $f(x) = \sin x,$

$$f'(x) = \cos x.$$

Hence also $d.\sin f(x) = \cos f(x).f'(x)\,dx.$ (18)

(β)

$$y = \cos x,$$

$$y+\Delta y = \cos(x+\Delta x);$$

$$\therefore \quad \Delta y = \cos(x+\Delta x)-\cos x$$

$$= -2\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2},$$

because $\cos A-\cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2};$

$$\therefore \quad dy = -2\sin x\frac{dx}{2};$$

$$\therefore \quad d.\cos x = -\sin x\,dx. \tag{19}$$

If therefore $f(x) = \cos x,$

$$f'(x) = -\sin x.$$

Hence also $d.\cos f(x) = -\sin f(x).f'(x)\,dx.$ (20)

(γ)

$$y = \tan x,$$

$$y+\Delta y = \tan(x+\Delta x);$$

$$\therefore \quad \Delta y = \tan(x+\Delta x)-\tan x$$

$$= \frac{\tan x+\tan\Delta x}{1-\tan x\tan\Delta x}-\tan x$$

$$= \frac{\tan\Delta x\,\{1+(\tan x)^2\}}{1-\tan x\tan\Delta x}.$$

Let the increments be infinitesimal, in which case $\tan dx$ must be replaced by dx by reason of Lemma II, Art. 22; therefore

$$dy = \frac{\{1+(\tan x)^2\}\,dx}{1-\tan x\,dx}$$

$$= \{1+(\tan x)^2\}\,dx, \tag{21}$$

omitting the term of the denominator involving dx, because it is an infinitesimal subtracted from a finite quantity;

$$\therefore \quad d.\tan x = \{1+(\tan x)^2\}\,dx = (\sec x)^2\,dx.$$

Hence if $\quad f(x) = \tan x, \qquad f'(x) = (\sec x)^2.$

Also $\quad d.\tan f(x) = \{\sec f(x)\}^2.f'(x)\,dx. \qquad (22)$

(δ) $\quad y = \sec x = \dfrac{1}{\cos x} = (\cos x)^{-1};$

$$\therefore \quad dy = (\cos x)^{-2}\sin x\,dx, \text{ by Art. 30,}$$

$$\therefore \quad dy = d.\sec x = \frac{\sin x\,dx}{(\cos x)^2} = \sec x\tan x\,dx.$$

If therefore $\quad f(x) = \sec x,$

$$f'(x) = \sec x.\tan x;$$

and $\quad d.\sec f(x) = \sec f(x).\tan f(x).f'(x)\,dx. \qquad (23)$

(ε) $\quad y = \operatorname{versin} x = 1-\cos x;$

$$\therefore \quad dy = d.(1-\cos x) = \sin x\,dx.$$

If therefore $\quad f(x) = \operatorname{versin} x,$

$$f'(x) = \sin x;$$

and $\quad d.\operatorname{versin} f(x) = \sin f(x).f'(x)\,dx. \qquad (24)$

(ζ) $\quad y = \cot x,$

$$\begin{aligned} y+\Delta y &= \cot(x+\Delta x), \\ \Delta y &= \cot(x+\Delta x)-\cot x \\ &= \frac{1-\tan x\tan\Delta x}{\tan x+\tan\Delta x} - \frac{1}{\tan x} \\ &= \frac{-\{1+(\tan x)^2\}\tan\Delta x}{(\tan x+\tan\Delta x)\tan x}; \end{aligned}$$

$$\begin{aligned} \therefore \quad dy &= -\frac{(\sec x)^2\,dx}{(\tan x)^2} \\ &= -\frac{dx}{(\sin x)^2} = -(\operatorname{cosec} x)^2\,dx. \qquad (25) \end{aligned}$$

If therefore $\quad f(x) = \cot x,$

$$f'(x) = -(\operatorname{cosec} x)^2;$$

and $\quad d.\cot f(x) = -\{\operatorname{cosec} f(x)\}^2 f'(x)\,dx. \qquad (26)$

(η) $\quad y = \operatorname{cosec} x$

$$= \frac{1}{\sin x} = (\sin x)^{-1},$$

$$\begin{aligned} dy &= -(\sin x)^{-2}\cos x\,dx \\ &= -\frac{\cos x}{(\sin x)^2}\,dx, \\ d.\operatorname{cosec} x &= -\operatorname{cosec} x\cot x\,dx. \qquad (27) \end{aligned}$$

If therefore $f(x) = \operatorname{cosec} x,$

$$f'(x) = -\operatorname{cosec} x.\cot x;$$

and $d.\operatorname{cosec} f(x) = -\operatorname{cosec} f(x)\cot f(x).f'(x)\,dx. \qquad (28)$

39.] It is also manifest from the geometry of the figure, see fig. 8, that the increments of the trigonometrical functions due to the increment of the arc are such as have been deduced in the above formulæ. For let AP be the arc of a circle whose radius is unity, and PQ be any small arc added to it, which ultimately becomes infinitesimal:

Let $AP = x, \quad PQ = \Delta x.$

$$\begin{aligned} \text{Then } PM &= \sin x, & NQ &= \sin(x+\Delta x); & \therefore\ QR &= \Delta.\sin x. \\ CM &= \cos x, & CN &= \cos(x+\Delta x); & \therefore\ NM &= -\Delta.\cos x. \\ AT &= \tan x, & AT' &= \tan(x+\Delta x); & \therefore\ TT' &= \Delta.\tan x. \\ CT &= \sec x, & CT' &= \sec(x+\Delta x); & \therefore\ ST' &= \Delta.\sec x. \end{aligned}$$

$$ST : QP :: CT : CP, \quad \text{i. e. } ST : \Delta x :: \sec x : 1.$$

$$\therefore \quad ST = \Delta x \sec x.$$

When Δx becomes dx, PQ, ST become straight lines, and perpendicular to CT, and ST becomes $dx \sec x$: whence

$$\begin{aligned} d.\sin x &= QR = PQ \sin QPR = PQ \cos RPC = PQ \cos PCA \\ &= dx \cos x. \\ d.\cos x &= -NM = -PR = -PQ \cos QPR = -PQ \sin PCA \\ &= -dx \sin x. \\ d.\tan x &= TT' = ST \sec T'TS = ST \operatorname{cosec} CTA = ST \sec PCA \\ &= (\sec x)^2\,dx. \\ d.\sec x &= ST' = ST \tan T'TS = ST \tan PCA = \sec x \tan x\,dx. \end{aligned}$$

40.] The differentials of $\tan x$ and $\cot x$ may also be determined from those of $\sin x$ and $\cos x$ as follows:

$$y = \tan x = \frac{\sin x}{\cos x};$$

therefore, by Art. 29,

$$dy = \frac{\cos x\, d.\sin x - \sin x\, d.\cos x}{(\cos x)^2}$$

$$\therefore \quad d.\tan x = \frac{(\sin x)^2\, dx + (\cos x)^2\, dx}{(\cos x)^2}$$
$$= \frac{dx}{(\cos x)^2}$$
$$= (\sec x)^2\, dx.$$

And
$$\cot x = \frac{\cos x}{\sin x};$$
$$\therefore \quad d.\cot x = \frac{\sin x\, d.\cos x - \cos x\, d.\sin x}{(\sin x)^2}$$
$$= \frac{-(\sin x)^2\, dx - (\cos x)^2\, dx}{(\sin x)^2}$$
$$= \frac{-dx}{(\sin x)^2}$$
$$= -(\operatorname{cosec} x)^2\, dx.$$

41.] Ex. 1. $y = \sin x^2$, $\qquad dy = \cos x^2\, 2x\, dx.$

Ex. 2. $y = \sin \sin x$, $\qquad dy = \cos \sin x \cos x\, dx.$

Ex. 3. $y = a^{\cos x}$, $\qquad dy = -\log_e a . a^{\cos x} \sin x\, dx.$

Ex. 4. $y = (\sin x + \cos x)^n$,
$$dy = n(\sin x + \cos x)^{n-1} (\cos x - \sin x)\, dx.$$

Ex. 5. $y = e^{ax} \cos nx$,
$$dy = ae^{ax} \cos nx\, dx - ne^{ax} \sin nx\, dx$$
$$= e^{ax}\{a \cos nx - n \sin nx\}\, dx.$$

Ex. 6. $y = \dfrac{(\sin x)^n}{\sin nx}$,
$$dy = \frac{n(\sin x)^{n-1} \cos x \sin nx\, dx - n(\sin x)^n \cos nx\, dx}{(\sin nx)^2}$$
$$= \frac{n(\sin x)^{n-1} (\cos x \sin nx - \sin x \cos nx)\, dx}{(\sin nx)^2}$$
$$= \frac{n(\sin x)^{n-1} \sin (n-1) x\, dx}{(\sin nx)^2}.$$

42.] Differentiation of inverse circular functions.

(a) $y = \sin^{-1} \dfrac{x}{a}$, $\qquad y + \Delta y = \sin^{-1} \dfrac{x + \Delta x}{a}$;

$$\therefore \quad \Delta y = \sin^{-1} \frac{x + \Delta x}{a} - \sin^{-1} \frac{x}{a}$$
$$= \sin^{-1} \left\{ \frac{x + \Delta x}{a} \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} - \frac{x}{a} \left\{1 - \left(\frac{x + \Delta x}{a}\right)^2\right\}^{\frac{1}{2}} \right\}$$

$$\Delta y = \sin^{-1}\left\{\frac{x+\Delta x}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}-\frac{x}{a}\left(1-\frac{x^2}{a^2}-\frac{2x\,\Delta x}{a^2}\right)^{\frac{1}{2}}\right\}$$

$$= \sin^{-1}\left\{\frac{x+\Delta x}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}-\frac{x}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}+\frac{x^2\,\Delta x}{a^3}\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}\right\},$$

$$\therefore\quad dy = \sin^{-1}\left\{\frac{dx}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}+\frac{x^2\,dx}{a^3}\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}\right\}$$

$$= \frac{dx}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}+\frac{x^2\,dx}{a^3}\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}},$$

by reason of Cor. II, Lemma II, Art. 22;

$$\therefore\quad d.\sin^{-1}\frac{x}{a} = \frac{dx}{\{a^2-x^2\}^{\frac{1}{2}}}. \tag{29}$$

If therefore $f(x) = \sin^{-1}\dfrac{x}{a}$,

$$f'(x) = \frac{1}{\{a^2-x^2\}^{\frac{1}{2}}}. \tag{30}$$

The differential may also be found as follows:

$$y = \sin^{-1}\frac{x}{a};$$

$$\therefore\quad x = a\sin y, \qquad dx = a\cos y\,dy;$$

but $\cos y = \{1-(\sin y)^2\}^{\frac{1}{2}} = \left\{1-\dfrac{x^2}{a^2}\right\}^{\frac{1}{2}}$:

$$\therefore\quad dx = a\left\{1-\frac{x^2}{a^2}\right\}^{\frac{1}{2}}dy = \{a^2-x^2\}^{\frac{1}{2}}\,dy,$$

$$\therefore\quad dy = d.\sin^{-1}\frac{x}{a} = \frac{dx}{\{a^2-x^2\}^{\frac{1}{2}}}; \tag{31}$$

and as this latter process is the shorter, we shall use it in the following similar problems, although the former is more direct and elementary and is equally applicable.

(β) $\quad y = \cos^{-1}\dfrac{x}{a}; \qquad \therefore\quad \cos y = \dfrac{x}{a};$

$$\therefore\quad d.\cos y = -\sin y\,dy = \frac{dx}{a};$$

but $\sin y = \{1-(\cos y)^2\}^{\frac{1}{2}} = \left\{1-\dfrac{x^2}{a^2}\right\}^{\frac{1}{2}} = \dfrac{1}{a}\{a^2-x^2\}^{\frac{1}{2}};$

$$\therefore\quad -\{a^2-x^2\}^{\frac{1}{2}}\,dy = dx,$$

$$\therefore \quad dy = d.\cos^{-1}\frac{x}{a} = \frac{-dx}{\{a^2-x^2\}^{\frac{1}{2}}}.$$

If therefore $$f(x) = \cos^{-1}\frac{x}{a},$$

$$f'(x) = \frac{-1}{\{a^2-x^2\}^{\frac{1}{2}}}. \tag{32}$$

(γ) $$y = \tan^{-1}\frac{x}{a}, \qquad \therefore \quad x = a\tan y;$$

$$\begin{aligned} dx &= a\{1+(\tan y)^2\}\,dy \\ &= a\left(1+\frac{x^2}{a^2}\right)dy, \end{aligned}$$

$$\therefore \quad dy = d.\tan^{-1}\frac{x}{a} = \frac{a\,dx}{a^2+x^2}. \tag{33}$$

If therefore $$f(x) = \tan^{-1}\frac{x}{a},$$

$$f'(x) = \frac{a}{a^2+x^2}.$$

(δ) $$y = \sec^{-1}\frac{x}{a}, \qquad \therefore \quad x = a\sec y;$$

$$\begin{aligned} dx &= a\sec y\tan y\,dy \\ &= a\frac{x}{a}\left\{\frac{x^2}{a^2}-1\right\}^{\frac{1}{2}}dy, \\ &= \frac{x}{a}\{x^2-a^2\}^{\frac{1}{2}}\,dy; \end{aligned}$$

$$\therefore \quad dy = d.\sec^{-1}\frac{x}{a} = \frac{a\,dx}{x\{x^2-a^2\}^{\frac{1}{2}}}. \tag{34}$$

If therefore $$f(x) = \sec^{-1}\frac{x}{a},$$

$$f'(x) = \frac{a}{x\{x^2-a^2\}^{\frac{1}{2}}}.$$

(ϵ) $$y = \text{versin}^{-1}\frac{x}{a}, \qquad \therefore \quad x = a\,\text{versin}\,y;$$

$$dx = a\sin y\,dy;$$

but $$\begin{aligned} \sin y &= \{1-(\cos y)^2\}^{\frac{1}{2}} = \{1-(1-\text{versin}\,y)^2\}^{\frac{1}{2}} \\ &= \left\{1-\left(1-\frac{x}{a}\right)^2\right\}^{\frac{1}{2}} \\ &= \frac{1}{a}\{2ax-x^2\}^{\frac{1}{2}}; \end{aligned}$$

$$\therefore \quad dx = \{2ax-x^2\}^{\frac{1}{2}}\,dy,$$

$$\therefore \quad dy = d.\text{versin}^{-1}\frac{x}{a} = \frac{dx}{\{2ax-x^2\}^{\frac{1}{2}}}. \qquad (35)$$

If therefore
$$f(x) = \text{versin}^{-1}\frac{x}{a},$$
$$f'(x) = \frac{1}{\{2ax-x^2\}^{\frac{1}{2}}}.$$

(ζ) Similarly, if $y = \cot^{-1}\dfrac{x}{a}$,

$$dy = d.\cot^{-1}\frac{x}{a} = \frac{-a\,dx}{a^2+x^2}. \qquad (36)$$

(η)
$$y = \text{cosec}^{-1}\frac{x}{a},$$
$$dy = d.\text{cosec}^{-1}\frac{x}{a} = \frac{-a\,dx}{x\{x^2-a^2\}^{\frac{1}{2}}}. \qquad (37)$$

In respect of the above differentials it is to be observed, that the sum of $d.\sin^{-1}\frac{x}{a}$ and $d.\cos^{-1}\frac{x}{a}$ is zero; which arises from the fact that the sum of two such arcs is $\frac{\pi}{2}$, which is constant, and whose differential therefore is equal to zero. The same thing, and for the same reason, is true of $d.\tan^{-1}\frac{x}{a}+d.\cot^{-1}\frac{x}{a}$, and of $d.\sec^{-1}\frac{x}{a}+d.\text{cosec}^{-1}\frac{x}{a}$.

43.] Hence, if the radius is unity, we have

$$d.\sin^{-1}x = \frac{dx}{\{1-x^2\}^{\frac{1}{2}}}, \qquad d.\cos^{-1}x = \frac{-dx}{\{1-x^2\}^{\frac{1}{2}}},$$

$$d.\tan^{-1}x = \frac{dx}{1+x^2}, \qquad d.\cot^{-1}x = \frac{-dx}{1+x^2},$$

$$d.\sec^{-1}x = \frac{dx}{x\{x^2-1\}^{\frac{1}{2}}}, \qquad d.\text{cosec}^{-1}x = \frac{-dx}{x\{x^2-1\}^{\frac{1}{2}}},$$

$$d.\text{versin}^{-1}x = \frac{dx}{\{2x-x^2\}^{\frac{1}{2}}}.$$

Hence also, by reason of Art. 31,

$$d.\sin^{-1}f(x) = \frac{f'(x)\,dx}{\{1-(f(x))^2\}^{\frac{1}{2}}}; \qquad (38)$$

$$d.\tan^{-1}f(x) = \frac{f'(x)\,dx}{1+(f(x))^2}. \tag{39}$$

Similar results are true of the other functions.

44.] These results may also be obtained from geometry, by a process similar to that employed in Art. 39. See fig. 8.

To find $d.\sin^{-1}x$.

$$\begin{aligned} &\text{Let } \mathrm{MP} = x, &&\therefore\ \mathrm{AP} = \sin^{-1}x = y, \\ &\mathrm{NQ} = x+\Delta x, &&\therefore\ \mathrm{AQ} = \sin^{-1}(x+\Delta x) = y+\Delta y. \end{aligned}$$

$$\therefore\ \mathrm{RQ} = \Delta x.$$

$$\Delta y = \Delta \sin^{-1}x = \text{the arc } \mathrm{PQ} = \mathrm{RQ} \sec \mathrm{PQR} = \mathrm{RQ} \sec \mathrm{PCA};$$

$$\therefore\ dy = d.\sin^{-1}x = dx \sec(\sin^{-1}x) = \frac{dx}{(1-x^2)^{\frac{1}{2}}}.$$

To find $d.\tan^{-1}x$.

$$\begin{aligned} &\mathrm{AT} = x, &&\therefore\ \mathrm{AP} = \tan^{-1}x = y, \\ &\mathrm{AT'} = x+\Delta x, &&\mathrm{AQ} = \tan^{-1}(x+\Delta x) = y+\Delta y, \end{aligned}$$

$$\therefore\ \mathrm{TT'} = \Delta x.$$

$$\therefore\ \Delta y = \Delta \tan^{-1}x = \text{the arc } \mathrm{PQ} = \mathrm{ST}\frac{\mathrm{CP}}{\mathrm{CT}},$$

because $\mathrm{PQ} : \mathrm{ST} :: \mathrm{CP} : \mathrm{CT}$;

but
$$\begin{aligned} \mathrm{ST} &= \mathrm{TT'} \cos \mathrm{STT'} = \Delta x \cos \tan^{-1}x = \frac{\Delta x}{\{1+x^2\}^{\frac{1}{2}}}, \\ \mathrm{CP} &= \mathrm{CA} = 1, \\ \mathrm{CT} &= \sec \tan^{-1}x = \{1+x^2\}^{\frac{1}{2}}; \end{aligned}$$

$$\therefore\ d.\tan^{-1}x = \frac{dx}{1+x^2};$$

and similarly may the other differentials be found.

45.] Ex. 1. $y = \sin^{-1}mx$, $\qquad dy = \dfrac{m\,dx}{\{1-m^2x^2\}^{\frac{1}{2}}}$.

Ex. 2. $y = e^{\tan^{-1}x}$, $\qquad dy = e^{\tan^{-1}x}\dfrac{dx}{1+x^2}$.

Ex. 3. $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, $\qquad dy = \dfrac{2\,dx}{1+x^2}$.

Ex. 4. $y = x^{\sin^{-1}x}$;

let us take the Napierian logarithms of both members of the equation; and we have

$$\log y = \sin^{-1}x \log x.$$

$$\frac{dy}{y} = \log x \frac{dx}{\{1-x^2\}^{\frac{1}{2}}} + \frac{\sin^{-1} x . dx}{x},$$

$$\therefore \quad dy = x^{\sin^{-1} x} \left\{ \frac{x \log x + (1-x^2)^{\frac{1}{2}} \sin^{-1} x}{x(1-x^2)^{\frac{1}{2}}} \right\} dx.$$

The number of examples which have been differentiated, including the types of all known algebraical, exponential, logarithmic, and circular forms, and all of which indicate the derived function to be a new function of x, is sufficient to justify the presumption of Art. 18, that the differential of $f(x)$ is a product of two factors, of which one is dx and the other is a finite function of x. It is in fact almost an inductive argument *per simplicem enumerationem.*

SECTION 2.—*The differentiation of functions of many variables.*

46.] On functions of many variables.

When many variables are involved in an equation, it is theoretically possible so to arrange them that one shall be a function of all the others, whereby it becomes an explicit function of many variables. And these latter variables may be independent of each other, or have such relations amongst them that a variation of one may involve a variation of one or more of the others, whereby the number of independent variables may be diminished; the former case, which is the more general, shall be first considered, and then such modifications shall be introduced into the result as shall adapt it to the latter, and as the mutual interdependence of the variables may require.

For the sake of illustration let us consider a tree, and let us suppose its growth to depend on three circumstances which are independent of each other, viz. the fertility of the soil, the rain that waters it, and the heat of the sun: then, if the relation or law of connexion of these four things is expressed mathematically and in an explicit form, we shall have the growth of the tree a function of three variables; thus,

Growth of tree = F (fertile soil, rain, solar heat).

And if the law is known which connects the single effect with the three producing causes, then F, the symbol of the form of the function, is known.

Now observing the independence of the three variables involved under the functional symbol, and that therefore any one may vary without necessitating a variation of the others, and that a variation of any one will cause a variation in the tree's growth, it follows that there may be three several variations of the effect, owing to the separate and several variations of the three producing causes. Thus, suppose the fertility of the soil to be increased, while the other two causes are unchanged, the tree's growth may be thereby increased; and similarly may it be increased by a change in either of the other two causes. Now although these three acting causes are so combined by means of the connecting equation, that when a finite change of one has taken place, a subsequent change of another may not have the same *absolute* effect on the tree's growth as if it had changed without the first having previously varied, yet such will not be the case when the changes are infinitesimal; because the infinitesimal variation of one acting subsequently to and on the back of the infinitesimal variation of the other, infinitesimals of a higher order, or products of infinitesimals, will be introduced; and these must be neglected by virtue of Theorem VI, Art. 9; and thus the absolute infinitesimal change in the tree's growth, due to the infinitesimal variation of any one of the three acting causes, will be the same, whether the other two causes have changed or not. And therefore, when all three have changed, the resulting change of the effect will be the sum of the three effects due to each of the three producing causes; of this however a mathematical and rigorous proof will be given subsequently. Thus we have variations of the growth of four distinct kinds: a variation due to each of the three causes acting separately, and the whole variation which is the sum of these three. This latter is called a *total variation* or a *total differential,* and the former are called *partial variations* or *partial differentials;* each of which, it is plain, is to be calculated on the supposition, that one variable only changes and that the others do not change.

Or again: consider a plane rectangle OPRS, see fig. 9, of which one side $OP = x$, and the other $OS = y$; and let u represent its area. Therefore,

$$u = xy.$$

Now the area may vary owing to a change in the length of either side, or to changes of the lengths of both. Suppose x to be increased by $PQ = \Delta x$, see Article 19; then the area is increased by the rectangle $RPQW = y\Delta x$; which, when Δx becomes dx, is $y\,dx$. Or suppose y to receive a finite increment, that is, OS to be increased by $\text{ST} = \Delta y$; then the increase of the area is the rectangle $\text{TR} = x\,\Delta y$; which, when the increase is infinitesimal, is $x\,dy$; hence the partial changes or increments of the rectangle are $y\,dx$ and $x\,dy$. But suppose both sides to have received infinitesimal variations; that is, x to become $x+dx$, and y to become $y+dy$, then the rectangle becomes

$$(x+dx)(y+dy) = xy + y\,dx + x\,dy + dx\,dy;$$

that is, the increment of the rectangle is $y\,dx + x\,dy + dx\,dy$, which terms severally represent the rectangles RQ, TR, UW; and therefore, when the increments are infinitesimal, the first two are rectangles of finite length but of infinitesimal breadth, and the latter, being a rectangle with infinitesimal sides, is but a *point*, which must therefore be neglected; and we have the total differential of u

$$= x\,dy + y\,dx$$
$$= \text{the sum of the partial differentials.}$$

Bearing in mind what was said in Art. 19, and that Greek letters are used to signify finite changes or differences, and English letters infinitesimal variations or differentials, we shall find the following symbolization convenient.

Let $u = \text{F}(x, y, z, \ldots)$ be a function of many variables, and let $\text{D}u$ or DF represent the total differential of u due to the variations of all the variables, and let

$d_x u$, or $d_x \text{F}$ express the partial change of u due to a change of x,
$d_y u$, or $d_y \text{F}$ - - - - - - - - - - - - - - - - - y,
$d_z u$, or $d_z \text{F}$ - - - - - - - - - - - - - - - - - z,
. . . .

and let $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$, $\left(\frac{du}{dz}\right)$, ... omitting the subscript letters, represent the ratios to the increments of the variables, of the several variations of the function due to the variation of the variables separately; that is, let them represent *partial* differential coefficients, or *partial* derived functions, the brackets indicating that they do so, and the variable in the denominator of the fractions being that, due to the change of which, the partial variation of u is calculated.

Thus, although the numerators involve the same symbol du, yet the same thing is not represented by it; for the du arising

If in a given time x increases dx: $\frac{du}{dx}$ is the rate of change of u caused by x hence $\left(\frac{du}{dx}\right)dx$ is the amount of change of u while x is increasing dx — which amt. is due to x alone.

from the growth of x into $x+dx$, may be a wholly different function from the du which comes from the growth of y into $y+dy$. If then we have occasion to use these symbols together, a mark of distinction is necessary, such as we have in $d_x u$, $d_y u, \dots$; but when the ratios of the changes are required, the denominator in addition to the bracket is sufficient to indicate the origin of the function.

Dy is the amt. which y changes while x changes dx.

That we may have distinctive names for the differentials which arise in processes so different, I shall call them the x-, the y-, ... differentials, according as they arise from the infinitesimal variation of x, of $y, \dots$; and I shall apply similar names to the several partial derived functions or differential coefficients.

Thus much as to the symbols, their nomenclature, and their meanings. Let us now consider the most simple case of a function of two independent variables.

47.] Differentiation of a function of two independent variables; and of the form $u = \text{F}(x, y)$.

Let x and y receive the increments Δx and Δy; and let the corresponding increment of u be Δu; so that

$$\Delta u = \text{F}(x+\Delta x, y+\Delta y) - \text{F}(x, y);$$

$$\therefore \quad \Delta u = \text{F}(x+\Delta x, y+\Delta y) - \text{F}(x, y+\Delta y) + \text{F}(x, y+\Delta y) - \text{F}(x, y);$$

let the variations of x and y be infinitesimal, then Δu becomes $\text{D}u$, which is the total change in u; and $\text{F}(x+\Delta x, y+\Delta y) - \text{F}(x, y+\Delta y)$ becomes $d_x u$; because whatever difference there is between the first and second of these quantities, it is solely due to the change of x, $y+\Delta y$ being the value of y in both of them; and for a similar reason, $\text{F}(x, y+\Delta y) - \text{F}(x, y)$ becomes $d_y u$; and therefore

$$\text{D}u = d_x u + d_y u; \tag{40}$$

that is, the total differential of a function of two independent variables is equal to the sum of the partial differentials.

Also we have,

$$\Delta u = \frac{\text{F}(x+\Delta x, y+\Delta y) - \text{F}(x, y+\Delta y)}{\Delta x}\Delta x + \frac{\text{F}(x, y+\Delta y) - \text{F}(x, y)}{\Delta y}\Delta y;$$

and therefore if the increments are infinitesimal,

$$\text{D}u = {}^{*}\left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy; \tag{41}$$

in which formula dx and dy are the infinitesimal increments of

x and y. Hence we have the ratio of the total change of the function to the change of either variable: thus

$$\frac{\mathrm{D}u}{dx} = \left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right)\frac{dy}{dx}; \qquad (42)$$

$$\frac{\mathrm{D}u}{dy} = \left(\frac{du}{dx}\right)\frac{dx}{dy} + \left(\frac{du}{dy}\right). \qquad (43)$$

Ex. 1.

$$\begin{aligned} u &= ay^2 + bxy + cx^2 + ey + gx + k, \\ d_x u &= by\,dx + 2cx\,dx + g\,dx \\ &= (by + 2cx + g)\,dx; \\ d_y u &= 2ay\,dy + bx\,dy + e\,dy \\ &= (2ay + bx + e)\,dy; \\ \therefore\quad \mathrm{D}u &= d_x u + d_y u \\ &= (by + 2cx + g)\,dx + (bx + 2ay + e)\,dy. \end{aligned}$$

Or thus,

$$\left(\frac{du}{dx}\right) = by + 2cx + g, \qquad \left(\frac{du}{dy}\right) = bx + 2ay + e;$$

$$\begin{aligned} \therefore\quad \mathrm{D}u &= \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy \\ &= (by + 2cx + g)\,dx + (bx + 2ay + e)\,dy. \end{aligned}$$

Ex. 2.

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2};$$

$$\left(\frac{du}{dx}\right) = \frac{2x}{a^2}, \qquad \left(\frac{du}{dy}\right) = \frac{2y}{b^2};$$

$$\therefore\quad \mathrm{D}u = \frac{2x}{a^2}\,dx + \frac{2y}{b^2}\,dy.$$

Ex. 3.

$$u = \sin^{-1}\frac{x}{a} + \sin^{-1}\frac{y}{b};$$

$$\left(\frac{du}{dx}\right) = \frac{1}{\{a^2 - x^2\}^{\frac{1}{2}}}, \qquad \left(\frac{du}{dy}\right) = \frac{1}{\{b^2 - y^2\}^{\frac{1}{2}}};$$

$$\therefore\quad \mathrm{D}u = \frac{dx}{\{a^2 - x^2\}^{\frac{1}{2}}} + \frac{dy}{\{b^2 - y^2\}^{\frac{1}{2}}}.$$

Instead however of formally going through the processes of partial differentiation, and then of combining them, as I have above, I may take the result as exhibited in (40) or (41), and calculate the total differential immediately. Thus let

$$\begin{aligned} u &= ay^2 + bxy + cx^2 + ey + gx + k, \\ \mathrm{D}u &= 2ay\,dy + b(x\,dy + y\,dx) + 2cx\,dx + e\,dy + g\,dx; \end{aligned}$$

and thus for other examples.

Ex. 4.
$$u = \frac{x+y}{x-y},$$
$$\text{D}u = \frac{(x-y)(dx+dy)-(x+y)(dx-dy)}{(x-y)^2}$$
$$= \frac{2(x\,dy-y\,dx)}{(x-y)^2}.$$

Ex. 5.
$$u = \tan^{-1}\frac{y}{x}.$$
$$\text{D}u = \frac{x\,dy-y\,dx}{x^2+y^2}.$$

48.] Differentiation of an implicit function of two independent variables.

The principles applied in the last and preceding Articles enable us to differentiate an implicit function of two variables, and thereby to determine the ratio of the corresponding differentials of the variables, without the expression being put in the form of an explicit function of one variable *.

---

* To the above method of differentiating implicit functions it may be objected, that if
$$u = f(x,y) = c,$$
there can be no change in u due to a partial change in x, because u is constant; and for a similar reason there can be no variation in u due to a variation of y only; and therefore that x and y cannot vary separately, but must vary simultaneously; that one cannot change without the other, and therefore that there can be no partial variations of the function. To this it is replied by asking, What do we mean by a constant such as c on the right-hand side of the equation? We mean that whose total variation is zero. If therefore we consider c to be the sum of two quantities c_1 and c_2, such that when x varies c_1 varies, and c_2 varies when y varies, but that c_1 and c_2 are so related that $dc_1 + dc_2 = 0 = dc$; then the total variation of $f(x,y)$, being the sum of the partial variations, will be equal to 0, and the condition of $f(x,y)$ being equal to c will be satisfied. Whereas then each partial variation taken separately is a violation of the condition expressed by the equation, yet the relation of the two when added together is such, as to be in accordance with the equation. The partial variation of the second neutralises the inconsistency of the partial variation of the first with the equation. This may be geometrically explained as follows. Suppose
$$f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = c,$$
which equation represents an ellipse the size of which depends on the value of the constant c, as well as on a and b. Suppose we consider y alone to vary, while x is constant; the point corresponding to x, $y+dy$, is no longer

Let $f(x, y) = c$ be the implicit function of two variables, which may be put in the form

$$u = f(x, y) = c.$$

Then, since u is equal to a constant, $\mathrm{D}u = 0$; and therefore by the last Article,

$$\left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy = 0;$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)}. \tag{44}$$

Ex. 1. $\quad u = x^3 + 3axy + y^3 = a^3;$

$$\left(\frac{du}{dx}\right) = 3x^2 + 3ay, \qquad \left(\frac{du}{dy}\right) = 3y^2 + 3ax;$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}.$$

Ex. 2. $\quad u = x^y + y^x = a;$

$$\left(\frac{du}{dx}\right) = yx^{y-1} + y^x \log_e y, \qquad \left(\frac{du}{dy}\right) = x^y \log_e x + xy^{x-1}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{xy^{x-1} + x^y \log x}.$$

And without going through the intermediate stage of partial differentiation, we may take (40) or (41) of the preceding Article, and differentiate immediately. Thus

Ex. 3. $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$

$$\frac{2x\,dx}{a^2} + \frac{2y\,dy}{b^2} = 0, \qquad \therefore \quad \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}.$$

---

on the original ellipse, but lies a little above it or below it, according to the sign of dy; i. e. it is a point of another ellipse, which second ellipse depends on the variation of the part of c corresponding to the variation of y. And now suppose x to vary, y being constant, in consequence of which the other part of c, which has a variation due to the variation of x, changes, but changes in such a manner as to bring back the point to the original ellipse, the variations of the two parts of the constant being such that the sum of them is zero; whereas then the partial variation of the first carried the point off the ellipse, that of the second brought it back again. This case is exactly that of the total variation of a function of two variables, with the condition that the two partial variations are so related as to render the total variation equal to zero.

Ex. 4. $y^2(2a-x) = x^3;$

$$2y\,dy\,(2a-x) - y^2\,dx = 3x^2\,dx,$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2+y^2}{2y\,(2a-x)}.$$

Ex. 5. $y^2 = 4ax;$

$$2y\,dy = 4a\,dx,$$

$$\therefore \quad \frac{dy}{dx} = \frac{2a}{y}.$$

49.] **Differentiation of a function of many variables, all of which are independent of each other.**

Let $u = \mathrm{F}(x, y, z, \ldots\ldots)$ be a function of many variables, $x, y, z, \ldots\ldots$ all of which are independent of each other; then

$$
\begin{aligned}
u + \Delta u &= \mathrm{F}(x+\Delta x, y+\Delta y, z+\Delta z, \ldots);\\
\therefore \quad \Delta u &= \mathrm{F}(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) - \mathrm{F}(x, y, z, \ldots)\\
&= \mathrm{F}(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) - \mathrm{F}(x, y+\Delta y, z+\Delta z, \ldots)\\
&\quad + \mathrm{F}(x, y+\Delta y, z+\Delta z, \ldots) \quad - \mathrm{F}(x, y, z+\Delta z, \ldots)\\
&\quad + \mathrm{F}(x, y, z+\Delta z, \ldots) \qquad - \;.\;.\;.\;.\;.\\
&\quad + \;.\;.\;.\;.\;.\;. \qquad\qquad\qquad - \;.\;.\;.\;.\;.\\
&\quad + \;.\;.\;.\;.\;.\;.\;. \qquad\qquad\quad - \mathrm{F}(x, y, z, \ldots),
\end{aligned}
$$

by the introduction of quantities which taken through the whole expression cancel each other. Now let the increments be infinitesimal; then the lines in the right-hand member of the equation become severally the x-, the y-, the z-, ... partial differentials of u; and therefore

$$\mathrm{D}u = \mathrm{DF} = d_x u + d_y u + d_z u + \ldots; \tag{45}$$

that is, the total differential is equal to the sum of the partial differentials.

And if the result is expressed in terms of partial derived functions, then

$$\mathrm{D}u = \mathrm{DF} = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy + \left(\frac{du}{dz}\right)dz + \ldots \tag{46}$$

Whence, as in Art. 47, (42) and (43), the ratio of the total differential to that of any one of the variables may be found.

Ex. 1. $u = \sin(yz + zx + xy);$

$$\left(\frac{du}{dx}\right) = (z+y)\cos(yz+zx+xy),$$

$$\left(\frac{du}{dy}\right) = (x+z)\cos(yz+zx+xy),$$

$$\left(\frac{du}{dz}\right) = (y+x)\cos(yz+zx+xy);$$

$$\therefore\quad \mathrm{D}u = \cos(yz+zx+xy)\,\{(y+z)\,dx+(z+x)\,dy+(x+y)\,dz\}.$$

Ex. 2. $u = (x^2+y^2+z^2)^{\frac{1}{2}}+\tan^{-1}\dfrac{x}{z}+ax+by+cz;$

$$\therefore\quad \mathrm{D}u = \frac{x\,dx+y\,dy+z\,dz}{(x^2+y^2+z^2)^{\frac{1}{2}}}+\frac{z\,dx-x\,dz}{x^2+z^2}+a\,dx+b\,dy+c\,dz.$$

Ex. 3. $u = \dfrac{x^2y}{z^2-a^2};$

$$\left(\frac{du}{dx}\right) = \frac{2xy}{z^2-a^2},\qquad \left(\frac{du}{dy}\right) = \frac{x^2}{z^2-a^2},\qquad \left(\frac{du}{dz}\right) = -\frac{2x^2yz}{(z^2-a^2)^2};$$

$$\therefore\quad \mathrm{D}u = \frac{x(z^2-a^2)\,(2y\,dx+x\,dy)-2x^2yz\,dz}{(z^2-a^2)^2}.$$

These results might have been deduced immediately in the same way as those of Ex. 4 and 5 in Art. 47.

50.] If the variables are so combined in an equation that no one is expressed explicitly in terms of the others, then we have an implicit function of many variables of the form

$$\mathrm{F}(x,y,z,\ldots\ldots) = c;$$

and therefore $\qquad \mathrm{DF} = 0,$

and $$\left(\frac{d\mathrm{F}}{dx}\right)dx+\left(\frac{d\mathrm{F}}{dy}\right)dy+\left(\frac{d\mathrm{F}}{dz}\right)dz+\ldots\ldots = 0; \tag{47}$$

whence may be determined the absolute variation of any one of them which is due to the variations of all the others; thus

$$dx = -\frac{\left(\frac{d\mathrm{F}}{dy}\right)dy+\left(\frac{d\mathrm{F}}{dz}\right)dz+\ldots\ldots}{\left(\frac{d\mathrm{F}}{dx}\right)}.$$

Similarly may be found the variation of any one of the variables which is due to the variation of any one of the others; thus, for instance, let it be required to find the relation between the corresponding variations of x and y in the equation

$$\mathrm{F}(x,y,z,\ldots\ldots) = c,$$

when the other variables do not change value; then

$$\left(\frac{dF}{dx}\right) dx + \left(\frac{dF}{dy}\right) dy = 0;$$

51.] Differentiation of a function of many variables, some which are dependent on the others.

The principles of differentiation of functions of many variables

$$\left(\frac{du}{dy}\right) = (x+z)\cos(yz+zx+xy),$$

If we had an equation of the form

$$(1)\ p\left(\frac{df}{dx}\right) + q\left(\frac{df}{dy}\right) + r\left(\frac{df}{dz}\right) = 0$$

where $f(x, y, z) = 0$ (2) what would (1) become when we wrote (2) thus $x = \phi(y, z)$. (3) ?

$$\therefore \left(\frac{df}{dx}\right)dx + \left(\frac{df}{dy}\right)dy + \left(\frac{df}{dz}\right)dz = 0$$

from which

$$\left(\frac{df}{dy}\right) = -\left(\frac{dx}{dy}\right)\left(\frac{df}{dx}\right)$$

also

$$\left(\frac{df}{dz}\right) = -\left(\frac{dx}{dz}\right)\left(\frac{df}{dx}\right)$$

subs. these values in (1) and reducing

$$q\left(\frac{dx}{dy}\right) + r\left(\frac{dx}{dz}\right) = p$$

which is the relation required.

the corresponding variations of x and y in the equation

$$F(x, y, z, \ldots\ldots) = c,$$

when the other variables do not change value; then

$$\left(\frac{d\text{F}}{dx}\right)dx + \left(\frac{d\text{F}}{dy}\right)dy = 0;$$

$$\therefore \quad \frac{dx}{dy} = -\frac{\left(\frac{d\text{F}}{dy}\right)}{\left(\frac{d\text{F}}{dx}\right)}. \tag{48}$$

Similarly the ratio of the corresponding variations of x and z are given by the equation

$$\frac{dx}{dz} = -\frac{\left(\frac{d\text{F}}{dz}\right)}{\left(\frac{d\text{F}}{dx}\right)}. \tag{49}$$

Now these results are useful in questions like the following: Suppose that there is an expression,

$$\text{P}\left(\frac{dz}{dx}\right) + \text{Q}\left(\frac{dz}{dy}\right) = \text{R}, \tag{50}$$

where P, Q, R are functions of x, y and z; and that it expresses a property of the explicit function

$$z = f(x, y), \tag{51}$$

where $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$ are the partial derived-functions of $f(x, y)$; what is the equivalent form of (50), when (51) is put into the form

$$\text{F}(x, y, z) = 0\,? \tag{52}$$

From (52) we have

$$\left(\frac{d\text{F}}{dx}\right)dx + \left(\frac{d\text{F}}{dy}\right)dy + \left(\frac{d\text{F}}{dz}\right)dz = 0;$$

$$\therefore \quad \left(\frac{dz}{dx}\right) = -\frac{\left(\frac{d\text{F}}{dx}\right)}{\left(\frac{d\text{F}}{dz}\right)}, \qquad \left(\frac{dz}{dy}\right) = -\frac{\left(\frac{d\text{F}}{dy}\right)}{\left(\frac{d\text{F}}{dz}\right)};$$

and substituting these in (50), we have

$$\text{P}\left(\frac{d\text{F}}{dx}\right) + \text{Q}\left(\frac{d\text{F}}{dy}\right) + \text{R}\left(\frac{d\text{F}}{dz}\right) = 0; \tag{53}$$

and this is the equivalent expression. The converse operation is performed by a similar process. It will be observed that we have hereby passed from (50) which is an unsymmetrical expression, to (53) which is symmetrical.

51.] Differentiation of a function of many variables, some of which are dependent on the others.

The principles of differentiation of functions of many variables which have been explained in the five immediately preceding Articles are general, whatever are the values of the variables; and therefore *a fortiori* they are applicable to the less general case in which the variables are connected with each other by certain given relations: for what is true in the most general case cannot but be true in the particular.

Firstly let us take the most simple case, where u is a function of two variables, and of the form

$$u = \mathrm{F}(x, y);$$

and let us suppose y to be a function of x, and of the form

$$y = f(x);$$

then we have

$$\mathrm{D}u = \left(\frac{d\mathrm{F}}{dx}\right)dx + \left(\frac{d\mathrm{F}}{dy}\right)dy, \quad \text{and} \quad dy = f'(x)\,dx;$$

$$\therefore \quad \mathrm{D}u = \left(\frac{d\mathrm{F}}{dx}\right)dx + \left(\frac{d\mathrm{F}}{dy}\right)f'(x)\,dx;$$

$$\therefore \quad \frac{\mathrm{D}u}{dx} = \left(\frac{d\mathrm{F}}{dx}\right) + \left(\frac{d\mathrm{F}}{dy}\right)f'(x);$$

the meaning of the several terms of which expression is sufficiently obvious from their symbols.

Ex. 1. $u = \tan^{-1}\dfrac{x}{y}$, and $y = \{a^2-x^2\}^{\frac{1}{2}}$;

$$\therefore \quad \mathrm{D}u = \frac{y\,dx - x\,dy}{x^2+y^2}, \qquad dy = -\frac{x\,dx}{\{a^2-x^2\}^{\frac{1}{2}}};$$

whence, substituting

$$\mathrm{D}u = \frac{dx}{\{a^2-x^2\}^{\frac{1}{2}}};$$

a result which is manifestly as it ought to be, if we consider that the relation between u and x, after the elimination of y, is

$$u = \sin^{-1}\frac{x}{a}.$$

Again, consider the less simple case, $u = \mathrm{F}(x, y, z)$, where

$$y = f(x), \quad \text{and} \quad z = \phi(x);$$

$$\therefore \quad \mathrm{D}u = \left(\frac{d\mathrm{F}}{dx}\right)dx + \left(\frac{d\mathrm{F}}{dy}\right)dy + \left(\frac{d\mathrm{F}}{dz}\right)dz;$$

but $dy = f'(x)\,dx$, and $dz = \phi'(x)\,dx$;

whence, substituting and dividing by dx,

$$\frac{\mathrm{D}u}{dx} = \left(\frac{d\mathrm{F}}{dx}\right) + \left(\frac{d\mathrm{F}}{dy}\right)f'(x) + \left(\frac{d\mathrm{F}}{dz}\right)\phi'(x).$$

Ex. 2.
$$u = \sin ax + \sin by + \tan^{-1}\frac{y}{z},$$
$$y = \sec x, \qquad z = \operatorname{cosec} x;$$
$$\therefore \quad \mathrm{D}u = a\cos ax\,dx + b\cos by\,dy + \frac{z\,dy - y\,dz}{y^2+z^2};$$
$$dy = \sec x\tan x\,dx, \qquad dz = -\operatorname{cosec} x\cot x\,dx;$$
$$\therefore \quad \mathrm{D}u = a\cos ax\,dx + b\cos(b\sec x)\sec x\tan x\,dx + dx.$$

Or again, suppose $u = \mathrm{F}(x,y)$,

and $x = f(t), \qquad y = \phi(t),$

t being a third and different variable. Then
$$\mathrm{D}u = \left(\frac{d\mathrm{F}}{dx}\right)dx + \left(\frac{d\mathrm{F}}{dy}\right)dy,$$
$$= \left(\frac{d\mathrm{F}}{dx}\right)f'(t)\,dt + \left(\frac{d\mathrm{F}}{dy}\right)\phi'(t)\,dt;$$
$$\therefore \quad \frac{\mathrm{D}u}{dt} = \left(\frac{d\mathrm{F}}{dx}\right)f'(t) + \left(\frac{d\mathrm{F}}{dy}\right)\phi'(t).$$

Or again, suppose
$$u = \mathrm{F}(x,y,z), \qquad \text{and} \qquad z = f(x,y);$$
then
$$\mathrm{D}u = \left(\frac{d\mathrm{F}}{dx}\right)dx + \left(\frac{d\mathrm{F}}{dy}\right)dy + \left(\frac{d\mathrm{F}}{dz}\right)dz;$$
but
$$\mathrm{D}z \text{ (or } dz) = \left(\frac{df}{dx}\right)dx + \left(\frac{df}{dy}\right)dy;$$
$$\therefore \quad \mathrm{D}u = \left\{\left(\frac{d\mathrm{F}}{dx}\right) + \left(\frac{d\mathrm{F}}{dz}\right)\left(\frac{df}{dx}\right)\right\}dx + \left\{\left(\frac{d\mathrm{F}}{dy}\right) + \left(\frac{d\mathrm{F}}{dz}\right)\left(\frac{df}{dy}\right)\right\}dy.$$

Ex. 3.
$$u = x^{yz}, \qquad \text{and} \qquad z = \sin(xy);$$
$$\therefore \quad \mathrm{D}u = yz\,x^{yz-1}\,dx + x^{yz}\log_e x\,(y\,dz + z\,dy);$$
$$\mathrm{D}z \text{ (or } dz) = \cos(xy)\,(y\,dx + x\,dy);$$
$$\therefore \quad \mathrm{D}u = yx^{yz}\left\{\frac{z}{x} + y\log_e x\cos(xy)\right\}dx$$
$$+ \log_e x\,x^{yz}\{z + xy\cos(xy)\}\,dy.$$

The cases of differentiation of functions of the preceding kind are so various and numerous, that it is impossible to discuss all; but the principles explained and illustrated as above are applicable universally, and from the examples given the student ought to find no difficulty in solving other similar ones.

52.] In certain cases the subject variables enter under the functional symbols in particular combinations; and although the general principles unfolded above are applicable, yet the results take special forms; and certain relations exist between the several partial differentials of the given function by reason of the particular combination of the variables. Many cases of such functions will occur in the sequel, and we shall then have to consider them from a geometrical point of view: here I propose to exhibit their properties from an analytical point. And for this purpose some particular examples will serve best.

Let u be a function of two independent variables x and y; and suppose that x and y enter under the functional symbol in a certain combination, such as x divided by y: then

$$u = f\left(\frac{x}{y}\right); \tag{54}$$

that is, u is a homogeneous function of x and y of 0 dimensions.

Now whatever is the form of function which enters into the x-differential of u, the same function enters into the y-differential; although there will be different differential factors in each. Thus, for instance, if

$$u = \sin\frac{x}{y},$$

$$d_x u = \frac{dx}{y}\cos\frac{x}{y}; \qquad \left(\frac{du}{dx}\right) = \frac{1}{y}\cos\frac{x}{y};$$

$$d_y u = -\frac{x\,dy}{y^2}\cos\frac{x}{y}; \qquad \left(\frac{du}{dy}\right) = -\frac{x}{y^2}\cos\frac{x}{y}.$$

And generally in (54), if we replace $\frac{x}{y}$ by z, so that $u = f(z)$, and if $\frac{du}{dz} = f'(z)$; we have

$$\left(\frac{du}{dx}\right) = \frac{1}{y}f'\left(\frac{x}{y}\right), \qquad \left(\frac{du}{dy}\right) = -\frac{x}{y^2}f'\left(\frac{x}{y}\right); \tag{55}$$

so that, as we remarked above, the same form of function enters into both the partial derived functions; but the factors are different; and the relation between the partial differentials is, as deduced from (55),

$$x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = 0. \tag{56}$$

Again, suppose the function to be

$$u = f(xy), \tag{57}$$

and suppose $f'(z)$ to be the derived function of $f(z)$; then

$$\left(\frac{du}{dx}\right) = y\,f'(xy), \qquad \left(\frac{du}{dy}\right) = x\,f'(xy); \tag{58}$$

$$\therefore \quad x\left(\frac{du}{dx}\right) - y\left(\frac{du}{dy}\right) = 0; \tag{59}$$

(58) being the partial derived functions; and (59) exhibiting the relation between them.

Again, suppose the function to be

$$u = f(ax - by), \tag{60}$$

and suppose $f'(z)$ to be the derived function of $f(z)$; then from (60) we have

$$\left(\frac{du}{dx}\right) = a\,f'(ax-by), \qquad \left(\frac{du}{dy}\right) = -b\,f'(ax-by), \tag{61}$$

which are the partial differentials; and from them we have

$$b\left(\frac{du}{dx}\right) + a\left(\frac{du}{dy}\right) = 0; \tag{62}$$

which gives the relation between the partial derived functions.

Or again, suppose the function to be of three variables; and let us assume it to be of the form

$$u = f\left(\frac{xy}{z}\right); \tag{63}$$

then if $\frac{xy}{z} = \xi$, and if the derived function of $f(\xi)$ is $f'(\xi)$, we have

$$\left(\frac{du}{dx}\right) = \frac{y}{z}f'\left(\frac{xy}{z}\right), \quad \left(\frac{du}{dy}\right) = \frac{x}{z}f'\left(\frac{xy}{z}\right), \quad \left(\frac{du}{dz}\right) = -\frac{xy}{z^2}f'\left(\frac{xy}{z}\right),$$

which are the three partial derived functions of (63).

The following are two other examples of the same kind.

Ex. 1. $u = xy\,\phi\left(\frac{y}{x}\right)$;

$$\left(\frac{du}{dx}\right) = y\,\phi\left(\frac{y}{x}\right) - \frac{y^2}{x}\phi'\left(\frac{y}{x}\right),$$

$$\left(\frac{du}{dy}\right) = x\,\phi\left(\frac{y}{x}\right) + y\,\phi'\left(\frac{y}{x}\right);$$

$$\therefore \quad \mathrm{D}u = (y\,dx + x\,dy)\,\phi\left(\frac{y}{x}\right) + \frac{y\,(x\,dy - y\,dx)}{x}\,\phi'\left(\frac{y}{x}\right).$$

Ex. 2. $u = x\,\phi(xy)$;

$$\left(\frac{du}{dx}\right) = \phi(xy) + xy\,\phi'(xy),$$

$$\left(\frac{du}{dy}\right) = x^2\,\phi'(xy);$$

$$\therefore \quad \mathrm{D}u = \phi(xy)\,dx + x\,(y\,dx + x\,dy)\,\phi'(xy).$$

53.] Thus far the functions which have been considered are explicit. Let us briefly consider one or two cases in which the variables are implicitly involved; and firstly the following, which is of importance in a future problem.

Ex. 1. Given $z = \text{F}\{y+xf(z)\}$, where y and z are independent variables, it is required to find $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$, and the relation between them.

If $y+xf(z) = \xi$, let us symbolize the derived function of $\text{F}(\xi)$ by F'; then we have

$$\left(\frac{dz}{dx}\right) = \left\{f(z)+xf'(z)\left(\frac{dz}{dx}\right)\right\}\text{F}',$$

$$\left(\frac{dz}{dy}\right) = \left\{1+xf'(z)\left(\frac{dz}{dy}\right)\right\}\text{F}';$$

$$\therefore \quad \left(\frac{dz}{dx}\right) = \frac{f(z)\,\text{F}'}{1-xf'(z)\,\text{F}'}, \qquad \left(\frac{dz}{dy}\right) = \frac{\text{F}'}{1-xf'(z)\,\text{F}'};$$

$$\therefore \quad \left(\frac{dz}{dx}\right) = f(z)\left(\frac{dz}{dy}\right).$$

Ex. 2. Again, suppose

$$\frac{x-a}{z-c} = f\left(\frac{y-b}{z-c}\right); \tag{64}$$

and let it be required to find $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$.

If $\frac{y-b}{z-c} = \xi$, let us suppose the derived function of $f(\xi)$ to be $f'(\xi)$: then from (64) we have

$$\frac{1}{z-c} - \frac{x-a}{(z-c)^2}\left(\frac{dz}{dx}\right) = -\frac{y-b}{(z-c)^2}\left(\frac{dz}{dx}\right)f'\left(\frac{y-b}{z-c}\right)$$

$$-\frac{x-a}{(z-c)^2}\left(\frac{dz}{dy}\right) = \left\{\frac{1}{z-c} - \frac{y-b}{(z-c)^2}\left(\frac{dz}{dy}\right)\right\}f'\left(\frac{y-b}{z-c}\right);$$

whence we have the following relation between $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$,

$$(x-a)\left(\frac{dz}{dx}\right) + (y-b)\left(\frac{dz}{dy}\right) = z-c. \tag{65}$$

Or (64) might be put in the implicit form

$$u = \text{F}\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right). \tag{66}$$

Since $z-c$ enters into both of the subjects of the function, it is convenient for us at first to consider the $z-c$ which is combined with $y-b$ to be different to that which is combined with $x-a$; and let us therefore distinguish the second z by an accent, so that (66) becomes

$$u = \mathrm{F}\left(\frac{x-a}{z-c}, \frac{y-b}{z'-c}\right); \tag{67}$$

and because that which is true when z is accented, that is, in the general case, will also be true in the particular case, that is, when the accent is omitted, the general result of (67) will include that of (66).

Now by reason of the mode in which $x-a$ and $z-c$ enter into the right-hand side of (67), it is evident that the x- and z-partial differentials of u will contain the same form of function, but that their differential factors will be different; let the form of function which enters into both, whatever it is, be represented by $\phi\left(\frac{x-a}{z-c}, \frac{y-b}{z'-c}\right)$; so that

$$\left(\frac{d\mathrm{F}}{dx}\right) = \frac{1}{z-c}\,\phi\left(\frac{x-a}{z-c}, \frac{y-b}{z'-c}\right), \tag{68}$$

$$\left(\frac{d\mathrm{F}}{dz}\right) = -\frac{x-a}{(z-c)^2}\,\phi\left(\frac{x-a}{z-c}, \frac{y-b}{z'-c}\right). \tag{69}$$

In a similar way, the same form of function, whatever it is, will enter into the y- and the z'-partial differentials of u, but the differential factors will be different in the two differentials; let $\psi\left(\frac{x-a}{z-c}, \frac{y-b}{z'-c}\right)$ be the form of function; then

$$\left(\frac{d\mathrm{F}}{dy}\right) = \frac{1}{z'-c}\,\psi\left(\frac{x-a}{z-c}, \frac{y-b}{z'-c}\right), \tag{70}$$

$$\left(\frac{d\mathrm{F}}{dz'}\right) = -\frac{y-b}{(z'-c)^2}\,\psi\left(\frac{x-a}{z-c}, \frac{y-b}{z'-c}\right); \tag{71}$$

so that we have from (68) and (69),

$$(x-a)\left(\frac{d\mathrm{F}}{dx}\right) + (z-c)\left(\frac{d\mathrm{F}}{dz}\right) = 0; \tag{72}$$

and from (70) and (71),

$$(y-b)\left(\frac{d\mathrm{F}}{dy}\right) + (z'-c)\left(\frac{d\mathrm{F}}{dz'}\right) = 0. \tag{73}$$

Let us now consider the relation between these equations, when $z'=z$. Suppose a functional equation to be given in the form

$$\omega = f(z, z'); \tag{74}$$

then $$\mathrm{D}f = \left(\frac{df}{dz}\right) dz + \left(\frac{df}{dz'}\right) dz';$$

and if $z' = z$, $$\frac{\mathrm{D}f}{dz} = \left(\frac{df}{dz}\right) + \left(\frac{df}{dz'}\right); \tag{75}$$

of the latter term the accent is retained so as to indicate its origin.

Now in (72) and (73) let $z' = z$; then by addition we have

$$(x-a)\left(\frac{d\mathrm{F}}{dx}\right) + (y-b)\left(\frac{d\mathrm{F}}{dy}\right) + (z-c)\left\{\left(\frac{d\mathrm{F}}{dz}\right) + \left(\frac{d\mathrm{F}}{dz'}\right)\right\} = 0;$$

and by reason of (75)

$$(x-a)\left(\frac{d\mathrm{F}}{dx}\right) + (y-b)\left(\frac{d\mathrm{F}}{dy}\right) + (z-c)\left(\frac{d\mathrm{F}}{dz}\right) = 0. \tag{76}$$

I may observe that if $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$ in (65) are replaced by their equivalents given in Art. 50, the result is (76), as it ought to be.

And if (66) is cast into the more symmetrical form

$$u = \mathrm{F}\left(\frac{y-b}{z-c}, \frac{z-c}{x-a}, \frac{x-a}{y-b}\right), \tag{77}$$

we shall get the same result, viz. (76), by the following process.

Let (77) be expressed in the form

$$u = \mathrm{F}\left(\frac{y-b}{z-c}, \frac{z'-c}{x-a}, \frac{x'-a}{y'-b}\right);$$

then by a process the same as that by which (72) and (73) are found, we have

$$\left.\begin{aligned} (y-b)\left(\frac{d\mathrm{F}}{dy}\right) + (z-c)\left(\frac{d\mathrm{F}}{dz}\right) &= 0 \\ (z'-c)\left(\frac{d\mathrm{F}}{dz'}\right) + (x-a)\left(\frac{d\mathrm{F}}{dx}\right) &= 0 \\ (x'-a)\left(\frac{d\mathrm{F}}{dx'}\right) + (y'-b)\left(\frac{d\mathrm{F}}{dy'}\right) &= 0 \end{aligned}\right\}; \tag{78}$$

whence by addition, and by reason of equation (75), if $x' = x$, $y' = y$, $z' = z$, we have

$$(x-a)\left(\frac{d\mathrm{F}}{dx}\right) + (y-b)\left(\frac{d\mathrm{F}}{dy}\right) + (z-c)\left(\frac{d\mathrm{F}}{dz}\right) = 0; \tag{79}$$

a result the same as (76).

In the preceding process the relation between the partial differentials of the functions in (77) is found, when the problem is resolved into its most simple elements; an abridged form of it is more useful in application. And as the result thus obtained is applicable to other cases, it is desirable to insert it.

Let the functional form which occurs in the y- and the z-partial differentials of (77), as they arise from the variation of the first subject of the function, viz. $\frac{y-b}{z-c}$, be represented by F_1; and similarly let F_2 and F_3 represent those which arise from the second and the third subjects: then

$$\left(\frac{d\mathrm{F}}{dx}\right) = -\frac{z-c}{(x-a)^2}\mathrm{F}_2 + \frac{1}{y-b}\mathrm{F}_3,$$

$$\left(\frac{d\mathrm{F}}{dy}\right) = \frac{1}{z-c}\mathrm{F}_1 - \frac{x-a}{(y-b)^2}\mathrm{F}_3,$$

$$\left(\frac{d\mathrm{F}}{dz}\right) = -\frac{y-b}{(z-c)^2}\mathrm{F}_1 + \frac{1}{x-a}\mathrm{F}_2;$$

$$\therefore \quad (x-a)\left(\frac{d\mathrm{F}}{dx}\right) + (y-b)\left(\frac{d\mathrm{F}}{dy}\right) + (z-c)\left(\frac{d\mathrm{F}}{dz}\right) = 0.$$

Ex. 3. As another example of a similar process let us take the function

$$\frac{x}{a} - \frac{z}{c} = f\left(\frac{y}{b} - \frac{z}{c}\right); \tag{80}$$

$$\therefore \quad \frac{1}{a} - \frac{1}{c}\left(\frac{dz}{dx}\right) = -\frac{1}{c}\left(\frac{dz}{dx}\right) f'\left(\frac{y}{b} - \frac{z}{c}\right)$$

$$-\frac{1}{c}\left(\frac{dz}{dy}\right) = \left\{\frac{1}{b} - \frac{1}{c}\left(\frac{dz}{dy}\right)\right\} f'\left(\frac{y}{b} - \frac{z}{c}\right);$$

$$\therefore \quad a\left(\frac{dz}{dx}\right) + b\left(\frac{dz}{dy}\right) = c. \tag{81}$$

And if (80) is put into the form

$$u = \mathrm{F}\left(\frac{x}{a} - \frac{z}{c}, \frac{y}{b} - \frac{z}{c}\right),$$

by a process similar to that of the last example we shall find

$$a\left(\frac{d\mathrm{F}}{dx}\right) + b\left(\frac{d\mathrm{F}}{dy}\right) + c\left(\frac{d\mathrm{F}}{dz}\right) = 0; \tag{82}$$

and further, if (80) is put into the symmetrical implicit form,

$$u = \mathrm{F}\left(\frac{y}{b} - \frac{z}{c}, \frac{z}{c} - \frac{x}{a}, \frac{x}{a} - \frac{y}{b}\right); \tag{83}$$

and if, as in the preceding example, $\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3$ represent the functions which enter into the partial differentials, according as they arise from the differentiation of the first, second, and third of the subjects, we have

$$\left.\begin{aligned}\left(\frac{dF}{dx}\right) &= \frac{1}{a}\{F_3-F_2\}\\ \left(\frac{dF}{dy}\right) &= \frac{1}{b}\{F_1-F_3\}\\ \left(\frac{dF}{dz}\right) &= \frac{1}{c}\{F_2-F_1\}\end{aligned}\right\}; \tag{84}$$

$$\therefore \quad a\left(\frac{dF}{dx}\right)+b\left(\frac{dF}{dy}\right)+c\left(\frac{dF}{dz}\right) = 0. \tag{85}$$

Ex. 4. Similarly again, if x, y, z, t are all independent variables, and if

$$u = F\left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right);$$

and if $\left(\frac{dF}{dx}\right) = \frac{1}{t}F_1$, $\left(\frac{dF}{dy}\right) = \frac{1}{t}F_2$, $\left(\frac{dF}{dz}\right) = \frac{1}{t}F_3$,

then $\left(\frac{dF}{dt}\right) = -\frac{1}{t^2}\{xF_1+yF_2+zF_3\}$;

and therefore $$x\left(\frac{dF}{dx}\right)+y\left(\frac{dF}{dy}\right)+z\left(\frac{dF}{dz}\right)+t\left(\frac{dF}{dt}\right) = 0; \tag{86}$$

a theorem of which a general proof will be given hereafter.

CHAPTER III.

SUCCESSIVE DIFFERENTIATION, AND THEOREMS DEPENDENT ON IT.

SECTION 1.—*On successive differentiation of an explicit function of one variable.*

54.] IN the former part of the last Chapter rules have been constructed for differentiating explicit functions of one variable, and thus of deducing from $f(x)$, which was assumed to be the typical form, $d.f(x)$ or $f'(x)\,dx$; in this case x was made to increase or grow by the infinitesimal dx. Suppose now that x is increased again by an infinitesimal variation, and that it is our object to inquire what effect such a second increase will have on $f(x)$ and on $f'(x)\,dx$; this second increment may or may not be equal to the former one: doubtless the simpler case will be when it is equal, and therefore we will consider it first with this limitation, and subsequently discuss the general case when the successive increments of x are not equal. When x increases by equal increments, or *grows*, as we may say, at an uniform rate, I shall call it an *equicrescent variable* *.

Or we may consider the subject from another point of view; $f'(x)$ is in general a new function of x; and therefore as it was derived from $f(x)$, so by a similar process may another function be derived from it. This new function by an analogous notation is symbolized by $f''(x)$, and will in general be another function of x, and thus will admit of having another function derived from it by a similar process, which will be symbolized by $f'''(x)$, and so on. Thus may *derivation* be considered an algebraical artifice by which successive functions are formed, each from the preceding one; and these are called the derived-functions or *derivatives* of different orders of $f(x)$, viz.

---

* The variable, which I have ventured to call Equicrescent, and thus to coin a new word for, is by most writers called "Independent," and by some old ones "Principal Variable;" to the latter term the objection is, that it does not express the characteristic property of the thing to which it is applied, and has in fact no pretension to appropriateness of nomenclature; the former term is by all writers, and in the present treatise, used in a different signification, viz. to express that variable which first changes value, and due to the change of which the other variables change, and are therefore called *dependent*: see Art. 12; and as such an independent variable may or may not be equicrescent, it is inconvenient to use the same term in two different senses: and especially as the term does not express that character of the variable which renders a distinctive appellation desirable.

$$f(x),\ f'(x),\ f''(x),\ f'''(x), \ldots\ldots f^n(x),\ f^{n+1}(x) \ldots\ldots; \qquad (1)$$

$f(x)$ is called the primitive function; $f'(x)$ the first-derived; $f''(x)$ the second derived; $f^n(x)$ the nth derived function of $f(x)$. It is also to be observed, that each function in the above series is the first-derived of the immediately preceding function; and that $f^n(x) = f(x)$, when $n = 0$.

In the same way therefore as $d.f(x) = f'(x)\,dx$, so does $d.f'(x) = f''(x)\,dx$, and so on; whence we have the following series of equations:

$$\left.\begin{aligned} d.f(x) &= f'(x)\,dx, \\ d.f'(x) &= f''(x)\,dx, \\ d.f''(x) &= f'''(x)\,dx, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ d.f^{n-1}(x) &= f^n(x)\,dx. \end{aligned}\right\} \qquad (2)$$

The dx's, which are factors on the right-hand side of the above equations, are the several increments of x which give rise to the differentials of $f(x)$, $f'(x)$, …… and therefore are not necessarily all equal; but, as above, to consider the simpler case, let us assume them to be equal, so that the several functions vary by reason of *equal* variations of their variables: that is, let x be equicrescent; then, since

$$y = f(x), \qquad dy = d.f(x) = f'(x)\,dx.$$

And since $f'(x)$ is a function of x, we may differentiate again; whence, as dx is constant, we have

$$d.dy = d.f'(x)\,dx;$$

and since $d.dy$ signifies, that the operation symbolized by the character d is to be performed twice on y, and one operation to be on the back of the other, we may, in accordance with the notation of the index law, abbreviate $d.dy$ into d^2y; and replacing $d.f'(x)$ by $f''(x)\,dx$, and writing dx^2 for the square of dx, since the dx's are equal, we have

$$d^2y = f''(x)\,dx^2.$$

Similarly differentiating again, and writing d^3y for $d.d^2y$, and dx^3 for the cube of dx, we have

$$\begin{aligned} d^3y &= d.f''(x)\,dx^2 \\ &= f'''(x)\,dx^3, \end{aligned}$$

and so on; and the general expression becomes

$$d^ny = f^n(x)\,dx^n;$$

wherein dx^n signifies the nth power of dx, and would more correctly be expressed by the symbol $(dx)^n$; but as no ambiguity arises from the omission of the brackets, we have left them out for the relief of the notation.

The quantities $dy, d^2y, d^3y, \dots d^ny, \dots$ are severally called the first, second, third, ... nth, ... differentials of y; hence it follows, that the differential of the nth order of y or $f(x)$ is equal to the product of the nth derived-function and $(dx)^n$.

Also $f''(x), f'''(x), \dots f^n(x), \dots$ are severally called the second, third, ... nth, ... differential coefficients, in accordance with the principle of nomenclature given in Art. 18; because in the above equations they are the coefficients of $dx^2, dx^3, \dots dx^n, \dots$

Hence also we have,

$$\left.\begin{aligned} y &= f(x), \\ \frac{dy}{dx} &= f'(x), \\ \frac{d^2y}{dx^2} &= f''(x), \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \frac{d^ny}{dx^n} &= f^n(x). \end{aligned}\right\} \qquad (3)$$

By which form of writing the fractions on the left-hand side of the equations, it is indicated that x is *equicrescent*. Also let the difference of notation be observed between d^ny and dx^n.

Hence also it appears, that the nth derived-function of $f(x)$ is equal to the ratio of the nth differential of the function to the nth power of dx: dx being the quantity by which x has varied in each of the successive derivations; hence also it is plain that, if $f^n(x)$ is finite, d^ny is an infinitesimal of the nth order with respect to dx as a base.

It is also to be observed, that as $\frac{dy}{dx}$ represents the ratio of the variation of y, or $f(x)$ to that of x, so $\frac{d^2y}{dx^2} = f''(x) = \frac{d.f'(x)}{dx}$

represents the ratio of the variation of $f'(x)$, that is of $\frac{dy}{dx}$, to the variation of x; and so $\frac{d^3y}{dx^3}$ represents the ratio of the variation of $\frac{d^2y}{dx^2}$ to that of x; and similarly of the other derived-functions.

The rules which have been constructed in the preceding Chapter for the differentiation of $f(x)$, and the various specific forms of $f(x)$, are also applicable to $f'(x)$, $f''(x)$, ... $f^n(x)$, ... Thus, because it is shewn in Article 26 that

$$d.cf(x) = cd.f(x),$$

so

$$d^n.cf(x) = cd^n.f(x).$$

Also from Article 27,

$$d^n.\{f(x) \pm \phi(x)\} = d^n.f(x) \pm d^n.\phi(x).$$

Some examples of successive differentiation are subjoined, in which these principles are applied.

Ex. 1.

$$\begin{aligned} y &= x^n, \\ \frac{dy}{dx} &= nx^{n-1}, \\ \frac{d^2y}{dx^2} &= n(n-1)x^{n-2}, \\ \frac{d^3y}{dx^3} &= n(n-1)(n-2)x^{n-3}, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{d^ry}{dx^r} &= n(n-1)(n-2)\dots(n-r+2)(n-r+1)x^{n-r}. \end{aligned}$$

Ex. 2.

$$\begin{aligned} y &= \log_e x, \\ \frac{dy}{dx} &= \frac{1}{x} = x^{-1}, \\ \frac{d^2y}{dx^2} &= -x^{-2}, \\ \frac{d^3y}{dx^3} &= (-)^2\, 1.2\, x^{-3}, \\ \frac{d^4y}{dx^4} &= (-)^3\, 1.2.3\, x^{-4}, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{d^ry}{dx^r} &= (-)^{r-1}\, 1.2.3.\dots(r-2)(r-1)\, x^{-r}. \end{aligned}$$

Ex. 3.

$$y = a^x,$$

$$\frac{dy}{dx} = \log_e a\, a^x,$$

$$\frac{d^2y}{dx^2} = (\log_e a)^2 a^x,$$

. . . . . .

$$\frac{d^ry}{dx^r} = (\log_e a)^r a^x.$$

Ex. 4.

$$y = \sin nx,$$

$$\frac{dy}{dx} = n \cos nx = n \sin\left(nx + \frac{\pi}{2}\right),$$

$$\frac{d^2y}{dx^2} = n^2 \cos\left(nx + \frac{\pi}{2}\right) = n^2 \sin\left(nx + 2\frac{\pi}{2}\right),$$

. . . .

$$\frac{d^ry}{dx^r} = n^r \sin\left(nx + r\frac{\pi}{2}\right).$$

It appears therefore that the effect of one derivation on $\sin nx$ is to multiply the quantity by n and to increase the arc by $\frac{\pi}{2}$; and therefore it might at once be concluded, that the effect of r derivations is to multiply the quantity by n^r, and to increase the arc by $r\frac{\pi}{2}$.

Similarly, if $y = \cos nx$, $\quad \frac{d^ry}{dx^r} = n^r \cos\left(nx + r\frac{\pi}{2}\right).$

Ex. 5.

$$y = \frac{1}{x^2-a^2} = \frac{1}{2a}\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\}$$

$$= \frac{1}{2a}\{(x-a)^{-1} - (x+a)^{-1}\};$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2a}\{(x-a)^{-2} - (x+a)^{-2}\},$$

$$\frac{d^2y}{dx^2} = (-)^2\frac{1.2}{2a}\{(x-a)^{-3} - (x+a)^{-3}\},$$

$$\frac{d^3y}{dx^3} = (-)^3\frac{1.2.3}{2a}\{(x-a)^{-4} - (x+a)^{-4}\},$$

. . . .

$$\frac{d^ry}{dx^r} = (-)^r\frac{1.2.3\ldots r}{2a}\{(x-a)^{-(r+1)} - (x+a)^{-(r+1)}\}.$$

A similar method of decomposition into partial fractions may also be employed to find the general derived-function of $\frac{x}{x^2-a^2}$, $\frac{1}{x^2+a^2}$, $\frac{x}{x^2+a^2}$, and other functions of a similar form.

Ex. 6. $y = e^{ax} \sin nx,$

$$\frac{dy}{dx} = e^{ax}(a \sin nx + n \cos nx).$$

Let $a = k \cos \phi$, $\therefore$ $k^2 = a^2+n^2$,

$n = k \sin \phi$, $\tan \phi = \frac{n}{a}$;

on the legitimacy of which substitutions see Art. 60;

$$\therefore \quad \frac{dy}{dx} = k\, e^{ax} \sin (nx+\phi);$$

whence it is manifest, that

$$\begin{aligned} \frac{d^r y}{dx^r} &= k^r\, e^{ax} \sin (nx+r\phi) \\ &= (a^2+n^2)^{\frac{r}{2}}\, e^{ax} \sin (nx+r\phi). \end{aligned}$$

55.] If the function to be differentiated is a product of two functions of x, say of u and v, the following theorem, due to Leibnitz, and commonly called Leibnitz's Theorem, may be conveniently employed to find the general, viz. the rth, derived-function.

Let $f(x)$ be the function of x, of which let the two factors be u and v, so that

$$\begin{aligned} y = f(x) &= u \times v, \\ dy &= v\, du + dv\, u, \\ d^2 y &= v\, d^2 u + 2\, dv\, du + d^2 v\, u, \\ d^3 y &= v\, d^3 u + 3\, dv\, d^2 u + 3\, d^2 v\, du + d^3 v\, u; \end{aligned}$$

the law of the coefficients being, as it is manifest, that of the coefficients of $(1+x)^r$, when $r = 1, = 2, = 3, \ldots$; and, to shew that such is always the case, let us assume that

$$d^{r-1} y = v\, d^{r-1} u + \frac{r-1}{1}\, dv\, d^{r-2} u + \frac{(r-1)(r-2)}{1.2}\, d^2 v\, d^{r-3} u$$
$$+ \ldots + \frac{r-1}{1}\, du\, d^{r-2} v + u\, d^{r-1} v;$$

and let us differentiate; whereby

$$d^r y = v\,d^r u + \frac{r}{1}\,dv\,d^{r-1}u + \frac{r(r-1)}{1.2}\,d^2v\,d^{r-2}u$$
$$+ \ldots + r\,du\,d^{r-1}v + u\,d^r v.$$

Therefore, if the formula is true for $(r-1)$, it is true for r; but it is true for 2, therefore it is true for 3; and therefore it is true for *all positive and integral* values of r *;

$$\therefore\ d^r(u\times v) = d^r.f(x) = f^r(x)\,dx^r$$
$$= v\,d^r u + \frac{r}{1}\,dv\,d^{r-1}u + \frac{r(r-1)}{1.2}\,d^2v\,d^{r-2}u + \ldots \quad (4)$$

$$\therefore\ f^r(x) = v\frac{d^r u}{dx^r} + \frac{r}{1}\,\frac{dv}{dx}\,\frac{d^{r-1}u}{dx^{r-1}} + \frac{r(r-1)}{1.2}\,\frac{d^2v}{dx^2}\,\frac{d^{r-2}u}{dx^{r-2}} + \ldots \quad (5)$$

In applying the last formula to a given example, let the student be careful to take for u that function whose general derived-function is the more easily calculated.

Ex. 1. To find the rth derived-function of $e^{ax}\,x^n$.

$$u = e^{ax}, \qquad v = x^n,$$
$$\frac{du}{dx} = a\,e^{ax}, \qquad \frac{dv}{dx} = nx^{n-1},$$
$$\frac{d^2u}{dx^2} = a^2\,e^{ax}, \qquad \frac{d^2v}{dx^2} = n(n-1)\,x^{n-2},$$

. . . .

$$\frac{d^r u}{dx^r} = a^r\,e^{ax};$$

$$\therefore\quad \frac{d^r(e^{ax}\,x^n)}{dx^r} = a^r\,e^{ax}\,x^n + \frac{r}{1}\,a^{r-1}\,e^{ax}\,nx^{n-1}$$
$$+ \frac{r(r-1)}{1.2}\,a^{r-2}\,e^{ax}\,n(n-1)\,x^{n-2} + \ldots\ldots$$
$$= e^{ax}\,\{a^r x^n + \frac{r}{1}\,a^{r-1}\,n\,x^{n-1}$$
$$+ \frac{r(r-1)}{1.2}\,a^{r-2}\,n(n-1)\,x^{n-2} + \ldots\ldots\}$$

Ex. 2. To find the rth derived-function of $e^{ax}\cos nx\,x^m$.

$$u = e^{ax}\cos nx, \qquad v = x^m;$$

---

* On this mode of inductively proving the above theorem, see some remarks in a "System of Logic," by John Stuart Mill, book III, ch. ii, § 2, 2nd edition, London, 1846.

therefore by Ex. 6, Art. 54, if $\tan\phi = \frac{n}{a}$, and $k = \{n^2+a^2\}^{\frac{1}{2}}$,

$$\frac{du}{dx} = k\,e^{ax}\cos(nx+\phi), \qquad \frac{dv}{dx} = mx^{m-1},$$

$$\frac{d^2u}{dx^2} = k^2 e^{ax}\cos(nx+2\phi), \qquad \frac{d^2v}{dx^2} = m(m-1)\,x^{m-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^ru}{dx^r} = k^r e^{ax}\cos(nx+r\phi);$$

$$\begin{aligned}\therefore\ \frac{d^r(e^{ax}\cos nx\,x^m)}{dx^r} = e^{ax}\{&k^r\cos(nx+r\phi)\,x^m \\ &+\frac{r}{1}k^{r-1}\cos\{nx+(r-1)\phi\}\,mx^{m-1} \\ &+\frac{r(r-1)}{1.2}k^{r-2}\cos\{nx+(r-2)\phi\}\,m(m-1)\,x^{m-2} \\ &+\cdots\cdots\cdots\cdots\cdots\cdots\}.\end{aligned}$$

If the function of which the general, say the rth, derived-function is to be found, is the quotient of one function of x by another function of x, so that

$$f(x) = \frac{u}{v},$$

the preceding formula (5) is applicable, if we take u to be one factor and $\frac{1}{v}$ to be the other. The *general* formula for $f^r(x)$, as it stands, is however too complicated to be of any use.

And the formula (5) is also indirectly applicable to the discovery of $f^r(x)$, even when the primitive function is not the product of two given functions of the variable x. The mode of its application will be best exhibited by examples.

Ex. 1. If $f(x) = \sec x$, then $\cos x\,f(x) = 1$; and from (5),

$$\begin{aligned}\cos x\,f^n(x) - n\sin x\,f^{n-1}(x) - \frac{n(n-1)}{1.2}\cos x\,f^{n-2}(x)& \\ +\frac{n(n-1)(n-2)}{1.2.3}\sin x\,f^{n-3}(x) + \ldots &= 0,\end{aligned}$$

$$\begin{aligned}\therefore\ f^n(x) = &\left\{\frac{n}{1}f^{n-1}(x) - \frac{n(n-1)(n-2)}{1.2.3}f^{n-3}(x) + \ldots\right\}\tan x \\ &+\frac{n(n-1)}{1.2}f^{n-2}(x) - \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}f^{n-4}(x) + \ldots\ldots\end{aligned}$$

which gives the relation between any derived function and its preceding derived functions. Thus if

$$n = 1,\ \ f'(x) = \sec x \tan x,$$
$$n = 2,\ \ f''(x) = 2 \sec x \tan^2 x + \sec x,$$

. . . .

Ex. 2. If $f(x) = \tan^{-1} x$, $f'(x) = \dfrac{1}{1+x^2}$;

$$\therefore \quad (1+x^2) f'(x) = 1;$$

and applying to this the formula (5), we have

$$f^{n+1}(x)\,(1+x^2) + n f^n(x)\, 2x + \frac{n(n-1)}{1.2} f^{n-1}(x)\, 2 = 0;$$

$$\therefore \quad f^{n+1}(x) = -\frac{1}{1+x^2}\{2nx f^n(x) + n(n-1) f^{n-1}(x)\};$$

whence may $f'(x)$, $f''(x)$, ... be found.

56.] The following is another theorem concerning the relations between functions and their derived functions which holds good when the subject variable is equicrescent.

Since by (6), Article 18,

$$d.f(x) = f'(x)\,dx = f(x+dx) - f(x),$$
$$\therefore \quad d^2.f(x) = f''(x)\,dx^2 = f(x+2dx) - 2f(x+dx) + f(x),$$
$$d^3.f(x) = f'''(x)\,dx^3 = f(x+3dx) - 3f(x+2dx) + 3f(x+dx) - f(x),$$

and so on; whence by an inductive proof of the same kind as that of the preceding Article, we have

$$\begin{aligned} d^n.f(x) &= f^n(x)\,dx^n \\ &= f\{x+n\,dx\} - \frac{n}{1} f\{x+(n-1)dx\} + \frac{n(n-1)}{1.2} f\{x+(n-2)dx\} \\ &\qquad - \dots (-)^{n-1} \frac{n}{1} f(x+dx)\,(-)^n f(x). \quad (6) \end{aligned}$$

SECTION 2.—*Maclaurin's Theorem for the expansion of an explicit function of one variable.*

57.] One of the most useful applications of the preceding theory of successive differentiation is the means which we thence derive of expanding explicit functions of one variable in ascending powers of that subject variable; at least when such an expansion is possible. And although at the present stage of our treatise we are not able to investigate all the conditions of such a possibility, yet doubtless the expansion is possible under the following circumstances.

Suppose $f(x)$ to be a function of x, and to be capable of expansion in a series of ascending powers of x, of the form

$$f(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n + \dots \tag{7}$$

and suppose the series to be subject to the following conditions:

1st, $A_0, A_1, A_2, \dots A_n, \dots$ are constants, and to be determined.

2nd, $A_0, A_1, A_2, \dots A_n, \dots$ do not become infinite for any value of x for which the series is applied.

3rd, The series contains no terms involving negative or fractional powers of x.

Then, since

$$\begin{aligned} f(x) &= A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n + \dots \\ \therefore\ f'(x) &= A_1 + 2A_2 x + 3A_3 x^2 + \dots \\ f''(x) &= 2A_2 + 2.3A_3 x + \dots \\ f'''(x) &= 2.3A_3 + \dots \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f^n(x) &= 1.2.3 \dots (n-1) n A_n + \dots \end{aligned}$$

In these equations let $x = 0$; then, since by the second and third conditions none of the quantities assume the indeterminate form $\frac{0}{0}$, or become infinite, we have

$$\begin{aligned} f(0) &= A_0, & \therefore\ A_0 &= f(0), \\ f'(0) &= A_1, & \therefore\ A_1 &= \frac{f'(0)}{1}, \\ f''(0) &= 1.2A_2, & \therefore\ A_2 &= \frac{f''(0)}{1.2}, \\ f'''(0) &= 1.2.3A_3, & \therefore\ A_3 &= \frac{f'''(0)}{1.2.3}, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f^n(0) &= 1.2.3 \dots n A_n, & \therefore\ A_n &= \frac{f^n(0)}{1.2.3 \dots n}. \end{aligned}$$

Whence, substituting in equation (7), we have

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{1.2} + \dots + f^n(0)\frac{x^n}{1.2.3 \dots n} + \dots \tag{8}$$

$f(0), f'(0, f''(0) \dots$ being the values of $f(x), f'(x), f''(x), \dots$ when $x = 0$.

This series was discovered by Stirling, an English Mathematician of the early part of the last century; but having been introduced by Maclaurin into his Treatise of Fluxions, has been generally called by his name.

58.] For certain functions of x, such as those of the form $(a+x)^n$, where n is positive and integral, the derived-functions will vanish after a certain number of differentiations; and therefore the number of terms of the above series is limited in such cases. Generally the successively derived-functions are functions of x, and the series is continued to an infinite number of terms; but the sum of all the terms after the nth may be expressed as follows, by an algebraical formula. Since

$$\begin{aligned} f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{1.2} + \dots + f^{n-1}(0)\frac{x^{n-1}}{1.2.3\dots(n-1)} \\ + f^n(0)\frac{x^n}{1.2.3\dots n} + f^{n+1}(0)\frac{x^{n+1}}{1.2.3\dots(n+1)} \\ + f^{n+2}(0)\frac{x^{n+2}}{1.2.3\dots(n+2)} + \dots \quad (9)\end{aligned}$$

thė sum of all the terms after the nth

$$\begin{aligned} &= f^n(0)\frac{x^n}{1.2.3\dots n} + f^{n+1}(0)\frac{x^{n+1}}{1.2.3\dots(n+1)} + \dots \\ &= \frac{x^n}{1.2.3\dots n}\{f^n(0) + f^{n+1}(0)\frac{x}{n+1} \\ &\qquad + f^{n+2}(0)\frac{x^2}{(n+1)(n+2)} + \dots\} \quad (10) \\ &= \frac{x^n}{1.2.3\dots n}\{\text{a quantity} > f^n(0) \text{ and } < f^n(x)\}; \quad (11)\end{aligned}$$

the latter factor of (10) is, I say, greater than $f^n(0)$, because the sum of the series is algebraically greater than its first term; and is less than $f^n(x)$, because by reason of (9)

$$f^n(x) = f^n(0) + f^{n+1}(0)\frac{x}{1} + f^{n+2}(0)\frac{x^2}{1.2} + \dots$$

and with the exception of the first term of this series, each term of it is greater than the corresponding term of the series given in (10), inasmuch as the denominators are severally less. Hence representing by θ *some* positive and proper fraction, we may symbolize (11) by

$$\frac{x^n}{1.2.3\dots n}f^n(\theta x); \quad (12)$$

for the latter factor is too small when $\theta = 0$, and is too large when $\theta = 1$.

Hence we may write the series as follows:

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{1.2} + \ldots + \frac{x^n}{1.2.3\ldots n} f^n(\theta x); \quad (13)$$

which has a definite number of terms; and therefore the only apparent difference in the absolute equality of the two sides of the equation is that which arises from θ being an undetermined fraction greater than zero and less than unity.

As this series however is of great importance in the application of the Calculus, the proof of it must not rest on any fallacious assumption, or any vague limitations which may be too wide or too narrow, such as those of the stated conditions. Hence arises the need of a rigorous and exact proof of it, and of one which will limit the extent of its applicability, and which will be given hereafter; and therefore the explanation of the last two Articles is to be considered only as yielding a presumption that such a series as (13) is likely to be true.

59.] Examples of Maclaurin's Theorem.

Ex. 1. Let

$$\begin{aligned} f(x) &= (a+x)^n, & \therefore\ f(0) &= a^n, \\ f'(x) &= n(a+x)^{n-1}, & f'(0) &= na^{n-1}, \\ f''(x) &= n(n-1)(a+x)^{n-2}, & f''(0) &= n(n-1)a^{n-2}, \end{aligned}$$

. . . .

whence, if n is positive and integral,

$$f^n(x) = n(n-1)(n-2)\ldots 3.2.1, \quad f^n(0) = n(n-1)(n-2)\ldots 3.2.1;$$

$$\therefore\ f(x) = (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \ldots + nax^{n-1} + x^n; \quad (14)$$

the common Binomial Theorem, which is therefore a particular case of Maclaurin's Theorem.

Ex. 2.

$$\begin{aligned} f(x) &= e^x, & \therefore\ f(0) &= 1, \\ f'(x) &= e^x, & f'(0) &= 1, \\ &\ldots & &\ldots \\ f^n(x) &= e^x, & f^n(\theta x) &= e^{\theta x}; \end{aligned}$$

$$\therefore\ e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \ldots + \frac{x^{n-1}}{1.2.3\ldots(n-1)} + \frac{x^n}{1.2.3\ldots n}e^{\theta x}. \quad (15)$$

Ex. 3.

$$\begin{array}{ll}
f(x) = \sin x, & f(0) = 0, \\
f'(x) = \sin\left(x+\frac{\pi}{2}\right), & f'(0) = 1, \\
f''(x) = \sin\left(x+2\frac{\pi}{2}\right), & f''(0) = 0, \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot \\
f^n(x) = \sin\left(x+n\frac{\pi}{2}\right), & f^n(\theta x) = \sin\left(\theta x+n\frac{\pi}{2}\right);
\end{array}$$

$$\therefore \quad \sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \dots\dots$$

$$+ \frac{x^{n-1}}{1.2.3\dots(n-1)} \sin\frac{(n-1)\pi}{2} + \frac{x^n}{1.2.3\dots n}\sin\left(\theta x+n\frac{\pi}{2}\right). \quad (16)$$

Ex. 4.

$$\begin{array}{ll}
f(x) = \cos x, & \therefore \; f(0) = 1, \\
f'(x) = \cos\left(x+\frac{\pi}{2}\right), & f'(0) = 0, \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot \\
f^n(x) = \cos\left(x+n\frac{\pi}{2}\right), & f^n(\theta x) = \cos\left(\theta x+n\frac{\pi}{2}\right);
\end{array}$$

$$\therefore \quad \cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots\dots$$

$$+ \frac{x^{n-1}}{1.2.3\dots(n-1)} \cos\frac{(n-1)\pi}{2} + \frac{x^n}{1.2.3\dots n}\cos\left(\theta x+n\frac{\pi}{2}\right). \quad (17)$$

Ex. 5.

$$\begin{array}{ll}
f(x) = \log_e(1+x), & \therefore \; f(0) = 0, \\
f'(x) = (1+x)^{-1}, & f'(0) = 1, \\
f''(x) = -(1+x)^{-2}, & f''(0) = -1, \\
f'''(x) = (-)^2\, 1.2\,(1+x)^{-3}, & f'''(0) = 1.2, \\
f^{\mathrm{iv}}(x) = (-)^3\, 1.2.3\,(1+x)^{-4}, & f^{\mathrm{iv}}(0) = -1.2.3, \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot
\end{array}$$

$$f^n(x) = (-)^{n-1}\, 1.2.3\dots(n-1)\,(1+x)^{-n},$$

$$f^n(\theta x) = (-)^{n-1}\, 1.2.3\dots(n-1)\,(1+\theta x)^{-n};$$

$$\therefore \quad \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots\dots$$

$$(-)^{n-2}\frac{x^{n-1}}{n-1}(-)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n. \quad (18)$$

Ex. 6.

$$\begin{aligned}
f(x) &= \sin^{-1}x, & f(0) &= 0,\\
f'(x) &= \frac{1}{(1-x^2)^{\frac{1}{2}}} = (1-x^2)^{-\frac{1}{2}}, & f'(0) &= 1,\\
f''(x) &= x(1-x^2)^{-\frac{3}{2}}, & f''(0) &= 0,\\
f'''(x) &= (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}}, & f'''(0) &= 1,\\
f^{\text{iv}}(x) &= 9x(1-x^2)^{-\frac{5}{2}} + 3.5x^3(1-x^2)^{-\frac{7}{2}}, & f^{\text{iv}}(0) &= 0,\\
f^{\text{v}}(x) &= 9(1-x^2)^{-\frac{5}{2}} + \ldots\ldots & f^{\text{v}}(0) &= 9.
\end{aligned}$$

The general derived-function is of too complicated a form to be useful;

$$\therefore \quad \sin^{-1}x = x + \frac{x^3}{1.2.3} + \frac{9x^5}{1.2.3.4.5} + \ldots\ldots \tag{19}$$

But $\sin^{-1}x$ may be more easily expanded by the following artifice, which adapts itself to those functions of which a derived-function assumes an algebraical form.

Differentiating equation (8), Art. 57, we have

$$f'(x) = f'(0) + f''(0)\frac{x}{1} + f'''(0)\frac{x^2}{1.2} + f^{\text{iv}}(0)\frac{x^3}{1.2.3} + \ldots\ldots$$
$$+ f^{n+1}(0)\frac{x^n}{1.2.3\ldots n} + \ldots\ldots; \tag{20}$$

and from above,

$$\begin{aligned}
f'(x) &= (1-x^2)^{-\frac{1}{2}}\\
&= 1 + \frac{x^2}{2} + \frac{1.3}{2^2}\frac{x^4}{1.2} + \frac{1.3.5}{2^3}\frac{x^6}{1.2.3} + \ldots\ldots\\
&\qquad + \frac{1.3.5.7\ldots(2r-1)}{2^r}\frac{x^{2r}}{1.2.3\ldots r} + \ldots\ldots;
\end{aligned} \tag{21}$$

whence, equating coefficients of the same power of x in the two series, we have

$$f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = \frac{1.1.2}{2.1}, \quad f^{\text{iv}}(0) = 0,$$

$$f^{\text{v}}(0) = \frac{1.3}{2^2}\frac{1.2.3.4}{1.2},$$

$$f^{2r+1}(0) = \frac{1.3.5.7\ldots(2r-1)\,1.2.3\ldots(2r-1)\,2r}{2^r\,1.2.3\ldots(r-1)\,r};$$

whence we have

$$\sin^{-1}x = x + \frac{1}{2}\frac{1.2}{1}\frac{x^3}{1.2.3} + \frac{1.3}{2^2}\frac{1.2.3.4}{1.2}\frac{x^5}{1.2.3.4.5} + \ldots\ldots$$
$$+ \frac{1.3.5\ldots(2r-1)}{2^r}\frac{1.2.3.4\ldots(2r-1)\,2r}{1.2.3\ldots r}\frac{x^{2r+1}}{1.2.3\ldots 2r\,(2r+1)} + \ldots$$

$$\therefore \quad \sin^{-1}x = x + \frac{x^3}{6} + \frac{9x^5}{1.2.3.4.5} + \dots\dots$$

$$+ \frac{1.3.5\dots(2r-1)\,x^{2r+1}}{2.4.6\dots 2r\,(2r+1)} + \dots\dots \quad (22)$$

$\cos^{-1}x$, $\tan^{-1}x$, $\log(1+x)$, are other functions which may be conveniently expanded by this method; and which the student is recommended to apply the process to.

60.] As many properties of some series which have been expanded in the last article will be required in the sequel of the Treatise, it is most convenient to introduce them here, though they may more properly be considered to belong to analytical trigonometry.

By an imaginary or impossible quantity is meant, one of the form,

$$a+b\sqrt{-1}; \quad (23)$$

a and b being symbols of positive or negative possible quantities, and the symbol $\sqrt{-1}$ being that, which, when squared, is equal to -1. Two such expressions, which differ only in the sign of $\sqrt{-1}$, are said to be *conjugate* to each other; thus

$$a+b\sqrt{-1} \text{ and } a-b\sqrt{-1}$$

are called conjugate imaginary expressions; and it is to be observed, that the product of two such conjugate expressions, viz.

$$(a+b\sqrt{-1})(a-b\sqrt{-1}) = a^2+b^2. \quad (24)$$

Now such an expression as (23) may always be put under the form

$$r(\cos\theta + \sqrt{-1}\sin\theta); \quad (25)$$

in which case r is called the *modulus* of the expression $a+b\sqrt{-1}$. For let

$$\left.\begin{aligned} a &= r\cos\theta, \\ b &= r\sin\theta, \end{aligned}\right\} \qquad \therefore \quad \left.\begin{aligned} r^2 &= a^2+b^2, \\ \tan\theta &= \frac{b}{a}; \end{aligned}\right\} \quad (26)$$

and as a and b are possible quantities, a^2+b^2 is a positive quantity, and therefore r is possible; and as $\tan\theta$ passes through all values, from $-\infty$ through 0 to $+\infty$, as θ increases from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, whatever are the relative signs and magnitudes of a and b, there is always some angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ which will satisfy the equation, $\tan\theta = \frac{b}{a}$; therefore the substitutions made for a and b are possible, and therefore $a+b\sqrt{-1}$ may always be put in the form $r(\cos\theta+\sqrt{-1}\sin\theta)$. Also,

$$\frac{\sin\theta}{b}=\frac{\cos\theta}{a}=\frac{1}{\{a^2+b^2\}^{\frac{1}{2}}}, \text{ by Preliminary Theorem I.}$$

61.] In series (15), Art. 59, successively write for x, $x\sqrt{-1}$ and $-x\sqrt{-1}$; then

$$e^{x\sqrt{-1}}=1+x\sqrt{-1}-\frac{x^2}{1.2}-\sqrt{-1}\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\ldots \quad (27)$$

$$e^{-x\sqrt{-1}}=1-x\sqrt{-1}-\frac{x^2}{1.2}+\sqrt{-1}\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}-\ldots \quad (28)$$

$$\begin{aligned}\therefore\quad e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}&=2\left(1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\ldots\right)\\&=2\cos x, \text{ by (17), Art. 59;} \qquad (29)\end{aligned}$$

$$\begin{aligned}e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}&=2\sqrt{-1}\left(x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\ldots\right)\\&=2\sqrt{-1}\sin x, \text{ by (16), Art. 59;} \qquad (30)\end{aligned}$$

$$\therefore\quad e^{x\sqrt{-1}}=\cos x+\sqrt{-1}\sin x; \quad (31)\ *$$

$$e^{-x\sqrt{-1}}=\cos x-\sqrt{-1}\sin x; \quad (32)$$

whence, by division,

$$e^{2x\sqrt{-1}}=\frac{\cos x+\sqrt{-1}\sin x}{\cos x-\sqrt{-1}\sin x}=\frac{1+\sqrt{-1}\tan x}{1-\sqrt{-1}\tan x}; \quad (33)$$

and taking the Napierian logarithms of both sides of the equation, we have

$$\begin{aligned}2x\sqrt{-1}&=\log(1+\sqrt{-1}\tan x)-\log(1-\sqrt{-1}\tan x)\\&=\sqrt{-1}\tan x+\frac{(\tan x)^2}{2}-\sqrt{-1}\frac{(\tan x)^3}{3}\\&\qquad-\frac{(\tan x)^4}{4}+\sqrt{-1}\frac{(\tan x)^5}{5}+\ldots\ldots\\&\quad-\left\{-\sqrt{-1}\tan x+\frac{(\tan x)^2}{2}+\sqrt{-1}\frac{(\tan x)^3}{3}\right.\\&\qquad\left.-\frac{(\tan x)^4}{4}-\sqrt{-1}\frac{(\tan x)^5}{5}+\ldots\ldots\right\};\end{aligned}$$

the series being expanded by equation (18), Art. 59; whence, equating impossible parts, and dividing both sides by $2\sqrt{-1}$, we have

* Let $x=\frac{\pi}{2}$ ∴ $\epsilon^{\frac{\pi}{2}\sqrt{-1}}=\sqrt{-1}$ ∴ $\epsilon^{\frac{\pi}{2}}=(\sqrt{-1})^{-\sqrt{-1}}$.

$$x = \tan x - \frac{(\tan x)^3}{3} + \frac{(\tan x)^5}{5} - \dots\dots, \tag{34}$$

which is a series useful for the calculation of π.

Again, by equation (31),

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x;$$

$$\therefore\ e^{y\sqrt{-1}} = \cos y + \sqrt{-1}\sin y;$$

therefore by multiplication,

$$\begin{aligned} e^{(x+y)\sqrt{-1}} = {} & \cos x \cos y - \sin x \sin y \\ & + \sqrt{-1}(\cos x \sin y + \cos y \sin x), \end{aligned} \tag{35}$$

but
$$e^{(x+y)\sqrt{-1}} = \cos(x+y) + \sqrt{-1}\sin(x+y); \tag{36}$$

wherefore, equating the possible and impossible parts of the equal quantities (35) and (36), we have the fundamental trigonometrical formulæ,

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \tag{37}$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y. \tag{38}$$

Again,

$$\begin{aligned} (\cos x + \sqrt{-1}\sin x)&(\cos y + \sqrt{-1}\sin y)(\cos z + \sqrt{-1}\sin z)\dots \\ &= e^{x\sqrt{-1}} \times e^{y\sqrt{-1}} \times e^{z\sqrt{-1}} \dots = e^{(x+y+z+\dots)\sqrt{-1}} \\ &= \cos(x+y+z+\dots) + \sqrt{-1}\sin(x+y+z+\dots)\dots; \end{aligned} \tag{39}$$

whence, if $x = y = z = \dots$ to m quantities,

$$(\cos x + \sqrt{-1}\sin x)^m = \cos mx + \sqrt{-1}\sin mx;$$

in a similar way it may be shewn that

$$(\cos x - \sqrt{-1}\sin x)^m = \cos mx - \sqrt{-1}\sin mx;$$

and therefore generally,

$$(\cos x \pm \sqrt{-1}\sin x)^m = \cos mx \pm \sqrt{-1}\sin mx; \tag{40}$$

which is De Moivre's Theorem.

By these processes therefore the multiplication of a series of factors of the form $\cos x + \sqrt{-1}\sin x$, and therefore of all imaginary expressions, is reduced to the addition of the arcs under the circular functions; and the involution of such quantities to the multiplication of the arcs.

62.] Equivalent expression of $(\cos x)^n$, in terms of the cosines of the multiple arcs.

To abbreviate the notation, let us substitute as follows:

$$\left.\begin{aligned} e^{x\sqrt{-1}} &= z, \\ \therefore\ e^{-x\sqrt{-1}} &= \frac{1}{z}; \end{aligned}\right\} \qquad \left.\begin{aligned} e^{mx\sqrt{-1}} &= z^m, \\ \therefore\ e^{-mx\sqrt{-1}} &= \frac{1}{z^m}; \end{aligned}\right\} \tag{41}$$

$$\therefore\quad 2\cos x = z+\frac{1}{z},\qquad 2\cos mx = z^m+\frac{1}{z^m},\qquad (42)$$

$$2\sqrt{-1}\sin x = z-\frac{1}{z},\qquad 2\sqrt{-1}\sin mx = z^m-\frac{1}{z^m}.\qquad (43)$$

Since therefore $\quad 2\cos x = z+\frac{1}{z},$

$$\begin{aligned}\therefore\quad 2^n(\cos x)^n &= \left(z+\frac{1}{z}\right)^n\\ &= z^n+nz^{n-2}+\frac{n(n-1)}{1.2}z^{n-4}+\dots\\ &\qquad \dots+\frac{n(n-1)}{1.2}\frac{1}{z^{n-4}}+n\frac{1}{z^{n-2}}+\frac{1}{z^n}\\ &= z^n+\frac{1}{z^n}+n\left(z^{n-2}+\frac{1}{z^{n-2}}\right)\\ &\qquad +\frac{n(n-1)}{1.2}\left(z^{n-4}+\frac{1}{z^{n-4}}\right)+\dots \qquad (44)\\ &= 2\cos nx+2n\cos(n-2)x+2\frac{n(n-1)}{1.2}\cos(n-4)x+\dots;\end{aligned}$$

$$\therefore\quad (\cos x)^n = \frac{1}{2^{n-1}}\left\{\cos nx+n\cos(n-2)x + \frac{n(n-1)}{1.2}\cos(n-4)x+\dots\dots\right\}.\qquad (45)$$

If n is even, say $=2r$, there are $2r+1$ terms in series (44), $2r$ of which will give rise to r different cosines, viz. $\cos nx$, $\cos(n-2)x, \dots \cos 2x$; and the remaining term, which is the middle one of the series, is independent of x, viz.

$$\frac{n(n-1)(n-2)\dots\left(\frac{n}{2}+2\right)\left(\frac{n}{2}+1\right)}{1.2.3\dots\left(\frac{n}{2}-1\right)\frac{n}{2}}.$$

But if n is odd, say $=2r+1$, the series will have $2r+2$ terms, which may be combined into pairs; from each of which a cosine will arise, and there will be $\frac{n+1}{2}$ different cosines, viz. $\cos nx$, $\cos(n-2)x, \dots\dots \cos 3x, \cos x$.

Two examples are subjoined to illustrate the method of expansion.

Ex. 1. To expand $(\cos x)^6$ in terms of the cosines of the multiple arcs.

$$\because \quad 2\cos x = z + \frac{1}{z},$$

$$2^6(\cos x)^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$$= 2\cos 6x + 12\cos 4x + 30\cos 2x + 20;$$

$$\therefore \quad (\cos x)^6 = \frac{1}{2^5}\{\cos 6x + 6\cos 4x + 15\cos 2x + 10\}.$$

Ex. 2. To expand $(\cos x)^5$ in terms of the cosines of the multiple arcs.

$$\because \quad 2\cos x = z + \frac{1}{z},$$

$$2^5(\cos x)^5 = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

$$= 2\cos 5x + 10\cos 3x + 20\cos x,$$

$$\therefore \quad (\cos x)^5 = \frac{1}{2^4}\{\cos 5x + 5\cos 3x + 10\cos x\}.$$

63.] Equivalent expression of $(\sin x)^n$ in terms of the sines and cosines of the multiple arcs.

By (43), Art. 62, we have

$$2\sqrt{-1}\sin x = z - \frac{1}{z};$$

$$\therefore \quad 2^n(-1)^{\frac{n}{2}}(\sin x)^n = \left(z - \frac{1}{z}\right)^n$$

$$= z^n - nz^{n-2} + \frac{n(n-1)}{1.2}z^{n-4} - \dots\dots$$

$$(-)^{n-2}\frac{n(n-1)}{1.2}\frac{1}{z^{n-4}} (-)^{n-1} n \frac{1}{z^{n-2}} (-)^n \frac{1}{z^n}$$

$$= \left(z^n (-)^n \frac{1}{z^n}\right) - n\left(z^{n-2}(-)^{n-2}\frac{1}{z^{n-2}}\right)$$

$$+ \frac{n(n-1)}{1.2}\left(z^{n-4}(-)^{n-4}\frac{1}{z^{n-4}}\right) + \dots; \quad (46)$$

of which series there will be four cases according as n is of one or other of the forms $4r$, $4r+1$, $4r+2$, $4r+3$; but it is not worth while to write down the general terms of the series, as particular examples will be better solved independently. It is to be observed however that, if n is even, the series (46) involves cosines only; whereas, if n is odd, it involves sines only. Two examples are subjoined.

Ex. 1. To expand $(\sin x)^5$ in terms of the sines of the multiple arcs.

$$2\sqrt{-1}\sin x = z - \frac{1}{z};$$

$$\begin{aligned} 2^5\sqrt{-1}(\sin x)^5 &= z^5 - 5z^3 + 10z - 10\frac{1}{z} + 5\frac{1}{z^3} - \frac{1}{z^5} \\ &= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) \\ &= 2\sqrt{-1}\sin 5x - 10\sqrt{-1}\sin 3x + 20\sqrt{-1}\sin x; \end{aligned}$$

$$\therefore \quad (\sin x)^5 = \frac{1}{2^4}\{\sin 5x - 5\sin 3x + 10\sin x\}.$$

Ex. 2. To expand $(\sin x)^6$ in terms of the cosines of the multiple arcs.

$$\because \quad 2\sqrt{-1}\sin x = z - \frac{1}{z},$$

$$\begin{aligned} -2^6(\sin x)^6 &= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20 \\ &= 2\cos 6x - 12\cos 4x + 30\cos 2x - 20; \end{aligned}$$

$$\therefore \quad (\sin x)^6 = \frac{1}{2^5}\{-\cos 6x + 6\cos 4x - 15\cos 2x + 10\}.$$

64.] The solution of the equation $x^n - 1 = 0$, and the resolution of it into its factors.

By equation (40), Art. 61,

$$(\cos x \pm \sqrt{-1}\sin x)^m = \cos mx \pm \sqrt{-1}\sin mx; \qquad (47)$$

and this equation is true, whether m is positive or negative, integral or fractional. For m let $\frac{1}{n}$ be substituted, and for the general symbol x, let $x + 2k\pi$ be written, *k being a whole number*, so that (47) becomes

$$\{\cos(x + 2k\pi) \pm \sqrt{-1}\sin(x + 2k\pi)\}^{\frac{1}{n}}$$
$$= \cos\frac{x + 2k\pi}{n} \pm \sqrt{-1}\sin\frac{x + 2k\pi}{n}. \qquad (48)$$

In this equation let $x = 0$; whereby, as $\cos 2k\pi = 1$ and $\sin 2k\pi = 0$, we have

$$(1)^{\frac{1}{n}} = \cos\frac{2k\pi}{n} \pm \sqrt{-1}\sin\frac{2k\pi}{n}; \tag{49}$$

but if $x^n - 1 = 0$, $x = 1^{\frac{1}{n}}$; and therefore the several values of x, or roots of the equation $x^n - 1 = 0$, are the values which the right-hand member of (49) admits of. Now k may be any whole number; let therefore successively

$$\left.\begin{array}{lll} k = 0, & \text{then} & 1^{\frac{1}{n}} = \cos 0 \pm \sqrt{-1}\sin 0 = 1, \\ k = 1, & \text{- -} & 1^{\frac{1}{n}} = \cos\frac{2\pi}{n} \pm \sqrt{-1}\sin\frac{2\pi}{n}, \\ k = 2, & \text{- -} & 1^{\frac{1}{n}} = \cos\frac{4\pi}{n} \pm \sqrt{-1}\sin\frac{4\pi}{n}, \\ \ldots & \ldots & \ldots \\ \multicolumn{3}{l}{\text{and so on, until, if } n \text{ is even,}} \\ k = \frac{n}{2} - 1, & \text{-} & 1^{\frac{1}{n}} = \cos\frac{n-2}{n}\pi \pm \sqrt{-1}\sin\frac{n-2}{n}\pi, \\ k = \frac{n}{2}, & \text{- -} & 1^{\frac{1}{n}} = \cos\pi \pm \sqrt{-1}\sin\pi = -1; \end{array}\right\} \tag{50}$$

after which term the values recur, for the substitution of $\frac{n}{2} + 1$ for k gives the same values of $1^{\frac{1}{n}}$ as the substitution of $\frac{n}{2} - 1$.

And if n is odd, the substitutions for k must continue until $k = \frac{n-1}{2}$; in which case

$$(1)^{\frac{1}{n}} = \cos\frac{n-1}{n}\pi \pm \sqrt{-1}\sin\frac{n-1}{n}\pi, \tag{51}$$

and afterwards the values recur; and thus in both cases we have n, and only n, different quantities in the right-hand members of (50); and which are the values of $(1)^{\frac{1}{n}}$, and are the roots of the equation $x^n - 1 = 0$. Hence also it appears that $x^n - 1 = 0$ has n, and only n, different roots.

A complete discussion of the properties of these roots of unity would be out of place in this treatise; and would be unnecessary, inasmuch as more than we could insert is contained in Peacock's Algebra, Vol. II, Chapters XXIII, XXIV, XXV, and XLV, Cambridge, 1845. I must however observe on the form of the roots given in (50).

Since by (31) and (32),

$$\cos x \pm \sqrt{-1}\sin x = e^{\pm x\sqrt{-1}},$$

the roots given in the right-hand members of (50) are equal to

$$e^{\pm 0\sqrt{-1}}, \quad e^{\pm\frac{2\pi}{n}\sqrt{-1}}, \quad e^{\pm\frac{3\pi}{n}\sqrt{-1}}, \ldots\ldots e^{\pm\pi\sqrt{-1}},$$

if n is even; and to

$$e^{\pm 0\sqrt{-1}}, \quad e^{\pm\frac{2\pi}{n}\sqrt{-1}}, \ldots\ldots e^{\pm\frac{n-1}{n}\pi\sqrt{-1}},$$

if n is odd; the first terms of these series are evidently unity; and the last term of the former series is -1; and the roots in both series are in geometrical progression; and are the powers of $e^{\pm\frac{\pi}{n}\sqrt{-1}}$; the roots therefore are of the form

$$1, \ a, \ a^2, \ldots\ldots a^{n-1};$$

a being such that, if n is even, $a^{n-1} = -1$.

Let us take one or two examples.

Ex. 1. To find the roots of $x^3-1=0$.

In this case $n=3$; and we have from (50)

$$\begin{aligned}(1)^{\frac{1}{3}} &= 1 \\ &= \cos\frac{2\pi}{3} \pm \sqrt{-1}\sin\frac{2\pi}{3};\end{aligned}$$

and if the roots are expressed in the exponential form

$$\begin{aligned}(1)^{\frac{1}{3}} &= e^{\pm 0\sqrt{-1}} = 1 \\ &= e^{\pm\frac{2\pi}{3}\sqrt{-1}}.\end{aligned}$$

Ex. 2. To find the roots of $x^6-1=0$.

In this case we have $n=6$; and

$$\begin{aligned}(1)^{\frac{1}{6}} &= \cos 0 \pm \sqrt{-1}\sin 0 &&= e^{\pm 0\sqrt{-1}} = 1 \\ &= \cos\frac{\pi}{3} \pm \sqrt{-1}\sin\frac{\pi}{3} &&= e^{\pm\frac{\pi}{3}\sqrt{-1}} \\ &= \cos\frac{2\pi}{3} \pm \sqrt{-1}\sin\frac{2\pi}{3} &&= e^{\pm\frac{2\pi}{3}\sqrt{-1}} \\ &= \cos\pi \pm \sqrt{-1}\sin\pi &&= e^{\pm\pi\sqrt{-1}} = -1.\end{aligned}$$

Thus we may resolve x^n-1 into its factors; by reason of the roots in group (50), corresponding to $k=0$, there is a factor $x-1$; corresponding to $k=1$, there are two factors, viz.

$$x-\left(\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n}\right), \text{ and } x-\left(\cos\frac{2\pi}{n}-\sqrt{-1}\sin\frac{2\pi}{n}\right),$$

which may be compounded into the quadratic factor,

$$x^2-2x\cos\frac{2\pi}{n}+1;$$

similarly may each of the other pairs of simple factors be compounded into a quadratic factor; and the last factor, if n is even, is $x+1$. So that, if n is even,

$$x^n-1 = (x-1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)$$
$$\ldots\ldots\left(x^2-2x\cos\frac{n-2}{n}\pi+1\right)(x+1). \quad (52)$$

And, if n is odd,

$$x^n-1 = (x-1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)$$
$$\ldots\ldots\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right). \quad (53)$$

Thus $$x^3-1 = (x-1)\left(x^2-2x\cos\frac{2\pi}{3}+1\right);$$

$$x^6-1 = (x-1)\left(x^2-2x\cos\frac{\pi}{3}+1\right)\left(x^2-2x\cos\frac{2\pi}{3}+1\right)(x+1).$$

65.] The solution of the equation $x^n+1=0$, and the resolution of it into its factors.

In the general equation (48), let $x = \pi$; then, since $\cos(2k+1)\pi = -1$, and $\sin(2k+1)\pi = 0$,

$$(-1)^{\frac{1}{n}} = \cos\frac{2k+1}{n}\pi \pm \sqrt{-1}\sin\frac{2k+1}{n}\pi; \quad (54)$$

but if $x^n+1=0$, $x=(-1)^{\frac{1}{n}}$; and therefore the several values of x, or roots of the equation $x^n+1=0$, are the values which the right-hand member of equation (54) admits of. Now k may be any integral number; let therefore

$$\left.\begin{array}{lll}
k=0, & \text{then} & (-1)^{\frac{1}{n}} = \cos\frac{\pi}{n} \pm \sqrt{-1}\sin\frac{\pi}{n}, \\
k=1, & - \; - & (-1)^{\frac{1}{n}} = \cos\frac{3\pi}{n} \pm \sqrt{-1}\sin\frac{3\pi}{n}, \\
\ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
\text{until, if } n \text{ is even,} & & \\
k=\frac{n}{2}-1, & - & (-1)^{\frac{1}{n}} = \cos\frac{n-1}{n}\pi \pm \sqrt{-1}\sin\frac{n-1}{n}\pi;
\end{array}\right\} \quad (55)$$

after which the values recur, for the substitution of $\frac{n}{2}$ for k gives the same values of $(-1)^{\frac{1}{n}}$ as the substitution of $\frac{n}{2}-1$;

and the substitution of $\frac{n}{2}+1$ for k gives the same values of $(-1)^{\frac{1}{n}}$ as the substitution of $\frac{n}{2}-2$; and so on.

And if n is odd, the substitutions for k must continue until

$$\left.\begin{aligned} k &= \frac{n-3}{2}, \text{ then } (-1)^{\frac{1}{n}} = \cos\frac{n-2}{n}\pi \pm \sqrt{-1}\sin\frac{n-2}{n}\pi, \\ k &= \frac{n-1}{2}, \quad - \quad - \quad (-1)^{\frac{1}{n}} = \cos\pi \pm \sqrt{-1}\sin\pi = -1, \end{aligned}\right\} (56)$$

after which the values recur; and thus in both cases we have n, and only n, different values of $(-1)^{\frac{1}{n}}$; and therefore $x^n+1=0$ has n, and only n, different roots.

These roots may be expressed in the exponential form, in the same way as those considered in the preceding Article; but as the results are similar it is unnecessary to write them at length.

Let us consider one or two examples.

Ex. 1. Find the roots of $x^4+1=0$.

In this case $n=4$, and from (55) we have

$$\begin{aligned} (-1)^{\frac{1}{4}} &= \cos\frac{\pi}{4} \pm \sqrt{-1}\sin\frac{\pi}{4} \\ &= \cos\frac{3\pi}{4} \pm \sqrt{-1}\sin\frac{3\pi}{4}. \end{aligned}$$

Ex. 2. Find the roots of $x^5+1=0$.

In this case $n=5$, and from (56) we have

$$\begin{aligned} (-1)^{\frac{1}{5}} &= \cos\frac{\pi}{5} \pm \sqrt{-1}\sin\frac{\pi}{5} \\ &= \cos\frac{3\pi}{5} \pm \sqrt{-1}\sin\frac{3\pi}{5} \\ &= \cos\pi \pm \sqrt{-1}\sin\pi = -1. \end{aligned}$$

Thus also may $x^n+1=0$ be resolved into its factors; for by reason of the roots in group (55), corresponding to $k=0$, there are two factors, viz. $x-\left(\cos\frac{\pi}{n}+\sqrt{-1}\sin\frac{\pi}{n}\right)$, and $x-\left(\cos\frac{\pi}{n}-\sqrt{-1}\sin\frac{\pi}{n}\right)$, which may be compounded into a quadratic factor $x^2-2x\cos\frac{\pi}{n}+1$. Similarly may each of the other pairs of factors corresponding to $k=1$, $k=2$, ...

$k = \frac{n}{2} - 1$ be compounded into a quadratic factor; so that, if n is even,

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\ldots\ldots$$

$$\ldots\ldots\left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right). \quad (57)$$

Similarly, if n is odd, there are quadratic factors corresponding to $k = 0$, $k = 1$, $k = 2$, ... $k = \frac{n-3}{2}$; and when $k = \frac{n-1}{2}$, as appears by (56), there is a simple factor $x + 1$; so that

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\ldots\ldots$$

$$\ldots\ldots\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right)(x+1). \quad (58)$$

Thus

$$x^5 + 1 = \left(x^2 - 2x\cos\frac{\pi}{5} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{5} + 1\right)(x+1),$$

$$x^6 + 1 = \left(x^2 - 2x\cos\frac{\pi}{6} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{6} + 1\right)\left(x^2 - 2x\cos\frac{5\pi}{6} + 1\right).$$

66.] If the problem is to resolve $x^n - a^n$, and $x^n + a^n$ into their factors, then, since

$$x^n - a^n = 0, \qquad x = a(1)^{\frac{1}{n}};$$

$$\text{and} \qquad x^n + a^n = 0, \qquad x = a(-1)^{\frac{1}{n}};$$

so that, in both cases, there will be n different values of x, arising from the n different values of $(1)^{\frac{1}{n}}$ and of $(-1)^{\frac{1}{n}}$, which have been found above.

67.] On the imaginary logarithms of positive and negative numbers.

One of the most remarkable results of our admission into the subjects of the Calculus of such quantities and symbols as we are now considering, is the extension which they afford to the theory of logarithms. In ordinary treatises on logarithms it is shewn that the logarithm of 0 is $-\infty$; of 1 is 0; and of $+\infty$ is $+\infty$; so that the whole range of possible number from $-\infty$ to $+\infty$ is exhausted in the logarithms of number between 0 and $+\infty$; what then is the form of logarithm of negative numbers? The preceding theory enables us to give some answer to this question.

Since, by equation (31), Art. 61,

$$\cos x + \sqrt{-1}\sin x = e^{x\sqrt{-1}},$$

for x let us write $x+2k\pi$, k being any integer;

$$\therefore \quad \cos(x+2k\pi) + \sqrt{-1}\sin(x+2k\pi) = e^{(x+2k\pi)\sqrt{-1}}. \tag{59}$$

$$\text{Let} \quad x = 0, \qquad \therefore \quad 1 = e^{2k\pi\sqrt{-1}}; \tag{60}$$

$$x = \pi, \qquad -1 = e^{(2k+1)\pi\sqrt{-1}}; \tag{61}$$

in which remarkable results it must be remembered that e and π are severally the symbols of the arithmetical numbers 2.7182818 and 3.14159. Therefore from (60) it follows, that $2k\pi\sqrt{-1}$ is the *general* Napierian logarithm of 1, and $(2k+1)\pi\sqrt{-1}$ of -1; and therefore

$$2k\pi\sqrt{-1} = \log_e 1, \tag{62}$$

$$(2k+1)\pi\sqrt{-1} = \log_e(-1). \tag{63}$$

In (62), if $k=0$, $\log_e 1 = 0$, which is the common *arithmetical* logarithm of 1; but as k may be any integral number, it follows that $+1$ has an infinite number of Napierian logarithms, all of which, except 0, are affected with $\sqrt{-1}$; hence also it follows, that every positive number has an infinite number of logarithms to the same base. For suppose x to be the arithmetical Napierian logarithm of y, so that

$$y = e^x;$$

$$\text{then} \quad y = e^x \times 1 = e^x \times e^{2k\pi\sqrt{-1}} = e^{(x+2k\pi\sqrt{-1})}; \tag{64}$$

therefore $x+2k\pi\sqrt{-1}$ is the general Napierian logarithm of y, which we will represent by $\operatorname{Log}_e y$; and let us represent the arithmetical logarithm by $\log_e y$;

$$\therefore \quad \operatorname{Log}_e y = \log_e y + 2k\pi\sqrt{-1}; \tag{65}$$

hence the arithmetical logarithm is the particular value of the general logarithm corresponding to $k=0$.

$$\text{Also since} \qquad \operatorname{Log}_a y = \frac{\operatorname{Log}_e y}{\operatorname{Log}_e a}, \tag{65*}$$

$$\therefore \quad \operatorname{Log}_a y = \log_a y + \frac{2k\pi\sqrt{-1}}{\operatorname{Log}_e a}; \tag{66}$$

and therefore, whatever is the base, every number has an infinite number of logarithms to that base.

Again, in equation (63), since k is to be an integer, it follows, that $\log_e(-1)$ can never be a possible quantity, and therefore -1 has no arithmetical logarithm; yet, since k may be any

Price's Cal. Vol. I. Eq. (65)* et. seq. .

In general we have

$$y = a^{l'} = \varepsilon^{l} \quad (1).$$

then, $l'.\log a = l.\log \varepsilon$ (2). (to any base.)

hence $l'\,\mathrm{Log}_{\varepsilon}.a = l.\mathrm{Log}_{\varepsilon}.\varepsilon$ (3).

Now, from Price Eq. (65) we have, when $y = \varepsilon$;

$$\mathrm{Log}_{\varepsilon}.\varepsilon = \log_{\varepsilon}.\varepsilon + 2k\pi\sqrt{-1} = 1 + 2k\pi\sqrt{-1}.$$

$\therefore$ $l'.\mathrm{Log}_{\varepsilon}\,a = l.\{1 + 2k\pi\sqrt{-1}\}$ (4).

Now, $l' = \mathrm{Log}_{a}.y$ and $l = \mathrm{Log}_{\varepsilon}.y$

$\therefore$

$$\mathrm{Log}_{a}.y = \frac{\mathrm{Log}_{\varepsilon}.y\{1 + 2k\pi\sqrt{-1}\}}{\mathrm{Log}_{\varepsilon}.a} \quad (5)$$

$$= \frac{\{\log_{\varepsilon}.y + 2k\pi\sqrt{-1}\}\{1 + 2k\pi\sqrt{-1}\}}{\mathrm{Log}_{\varepsilon}.a} \quad (6,$$

$$= \frac{\log_{\varepsilon}.y\{1 + 2k\pi\sqrt{-1}\} + 2k\pi\sqrt{-1} - 4k^2\pi^2}{\mathrm{Log}_{\varepsilon}.a.} \quad (7)$$

integer, every negative number has an infinite number of Napierian logarithms, and therefore of logarithms to any other base, all of which are affected with $\sqrt{-1}$.

In equation (63), let $k = 0$,

$$\therefore \quad \pi\sqrt{-1} = \log_e(-1); \tag{67}$$

and more generally,

$$\pi = \frac{\log_e(-1)}{(2k+1)\sqrt{-1}}; \tag{68}$$

two of the most curious results in Analysis; in which however, and in all similar expressions, we must bear in mind that e and π are the symbolical representations of certain series, and are therefore to be interpreted with respect to them; and in an algebraical system of course, which admits them amongst those quantities whose laws and combinations it takes cognizance of.

SECTION 3.—*Theory of the Equicrescent Variable, and Taylor's Series.*

68.] In the two preceding Sections of this Chapter we have considered certain theorems which arise from an *algebraical* relation between $f(x)$ and its several and successive derived-functions; but as these have been formed on an unexplained hypothesis of x increasing by equal increments, by virtue of which x has been called the *equicrescent* variable, an adequate conception of the subject requires a discussion at greater length. We shall reserve other important *algebraical* relations to the succeeding Chapter.

Let $y = f(x)$ be an explicit function of x, finite and continuous for all values of x under which we consider it; whether any other values may make it infinite or discontinuous is what we are not concerned with.

Let x receive an increment dx, so that, in accordance with Art. 17, y becomes $y+dy$, and we have

$$y+dy = f(x+dx). \tag{69}$$

And let x receive another increment, not necessarily equal to dx; and let it be $dx+d.dx = dx+d^2x$; so that we have

$$y+dy+d(y+dy) = f\{x+dx+d(x+dx)\},$$

$$y+2\,dy+d^2y = f(x+2\,dx+d^2x). \tag{70}$$

And again, let x receive another increment not equal to the former ones; and so on, whereby the following series of equations will be formed;

$$\left.\begin{aligned}
y &= f(x),\\
y+dy &= f(x+dx),\\
y+2\,dy+d^2y &= f(x+2\,dx+d^2x),\\
y+3\,dy+3\,d^2y+d^3y &= f(x+3\,dx+3\,d^2x+d^3x),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
y+n\,dy+\frac{n(n-1)}{1.2}\,d^2y+\frac{n(n-1)(n-2)}{1.2.3}\,d^3y+\ldots\ldots&\\
&= f\left\{x+n\,dx+\frac{n(n-1)}{1.2}\,d^2x+\ldots\ldots\right\};
\end{aligned}\right\} \quad (71)$$

the last general expression in which group may be shewn to be true for all positive and integral values of n, by a method similar to that employed in Art. 55.

Hereby are determined the several and successive complete variations of an explicit function of x, corresponding to the successive values of the variable; but the general result is much shortened by making x equicrescent.

69.] Let all the dx's be equal, so that d^2x, which is the increment of one dx over the preceding dx, is equal to zero; and similarly

$$d^3x = d^4x = \ldots = d^nx = \ldots = 0;$$

whereby the last equation of (71) becomes

$$f(x+n\,dx) = y+n\,dy+\frac{n(n-1)}{1.2}\,d^2y+\frac{n(n-1)(n-2)}{1.2.3}\,d^3y+\ldots \quad (72)$$

Thus the distinguishing character of an equicrescent variable is that all its differentials after the first vanish. And as this condition is of the greatest importance in the application of Infinitesimal Calculus to questions of Geometry and Physics, it is good to illustrate it before we proceed to discuss its other properties.

Suppose that we are considering any function of x between the values x_n and x_0, x_n being the greater of the two, and the function remaining finite and continuous for all values of x between these limits; let us resolve the difference $x_n - x_0$ into small elements, the number of them being of course infinite, when each element is infinitesimal; let dx be the type of each element. It is at once manifest that all the elements need not be equal; that is, all the dx's are not necessarily equal. And if they are not, there will be an excess of one over another; that is, there will be a d^2x. Neither again need all the d^2x's be equal; that is, the dx's need not be equicrescent; but if they are not, there will be an increment of one d^2x over another d^2x; that is, there will be a $d.d^2x$ or a d^3x. And similarly with regard to the other differentials. But if we once introduce the condition that a differential of any order shall be resolved into elements which are all equal to one another, then all the subsequent differentials vanish; and thus, if all the dx's are equal, as above,

$$d^2x = d^3x = \dots = 0.$$

Or again, conceive a small body, as a billiard-ball, to move over a finite distance in a straight line in a finite time; consider the straight line to be the axis of x, and a certain point in it to be taken for the origin; let the body at the beginning of a certain time be at a distance x_0 from the origin, and at the end of the time be at a distance x_n, and conceive the time of its passing over the distance $x_n - x_0$ to be t; resolve this time into equal elements dt, and the space $x_n - x_0$ into corresponding elements, of each of which the type is dx. If the body moves through the whole space at the same rate, viz. with the same velocity, then, during equal times dt, equal spaces dx will be described; but if the velocity varies, equal spaces will not be passed over in equal times. On the first supposition all the dx's will be equal, therefore $d^2x = 0$, and x is an equicrescent variable; on the second, the dx's vary, and d^2x, which is the increment of one dx passed over in a time dt, over another dx passed over in the preceding or succeeding time dt, as the case may be, is the measure of the increase of the rate of motion. If then all the d^2x's are equal, we say that the velocity of motion is continually increasing, and at a constant rate; but if d^2x is not constant, then the rate of increase of the velocity of the ball is no longer constant, but varies according to some law on which the rate of increase depends. It will be observed, however, that if the whole time of motion is resolved into equal elements dt, the supposition of x being equicrescent is incompatible with a varying velocity.

Hence too it is clear, that generally we are not at liberty to assume more than one of the variables to increase or decrease by equal augments; for in the case above, if we resolve the time into equal elements, then, in general, unequal spaces will be passed over in equal times, and we cannot consider all the dx's to be equal, and therefore we cannot make $d^2x = 0$; and if we resolve the distance into equal parts, then, if the velocity varies, these equal spaces will be passed over in unequal times, and therefore all the dt's will not be equal, and we cannot put $d^2t = 0$. In general, however, we are at liberty to choose for an equicrescent variable whichever is most convenient.

70.] Let us now consider in what manner these assumptions modify the equations of derived-functions in Art. 54. In the series there given we have

$$y = f(x), \qquad dy = f'(x)\,dx.$$

Now considering $f'(x)\,dx$ to be the product of two variable quantities, and differentiating it as such; and, in accordance with the former notation, making $f''(x)\,dx$ to be the symbol for $d.f'(x)$ and $f'''(x)\,dx$ for $d.f''(x)$, and so on; we have

$$\begin{aligned}
d^2y &= f''(x)\,dx^2 + f'(x)\,d^2x, \\
d^3y &= f'''(x)\,dx^3 + 3f''(x)\,dx\,d^2x + f'(x)\,d^3x, \\
d^4y &= f^{\text{iv}}(x)\,dx^4 + 6f'''(x)\,dx^2\,d^2x + 3f''(x)\,(d^2x)^2 \\
&\qquad + 4f''(x)\,dx\,d^3x + f'(x)\,d^4x, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots ;
\end{aligned}$$

Let x be equicrescent; that is, let dx be constant; whence $d^2x = 0$, $d^3x = 0$, ...;

$$\begin{aligned}
\therefore \quad dy &= f'(x)\,dx, \\
d^2y &= f''(x)\,dx^2, \\
d^3y &= f'''(x)\,dx^3, \\
d^4x &= f^{\text{iv}}(x)\,dx^4, \\
&\cdots\cdots ;
\end{aligned}$$

$\therefore$ $f''(x)$, or its equivalent $\frac{d^2y}{dx^2}$, is derived from $f'(x)$, on the supposition that x is the equicrescent variable;

$f'''(x)$, or its equivalent $\frac{d^3y}{dx^3}$, is derived from $f''(x)$, on the same supposition;

and $f^n(x) = \frac{d^ny}{dx^n}$ is derived from $f^{n-1}(x)$, on the same supposition.

Whenever therefore we meet with these or similar symbols, it is to be borne in mind that they have been successively derived on the supposition that the variable x, that is the variable in the denominator, is equicrescent.

71.] To return to equation (72), Art. 69. Let the whole quantity by which x is increased be finite, and be h; so that $n\,dx = h$; and therefore as dx is an infinitesimal, n is an infinity of the same order, Art. 8; whence we have

$$n = \frac{h}{dx}; \tag{73}$$

and by substitution in equation (72),

$$\begin{aligned} f(x+h) = y + \frac{h}{dx}dy + \frac{\frac{h}{dx}\left(\frac{h}{dx}-1\right)}{1.2} d^2y \\ + \frac{\frac{h}{dx}\left(\frac{h}{dx}-1\right)\left(\frac{h}{dx}-2\right)}{1.2.3} d^3y + \ldots\ldots \end{aligned} \tag{74}$$

$$\begin{aligned} f(x+h) = y + h\frac{dy}{dx} + \frac{h(h-dx)}{1.2}\frac{d^2y}{dx^2} \\ + \frac{h(h-dx)(h-2dx)}{1.2.3}\frac{d^3y}{dx^3} + \ldots\ldots; \end{aligned} \tag{75}$$

and as dx is infinitesimal, we must omit in the several numerators the infinitesimals which are subtracted from the finite quantities; and if we replace $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots$ by $f(x), f'(x), f''(x), \ldots\ldots$, we have

$$\begin{aligned} f(x+h) = f(x) + hf'(x) + \frac{h^2}{1.2}f''(x) + \frac{h^3}{1.2.3}f'''(x) + \ldots \\ \ldots + \frac{h^n}{1.2.3\ldots n}f^n(x) + \ldots; \end{aligned} \tag{76}$$

a series known by the name of Taylor's Series, having been discovered by Dr. Brook Taylor, and first given in his "Methodus Incrementorum" in the year 1715. It is however of the utmost importance that the conditions and extent of its applicability should be accurately determined, and so another and more exact proof will be given hereafter; and the above may be considered in the light of a presumption, that such a relation as equation (76) is likely to be true.

72.] In certain cases wherein a derived-function becomes a constant, and therefore the subsequent derived-functions vanish, the number of terms of the series (76) is finite; but generally the successively derived-functions are functions of x, and the series is continued to an infinite number of terms; but the sum of all the terms after the nth may be expressed as follows by an algebraical formula, and thus proved to have a value contained within certain determinable limits: since

$$f(x+h) = f(x) + \frac{h}{1} f'(x) + \frac{h^2}{1.2} f''(x) + \dots + \frac{h^{n-1}}{1.2.3\dots(n-1)} f^{n-1}(x)$$
$$+ \frac{h^n}{1.2.3\dots n} f^n(x) + \frac{h^{n+1}}{1.2.3\dots(n+1)} f^{n+1}(x) + \dots; \quad (77)$$

therefore the sum of all the terms after the nth is equal to

$$\frac{h^n}{1.2.3\dots n} f^n(x) + \frac{h^{n+1}}{1.2.3\dots(n+1)} f^{n+1}(x) + \dots\dots$$
$$= \frac{h^n}{1.2.3\dots n} \left\{ f^n(x) + \frac{h}{n+1} f^{n+1}(x) + \frac{h^2}{(n+1)(n+2)} f^{n+2}(x) + \dots \right\} \quad (78)$$
$$= \frac{h^n}{1.2.3\dots n} \{\text{some quantity} > f^n(x) \text{ and } < f^n(x+h)\}; \quad (79)$$

the latter factor of (78) is I say greater than $f^n(x)$, because the algebraical sum of such a series of terms is greater than its first term; and it is less than $f^n(x+h)$, because if equation (77) is derivated n times, there results

$$f^n(x+h) = f^n(x) + \frac{h}{1} f^{n+1}(x) + \frac{h^2}{1.2} f^{n+2}(x) + \frac{h^3}{1.2.3} f^{n+3}(x) + \dots;$$

and with the exception of the first term of this series, every term is greater than the corresponding term of the series in the latter factor of (78), because, the numerator of the fractions being the same, the denominators are severally less. Hence representing by θ *some* positive and proper fraction, that is, some number greater than zero and less than unity, we may symbolize (79) by

$$\frac{h^n}{1.2.3\dots n} f^n(x+\theta h); \quad (80)$$

for (80) is too small when $\theta = 0$, and is too large when $\theta = 1$; hence *some* value of θ between 0 and 1 will give the correct value. Thus Taylor's Series becomes

$$f(x+h) = f(x) + \frac{h}{1} f'(x) + \frac{h^2}{1.2} f''(x) + \dots$$
$$\dots + \frac{h^n}{1.2.3\dots n} f^n(x+\theta h). \quad (81)$$

As the right-hand member of the equation has a determinate number of terms, the only difference in the absolute equality of the two sides of the equation is that which arises from θ being an undetermined fraction, *mean* between 0 and 1. The last term of (81) is called the *limit* of Taylor's Series.

73.] Examples of Taylor's Series.

Ex. 1.
$$\begin{aligned} f(x) &= \log_e x, \\ f'(x) &= \frac{1}{x} = x^{-1}, \\ f''(x) &= (-)\, x^{-2}, \\ f'''(x) &= (-)^2\, 1.2\, x^{-3}, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f^n(x) &= (-)^{n-1}\, 1.2.3\ldots(n-1)\, x^{-n}; \end{aligned}$$

$$\begin{aligned} \log_e(x+h) = \log_e x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots \\ (-)^{n-2}\frac{h^{n-1}}{(n-1)\,x^{n-1}}\,(-)^{n-1}\frac{h^n}{n}\,\frac{1}{(x+\theta h)^n}. \end{aligned}$$

Ex. 2.
$$f(x) = \sin x.$$

$$\begin{aligned} \sin(x+h) = \sin x + \cos x\,\frac{h}{1} - \sin x\,\frac{h^2}{1.2} + \ldots \\ \ldots + \sin\left\{x + (n-1)\frac{\pi}{2}\right\}\frac{h^{n-1}}{1.2.3\ldots(n-1)} \\ + \frac{h^n}{1.2.3\ldots n}\sin\left(x+\theta h + n\frac{\pi}{2}\right). \quad (82) \end{aligned}$$

Ex. 3.
$$f(x) = \tan^{-1} x.$$

$$\begin{aligned} \tan^{-1}(x+h) = \tan^{-1} x + \frac{1}{1+x^2}\,\frac{h}{1} - \frac{2x}{(1+x^2)^2}\,\frac{h^2}{1.2} \\ + \frac{2(3x^2-1)}{(1+x^2)^3}\,\frac{h^3}{1.2.3} + \ldots \quad (83) \end{aligned}$$

By the method of Ex. 5, Art. 54, if $x = \cot\phi$, the nth derived-function of $\tan^{-1} x$ is found to be

$$(-)^{n-1}\, 1.2.3\ldots(n-1)\,(\sin\phi)^n \sin n\phi,$$

by means of which the limit of (83) may be determined.

Ex. 4. Given $f(x+y) = f(x) + f(y)$, where x and y are independent of each other; it is required to find the form of the function.

Expanding $f(x+y)$ by Taylor's Series, we have

$$f(x) + f(y) = f(x+y)$$

$$f(x)+f(y) = f(x)+f'(x)\frac{y}{1}+f''(x)\frac{y^2}{1.2}+f'''(x)\frac{y^3}{1.2.3}+\dots;$$

$$\therefore\ f(y) = f'(x)\frac{y}{1}+f''(x)\frac{y^2}{1.2}+f'''(x)\frac{y^3}{1.2.3}+\dots;$$

now as x and y are independent of each other, the form of the function of y does not depend on x; therefore x and all functions of x are constant with respect to y. Hence we may put

$$f'(x) = \text{a constant} = a;$$

$$\therefore\ f''(x) = f'''(x) = \dots\dots = 0;$$

$$\therefore\ f(y) = ay;$$

whence also $f(x) = ax$, and $f(x+y) = a(x+y)$;
a result which palpably satisfies the required conditions.

Ex. 5. $f(x+y) = f(x)\times f(y)$, where x and y are independent; to determine the form of the function.

Expanding as before, we have

$$f(x)\times f(y) = f(x)+f'(x)\frac{y}{1}+f''(x)\frac{y^2}{1.2}+f'''(x)\frac{y^3}{1.2.3}+\dots,$$

$$f(y) = 1+\frac{f'(x)}{f(x)}\frac{y}{1}+\frac{f''(x)}{f(x)}\frac{y^2}{1.2}+\frac{f'''(x)}{f(x)}\frac{y^3}{1.2.3}+\dots;$$

for the same reason as in the last example, let

$$\frac{f'(x)}{f(x)} = a; \qquad \therefore\ f'(x) = af(x);$$

$$\therefore\ f''(x) = af'(x) = a^2f(x), \qquad \therefore\ \frac{f''(x)}{f(x)} = a^2;$$

similarly $\dfrac{f'''(x)}{f(x)} = a^3$, &c.

$$\therefore\ f(y) = 1+\frac{ay}{1}+\frac{a^2y^2}{1.2}+\frac{a^3y^3}{1.2.3}+\dots\dots$$

$$= e^{ay};$$

$$\therefore\ \text{also } f(x) = e^{ax}, \quad \text{and} \quad f(x+y) = e^{a(x+y)};$$

$$\therefore\ f(x+y) = e^{ax}\times e^{ay} = f(x)\times f(y),$$

which satisfies the required condition.

By a similar process let the student prove that, if

$f(x+y)+f(x-y) = 2f(x)\times f(y)$, $\quad f(x) = \cos ax$.

And if $f(xy) = f(x)+f(y)$, $\quad f(x) = \log_a x$;

in the last example he will find it convenient to assume $y = 1+\frac{h}{x}$, where h is undetermined.

74.] Also (81) may take another form. For h let $a-x$ be substituted; then we have

$$f(a) = f(x)+f'(x)\frac{a-x}{1}+f''(x)\frac{(a-x)^2}{1.2}+\dots$$
$$\dots+f^n\{x+\theta(a-x)\}\frac{(a-x)^n}{1.2\dots n};$$

and let a and x be mutually interchanged; then

$$f(x) = f(a)+f'(a)\frac{x-a}{1}+f''(a)\frac{(x-a)^2}{1.2}+\dots$$
$$\dots+f^n\{a+\theta(x-a)\}\frac{(x-a)^n}{1.2\dots n}, \quad (84)$$

whereby any function of x, which is capable of expansion in the form, is expanded in ascending powers of $x-a$.

As an example, let $f(x) = \log x$; then

$$\log x = \log a+\frac{x-a}{a}-\frac{1}{2}\left(\frac{x-a}{a}\right)^2+\frac{1}{3}\left(\frac{x-a}{a}\right)^3-\dots$$
$$(-)^{n-2}\frac{1}{n-1}\left(\frac{x-a}{a}\right)^{n-1}(-)^{n-1}\frac{1}{n}\left(\frac{x-a}{a+\theta(x-a)}\right)^n.$$

SECTION 4.—*Change of the equicrescent variable, and transformation of differential expressions.*

75.] From the supposition which we are at liberty to make that one of the variables involved in an equation should increase by equal increments, and therefore that the several differentials of it, after the first, should vanish, problems such as the following arise:

(1) Suppose $y = f(x)$, and that an expression is given involving x, y, and some of the derived-functions or differential coefficients which have been calculated on the supposition that one of the variables is equicrescent; to change the equation into its equivalent, when neither of the variables is equicrescent. Or

(2) To transform it into its equivalent, when the other variable is equicrescent. Or

(3) An expression being given involving a variable, which is either equicrescent or not, and its differentials, and also an equation being given connecting this variable with some other new variable; to eliminate the old variable and its differentials by means of these two equations, and to replace them in the original equation by their equivalents in terms of the new vari-

able: the new variable being equicrescent or not, as the case may be. Or

(4) It may be required to replace the variables and their differentials in a given differential expression, by their equivalents in terms of new variables, which are connected with the old variables by means of a sufficient number of given equations: the old and the new variables being equicrescent or not, as the case may be.

All these several processes involve transformations of differential expressions, and because such expressions commonly involve second and higher differentials, and one of the variables has been assumed to be equicrescent, they are called changes of the equicrescent variable. The method of effecting them is the same in all cases, viz.

To replace the expression, which has been simplified by the condition of a variable being equicrescent, by its complete value when no such modification has been made, and then to introduce the other conditions which the problem requires.

76.] Thus to solve the first two of the four cases above.

Let the given expression involve x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... ; the differential coefficients, as their form indicates, having been calculated on the supposition that x is equicrescent; it is required to replace these several differential coefficients by their equivalents, when x is not equicrescent.

Since by equations (2) and (3), Art. 54,

$$\frac{d^2y}{dx^2} = \frac{d.f'(x)}{dx}, \quad \text{and} \quad f'(x) = \frac{dy}{dx};$$

therefore if x is not equicrescent,

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y\,dx - d^2x\,dy}{dx^3}; \tag{85}$$

$$\frac{d^3y}{dx^3} = \frac{d\left(\frac{d^2y}{dx^2}\right)}{dx} = \frac{(d^3y\,dx - d^3x\,dy)\,dx - 3\,(d^2y\,dx - d^2x\,dy)\,d^2x}{dx^5}; \tag{86}$$

and similarly the equivalents of the other differential coefficients may be determined; the second members of the equations (85) and (86) being the complete values when neither x nor y is equicrescent.

If x is equicrescent, the equations are identical.

If y is equicrescent, then $d^2y = d^3y = \ldots = 0$; and the original quantities must be replaced by their equivalents, viz.

$$\frac{d^2y}{dx^2} \text{ must be replaced by } -\frac{d^2x\,dy}{dx^3}, \tag{87}$$

$$\frac{d^3y}{dx^3}\text{ - - - - - - - - } \frac{3\,(d^2x)^2\,dy - d^3x\,dy\,dx}{dx^5}, \tag{88}$$

$$\frac{d^4y}{dx^4}\text{ - - - - - - } \frac{10\,d^3x\,d^2x\,dx - (dx)^2\,d^4x - 15\,(d^2x)^3}{dx^7}\,dy; \tag{89}$$

. . . .

The form however of the nth equivalent is too complicated to be of any use in the solution of a given example.

Ex. 1. To transform $x\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 - \dfrac{dy}{dx} = 0$, into its equivalents, (α) when neither x nor y is equicrescent, (β) when y is equicrescent.

(α) Replace $\dfrac{d^2y}{dx^2}$ by its value as given in equation (85), and multiply by $(dx)^3$;

$$x\,(d^2y\,dx - d^2x\,dy) + dy^3 - dy\,dx^2 = 0.$$

(β) Let $d^2y = 0$, and we have

$$x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 - 1 = 0.$$

Ex. 2. To transform $(dy^2+dx^2)^{\frac{3}{2}} + a\,dx\,d^2y = 0$, when x is equicrescent, into its equivalents, (α) when neither x nor y is equicrescent, (β) when y is equicrescent.

(α) Replace d^2y by $\dfrac{d^2y\,dx - d^2x\,dy}{dx}$, and we have

$$(dy^2+dx^2)^{\frac{3}{2}} + a\,(d^2y\,dx - d^2x\,dy) = 0.$$

(β) And if y is equicrescent, $d^2y = 0$, and we have

$$(dy^2+dx^2)^{\frac{3}{2}} - a\,d^2x\,dy = 0,$$

$$\left\{1+\frac{dx^2}{dy^2}\right\}^{\frac{3}{2}} - a\frac{d^2x}{dy^2} = 0.$$

77.] In (3) and (4) of the cases of Art. 75, suppose the given expression to involve y, x, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, ... wherein x is the equicrescent variable, and suppose the new equation to be of the form $x = f(\theta)$, and the problem to be the elimination of x and its differentials between these two equations; we must first replace $\frac{d^2y}{dx^2}, \ldots\ldots$ by their complete values, and calculate dx, d^2x, d^3x, …… in terms of $d\theta$, $d^2\theta$, $d^3\theta$, ……, and then we are at liberty to make any supposition that may be convenient as to y or θ being equicrescent. And a similar method, as is shewn in Ex. 3, 4, 5 below, must be adopted when two equations are given connecting x and y with two new variables.

Ex. 1. Eliminate x between

$$\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0, \quad \text{and} \quad x = \cos\theta;$$

and simplify the result by making θ the equicrescent variable. Or the question may be put thus,

To transform $\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0$ into its equivalent, when θ $(= \cos^{-1}x)$ is the equicrescent variable.

The above equation when complete is

$$\frac{d^2y\,dx - d^2x\,dy}{dx^3} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0,$$

$$x = \cos\theta;$$

$$\therefore \quad dx = -\sin\theta\,d\theta, \qquad d^2x = -\cos\theta\,(d\theta)^2 - \sin\theta\,d^2\theta.$$

But since θ is equicrescent, $d^2\theta = 0$; whence, substituting, we have

$$\frac{-d^2y\sin\theta\,d\theta + \cos\theta\,(d\theta)^2\,dy}{-(\sin\theta)^3\,d\theta^3} + \frac{\cos\theta}{(\sin\theta)^2}\frac{dy}{\sin\theta\,d\theta} + \frac{y}{(\sin\theta)^2} = 0;$$

$$\therefore \quad \frac{d^2y}{d\theta^2} + y = 0.$$

Ex. 2. Eliminate x between

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0, \quad \text{and} \quad x^2 = 4\theta;$$

and express the result, (α) when neither y nor θ is equicrescent; (β) when θ is equicrescent; (γ) when y is equicrescent.

The complete expression of the above equation is

$$\frac{d^2y\,dx - d^2x\,dy}{dx^3} + \frac{1}{x}\frac{dy}{dx} + y = 0;$$

$$x = 2(\theta)^{\frac{1}{2}}; \qquad \therefore \quad dx = \theta^{-\frac{1}{2}}\,d\theta,$$

$$d^2x = -\frac{(d\theta)^2}{2\theta^{\frac{3}{2}}} + \frac{d^2\theta}{\theta^{\frac{1}{2}}};$$

whence, by substitution,

(α) $y\,(d\theta)^3 + dy\,(d\theta)^2 + \theta\, d^2y\, d\theta - \theta\, dy\, d^2\theta = 0.$

(β) And if θ is equicrescent, $d^2\theta = 0$; whereby we have

$$y\,(d\theta)^3 + dy\,(d\theta)^2 + \theta\, d^2y\, d\theta = 0;$$

or $$\theta \frac{d^2y}{d\theta^2} + \frac{dy}{d\theta} + y = 0.$$

(γ) And if y is equicrescent, $d^2y = 0$; whereby we have

$$\theta \frac{d^2\theta}{dy^2} - \left(\frac{d\theta}{dy}\right)^2 - y\left(\frac{d\theta}{dy}\right)^3 = 0.$$

Ex. 3. Eliminate x, y, dx and dy between

$$t = \frac{x\,dy - y\,dx}{y\,dy + x\,dx}, \quad x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta;$$

$$\therefore \quad dx = dr\cos\theta - r\sin\theta\, d\theta,$$

$$dy = dr\sin\theta + r\cos\theta\, d\theta;$$

$$\therefore \quad x\,dy - y\,dx = r^2\, d\theta, \qquad x\,dx + y\,dy = r\,dr;$$

$$\therefore \quad t = \frac{r\,d\theta}{dr}.$$

Ex. 4. Express in terms of r and θ, $\rho = \dfrac{\left\{1 + \dfrac{dy^2}{dx^2}\right\}^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}}$; having given $x = r\cos\theta$, and $y = r\sin\theta$; and state the result, (α) in the most general form; (β) when θ is equicrescent; (γ) when r is equicrescent.

The complete value of ρ is

$$\rho = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{d^2x\,dy - d^2y\,dx};$$

$$dx = dr\cos\theta - r\sin\theta\, d\theta,$$

$$dy = dr\sin\theta + r\cos\theta\, d\theta;$$

$$\therefore \quad d^2x = d^2r\cos\theta - 2\sin\theta\, dr\, d\theta - r\cos\theta\,(d\theta)^2 - r\sin\theta\, d^2\theta,$$

$$d^2y = d^2r\sin\theta + 2\cos\theta\, dr\, d\theta - r\sin\theta\,(d\theta)^2 + r\cos\theta\, d^2\theta;$$

$$\therefore \quad (dx)^2 + (dy)^2 = (dr)^2 + r^2\,(d\theta)^2,$$

$$d^2x\,dy - d^2y\,dx = r\, d^2r\, d\theta - 2\,(dr)^2\, d\theta - r^2\,(d\theta)^3 - r\, dr\, d^2\theta.$$

$$(\alpha) \qquad \therefore \quad \rho = \frac{\{(dr)^2 + r^2 (d\theta)^2\}^{\frac{3}{2}}}{r\,d^2r\,d\theta - 2\,(dr)^2\,d\theta - r^2\,(d\theta)^3 - r\,dr\,d^2\theta};$$

$$(\beta) \qquad = \frac{\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}^{\frac{3}{2}}}{r\frac{d^2r}{d\theta^2} - 2\left(\frac{dr}{d\theta}\right)^2 - r^2};$$

$$(\gamma) \qquad = -\frac{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{\frac{3}{2}}}{r\frac{d^2\theta}{dr^2} + r^2\left(\frac{d\theta}{dr}\right) + 2\frac{d\theta}{dr}}.$$

Ex. 5. Transform into its equivalent, when z is equicrescent, the expression

$$\text{P} = x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y\frac{dy}{dx},$$

if $x = e^z$, $y = e^z(v+2)$.

The complete expression for P is

$$\text{P} = x^2\frac{d^2y\,dx - d^2x\,dy}{dx^3} - (2x-y)\frac{dy}{dx};$$

and $\quad dx = e^z\,dz, \qquad dy = e^z(v+2)\,dz + e^z\,dv,$

$$d^2x = e^z\,dz^2, \qquad d^2y = e^z(v+2)\,dz^2 + 2\,e^z\,dz\,dv + e^z\,d^2v;$$

$$\therefore \quad \text{P} = e^z\left\{\frac{d^2v}{dz^2} + (v+1)\frac{dv}{dz} + v\,(v+2)\right\}.$$

SECTION 5.—*Successive differentiation of functions of two or more independent variables.*

78.] Thus far we have considered the successive differentials of explicit functions of the form $y = f(x)$. Let us now inquire into the form and the members of the several successive differentials of a function of many variables, all of which we will at first assume to be independent of each other: because we shall hereby treat of the most general case; and the particular one, when two or more of the variables are connected by an equation, will be included under it.

Let, as in Art. 49, the general form of the function be

$$u = \text{F}(x, y, z, \dots), \tag{90}$$

where $x, y, z, \dots$ are independent of each other; and are therefore such that a variation of one does not necessitate a variation of another. The symbols which we shall here adopt will be in principle the same as those which have heretofore been used in this Chapter; and they will be extended analogously to the peculiar circumstances of the present inquiry.

Now on referring to Article 46, it appears that a function of many independent variables is susceptible of variations of many kinds; thus only one of the variables may undergo a change; or two or more, or even all, may simultaneously change value; and consequently there may be either partial changes or a total change of the function. Hence have arisen partial and total variations. Similar changes take place in the successive variations of such a function, according as such changes are due to the variation of one, or more, or even all, of the variables; and, if the notation is adequate, it will indicate the particular kind of change which has taken place. To fix our thoughts, let us first take a function of two variables, and of the form

$$u = \mathrm{F}(x, y). \tag{91}$$

Then, by Art. 46, if $\mathrm{D}u$ represents the total change of u when x and y have both received infinitesimal increments dx and dy, and if $\left(\frac{du}{dx}\right) dx$ and $\left(\frac{du}{dy}\right) dy$ represent the changes of u when x and y have each singly and separately received infinitesimal increments dx and dy, then

$$\mathrm{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy. \tag{92}$$

It is evident that generally $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ are functions of both x and y; and that they, as well as $\mathrm{D}u$, admit of being differentiated again; and that of them there will be partial as well as total differentials. These several differentials must have distinctive symbols, and the distinction must be shewn in the symbols: if this object can be accomplished much confusion will be avoided.

Let the several successive and total differentials of $\mathrm{D}u$ be expressed by D^2u, D^3u, ... D^nu, ... Then since the x- and y-partial differentials of u have been represented by $\left(\frac{du}{dx}\right) dx$ and $\left(\frac{du}{dy}\right) dy$, it follows that the

x-partial differential of $\left(\frac{du}{dx}\right) = \frac{d}{dx}\left(\frac{du}{dx}\right)dx = \left(\frac{d^2u}{dx^2}\right)dx,$

y-partial differential of $\left(\frac{du}{dx}\right) = \frac{d}{dy}\left(\frac{du}{dx}\right)dy = \left(\frac{d^2u}{dy\,dx}\right)dy;$

and that the

x-partial differential of $\left(\frac{du}{dy}\right) = \frac{d}{dx}\left(\frac{du}{dy}\right)dx = \left(\frac{d^2u}{dx\,dy}\right)dx,$

y-partial differential of $\left(\frac{du}{dy}\right) = \frac{d}{dy}\left(\frac{du}{dy}\right)dy = \left(\frac{d^2u}{dy^2}\right)dy;$

the brackets being throughout employed to signify that the differentiations are partial. Similarly the

x-partial differential of $\left(\frac{d^2u}{dx^2}\right) = \frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)dx = \left(\frac{d^3u}{dx^3}\right)dx,$

y-partial differential of $\left(\frac{d^2u}{dx^2}\right) = \frac{d}{dy}\left(\frac{d^2u}{dx^2}\right)dy = \left(\frac{d^3u}{dy\,dx^2}\right)dy,$

x-partial differential of $\left(\frac{d^2u}{dx\,dy}\right) = \frac{d}{dx}\left(\frac{d^2u}{dx\,dy}\right)dx = \left(\frac{d^3u}{dx^2\,dy}\right)dx;$

and so on for the others. In accordance therefore with the principles and notation of Art. 54, $\left(\frac{d^2u}{dx^2}\right)$, $\left(\frac{d^2u}{dy^2}\right)$ will severally represent the second differential coefficients or derived functions of u, formed by separately making x and y to vary twice successively and by equal increments. And $\left(\frac{d^2u}{dx\,dy}\right)$ will represent the second derived-function of u, formed by making first y, and subsequently x, to vary by infinitesimal increments dy and dx. An analogous meaning belongs to symbols such as $\left(\frac{d^nu}{dx^n}\right)$, $\left(\frac{d^mu}{dy^m}\right)$, $\left(\frac{d^{m+n}u}{dx^m\,dy^n}\right)$, ... ; thus, for instance, by $\left(\frac{d^4u}{dx^2\,dy^2}\right)$ is meant the fourth derived-function of u, which is calculated by making y first to vary twice and by equal increments dy, and subsequently x to vary twice by equal increments dx. The order of factors in the denominator indicates the order in which the differentiations are performed: and in the next Article it will be proved that this order is indifferent. Hence also it follows that $\left(\frac{d^nu}{dx^n}\right)dx^n$ represents the nth differential of u, formed by making x to vary n times successively, the successive increments of x

being equal; and similarly $\left(\frac{d^m u}{dy^m}\right) dy^m$ represents the mth differential of u, formed by making y to increase m times successively by equal increments dy; and $\left(\frac{d^r u}{dx^m\, dy^{r-m}}\right) dx^m\, dy^{r-m}$ indicates the rth differential of u, formed by making x to increase m times successively by equal increments dx, and y to increase $(r-m)$ times successively by equal increments dy.

The principles on which this notation is constructed are so immediately applicable to functions of more than two variables, that it is unnecessary to do more than state the results as they occur in the following Articles.

Here however I must remark on a defect in the notation. If $u = \text{F}(x, y)$, for instance, the same symbol d^2u in the numerators of

$$\left(\frac{d^2u}{dx^2}\right), \quad \left(\frac{d^2u}{dx\,dy}\right), \quad \left(\frac{d^2u}{dy^2}\right),$$

means processes which, although of the same kind, may lead to results altogether different. In the first it means the second differential of u springing from two successive variations of x; in the second it represents d^2u springing from a variation of x, taking place on the back of a previous variation of y; and in the third, d^2u as originating in two successive variations of y. The brackets therefore are used in conjunction with the different denominators to indicate these variations of the same symbol, and are marks sufficiently distinctive to prevent confusion. But before we proceed further, we must prove the following proposition.

79.] If a function is differentiated many times in respect of independent variables which it contains, the result is the same, whatever is the order of the variables with respect to which it is differentiated: provided that it is differentiated the same number of times and with respect to the same variables.

$$\text{Let} \qquad u = \text{F}(x, y, z, \ldots\ldots);$$

$$\text{then} \qquad \left(\frac{d^2u}{dx\,dy}\right) = \left(\frac{d^2u}{dy\,dx}\right).$$

For the sake of convenience, using symbols of differentiation,

$$d_x \text{F}(x, y, z, \ldots) = \text{F}(x+dx, y, z, \ldots) - \text{F}(x, y, z, \ldots), \qquad (93)$$

$$d_y \text{F}(x, y, z, \ldots) = \text{F}(x, y+dy, z, \ldots) - \text{F}(x, y, z, \ldots); \qquad (94)$$

therefore from (93),

$$d_y d_x \mathrm{F}(x, y, z, \ldots) = \mathrm{F}(x+dx, y+dy, z, \ldots) - \mathrm{F}(x+dx, y, z, \ldots) - \mathrm{F}(x, y+dy, z, \ldots) + \mathrm{F}(x, y, z, \ldots);$$

and from (94),

$$d_x d_y \mathrm{F}(x, y, z, \ldots) = \mathrm{F}(x+dx, y+dy, z, \ldots) - \mathrm{F}(x, y+dy, z, \ldots) - \mathrm{F}(x+dx, y, z, \ldots) + \mathrm{F}(x, y, z, \ldots);$$

and as the two results are identical, it is manifest that

$$d_x d_y \mathrm{F}(x, y, z, \ldots) = d_y d_x \mathrm{F}(x, y, z, \ldots); \tag{95}$$

or, writing the result according to the notation of Art. 46,

$$\left(\frac{d^2u}{dx\,dy}\right) dx\,dy = \left(\frac{d^2u}{dy\,dx}\right) dy\,dx;$$

and dividing through by $dy\,dx$, we have

$$\left(\frac{d^2u}{dx\,dy}\right) = \left(\frac{d^2u}{dy\,dx}\right). \tag{96}$$

As the proof here given does not depend on the differentiations having been performed with respect to two variables only, it is plain that an analogous theorem is true for differentiations with respect to any number of variables; so that we may always interchange, in whatever manner it is convenient, the order in which the several differentiations are performed; as, for instance,

$$d_x d_y d_z u = d_y d_z d_x u = d_z d_x d_y u; \tag{97}$$

$$\text{or} \quad \left(\frac{d^3u}{dx\,dy\,dz}\right) = \left(\frac{d^3u}{dy\,dz\,dx}\right) = \left(\frac{d^3u}{dz\,dx\,dy}\right). \tag{98}$$

Hence also it follows, that if successive partial differentiations are performed on a function of many independent variables, by making x, y, and z to vary severally r times, s times, and t times, the order of these variations may be interchanged in any permutation, and the result is the same; thus if

$$u = \mathrm{F}(x, y, z, \ldots),$$

$$\frac{d^{r+s+t}u}{dx^r\,dy^s\,dz^t} = \frac{d^{s+t+r}u}{dy^s\,dz^t\,dx^r} = \frac{d^{t+r+s}u}{dz^t\,dx^r\,dy^s}$$

$$= \frac{d^{r+s+t}u}{dy\,dx^{r-1}\,dz^t\,dx\,dy^{s-1}} = \ldots\ldots \tag{99}$$

Of this property, thus proved in the general case, some particular examples are subjoined.

Ex. 1. $u = \dfrac{x^3-y^3}{x^2+y^2}$;

$$\left(\frac{du}{dx}\right) = \frac{4xy^2}{(x^2+y^2)^2}, \qquad \left(\frac{du}{dy}\right) = \frac{-4x^2y}{(x^2+y^2)^2},$$

$$\left(\frac{d^2u}{dy\,dx}\right) = \frac{8xy(x^2-y^2)}{(x^2+y^2)^3}, \qquad \left(\frac{d^2u}{dx\,dy}\right) = \frac{8xy(x^2-y^2)}{(x^2+y^2)^3}.$$

Ex. 2. $u = \dfrac{xy^2}{z^2-a^2}$;

$$\left(\frac{du}{dx}\right) = \frac{y^2}{z^2-a^2}, \qquad \left(\frac{du}{dy}\right) = \frac{2xy}{z^2-a^2}, \qquad \left(\frac{du}{dz}\right) = \frac{-2xy^2z}{(z^2-a^2)^2},$$

$$\left(\frac{d^3u}{dx\,dy\,dz}\right) = \left(\frac{d^3u}{dy\,dx\,dz}\right) = \left(\frac{d^3u}{dz\,dy\,dx}\right) = \frac{-4yz}{(z^2-a^2)^2}.$$

80.] **Differentiation of a function of two independent variables.**

Let $u = \text{F}(x, y)$;

then, by Art. 47, $\text{D}u = \left(\dfrac{du}{dx}\right) dx + \left(\dfrac{du}{dy}\right) dy$;

$$\therefore \quad \text{D.D}u = \text{D}\left(\frac{du}{dx}\right) dx + \text{D}\left(\frac{du}{dy}\right) dy + \left(\frac{du}{dx}\right) d^2x + \left(\frac{du}{dy}\right) d^2y.$$

Now in general, $\left(\dfrac{du}{dx}\right)$ and $\left(\dfrac{du}{dy}\right)$ are functions of x and y; so that by reason of (41), Art. 47,

$$\text{D}\left(\frac{du}{dx}\right) = \left(\frac{d\left(\frac{du}{dx}\right)}{dx}\right) dx + \left(\frac{d\left(\frac{du}{dx}\right)}{dy}\right) dy,$$

$$= \left(\frac{d^2u}{dx^2}\right) dx + \left(\frac{d^2u}{dy\,dx}\right) dy;$$

$$\text{D}\left(\frac{du}{dy}\right) = \left(\frac{d\left(\frac{du}{dy}\right)}{dx}\right) dx + \left(\frac{d\left(\frac{du}{dy}\right)}{dy}\right) dy,$$

$$= \left(\frac{d^2u}{dx\,dy}\right) dx + \left(\frac{d^2u}{dy^2}\right) dy;$$

whence, taking the most general case, in which neither x nor y is equicrescent, we have

$$\mathrm{D}^2u = \left(\frac{d^2u}{dx^2}\right)dx^2 + 2\left(\frac{d^2u}{dx\,dy}\right)dx\,dy + \left(\frac{d^2u}{dy^2}\right)dy^2 + \left(\frac{du}{dx}\right)d^2x + \left(\frac{du}{dy}\right)d^2y; \quad (100)$$

the brackets indicating that the derived-functions within them are partial. Similarly,

$$\mathrm{D}^3u = \left(\frac{d^3u}{dx^3}\right)dx^3 + 3\left(\frac{d^3u}{dx^2\,dy}\right)dx^2\,dy + 3\left(\frac{d^3u}{dx\,dy^2}\right)dx\,dy^2 + \left(\frac{d^3u}{dy^3}\right)dy^3$$
$$+3\left[\left(\frac{d^2u}{dx^2}\right)dx\,d^2x + \left(\frac{d^2u}{dx\,dy}\right)\{d^2x\,dy + d^2y\,dx\} + \left(\frac{d^2u}{dy^2}\right)dy\,d^2y\right]$$
$$+ \left(\frac{du}{dx}\right)d^3x + \left(\frac{du}{dy}\right)d^3y. \quad (101)$$

Similarly may other total differentials be found; but the general term is too complicated to be of any practical use.

Now let the results be simplified by making x and y equicrescent, neither of which assumptions is inconsistent with the given equation; then

$$d^2x = d^3x = \ldots = 0; \qquad d^2y = d^3y = \ldots = 0;$$

and we have the following series of equations,

$$u = \mathrm{F}(x, y),$$

$$\mathrm{D}u = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy,$$

$$\mathrm{D}^2u = \left(\frac{d^2u}{dx^2}\right)dx^2 + 2\left(\frac{d^2u}{dx\,dy}\right)dx\,dy + \left(\frac{d^2u}{dy^2}\right)dy^2, \quad (102)$$

$$\mathrm{D}^3u = \left(\frac{d^3u}{dx^3}\right)dx^3 + 3\left(\frac{d^3u}{dx^2\,dy}\right)dx^2\,dy + 3\left(\frac{d^3u}{dx\,dy^2}\right)dx\,dy^2 + \left(\frac{d^3u}{dy^3}\right)dy^3; \quad (103)$$

and so on; the law of the coefficients being the same as that of $(1+x)^n$; which may be proved to be true for positive integral values of the exponent, by a train of reasoning similar to that in Art. 55; whence the nth differential is

$$\mathrm{D}^nu = \left(\frac{d^nu}{dx^n}\right)dx^n + n\left(\frac{d^nu}{dx^{n-1}\,dy}\right)dx^{n-1}\,dy + \frac{n(n-1)}{1.2}\left(\frac{d^nu}{dx^{n-2}\,dy^2}\right)dx^{n-2}\,dy^2 + \ldots + \left(\frac{d^nu}{dy^n}\right)dy^n. \quad (104)$$

Ex. 1. $u = (x^2+y^2)^{\frac{1}{2}}$;

$$\left(\frac{du}{dx}\right) = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}, \quad \left(\frac{d^2u}{dx^2}\right) = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}, \quad \left(\frac{d^3u}{dx^3}\right) = \frac{-3xy^2}{(x^2+y^2)^{\frac{5}{2}}},$$

$$\left(\frac{du}{dy}\right) = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}, \quad \left(\frac{d^2u}{dx\,dy}\right) = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}, \quad \left(\frac{d^3u}{dx^2\,dy}\right) = \frac{y(2x^2-y^2)}{(x^2+y^2)^{\frac{5}{2}}},$$

$$\left(\frac{d^2u}{dy^2}\right) = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}, \quad \left(\frac{d^3u}{dx\,dy^2}\right) = \frac{x(2y^2-x^2)}{(x^2+y^2)^{\frac{5}{2}}},$$

$$\left(\frac{d^3u}{dy^3}\right) = \frac{-3yx^2}{(x^2+y^2)^{\frac{5}{2}}};$$

$$\text{D}^3u = \{-3xy^2\,dx^3 + 3y(2x^2-y^2)\,dx^2\,dy + 3x(2y^2-x^2)\,dx\,dy^2 - 3yx^2\,dy^3\}\frac{1}{(x^2+y^2)^{\frac{5}{2}}}.$$

Ex. 2. $u = e^{ax+by}$;

$$\left(\frac{du}{dx}\right) = a\,e^{ax+by}, \qquad \left(\frac{d^2u}{dx^2}\right) = a^2\,e^{ax+by},$$

$$\left(\frac{du}{dy}\right) = b\,e^{ax+by}, \qquad \left(\frac{d^2u}{dx\,dy}\right) = ab\,e^{ax+by},$$

$$\left(\frac{d^2u}{dy^2}\right) = b^2\,e^{ax+by};$$

$$\therefore \quad \text{D}^2u = \{a^2dx^2 + 2ab\,dx\,dy + b^2dy^2\}\,e^{ax+by},$$
$$= (a\,dx + b\,dy)^2\,e^{ax+by}.$$

81.] By a similar process, if u is a function of many independent variables, $x, y, z, \ldots$ all of which are equicrescent, we have the following series of equations:

$$u = \text{F}(x, y, z, \ldots),$$

$$\text{D}u = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy + \left(\frac{du}{dz}\right)dz + \ldots\ldots, \tag{105}$$

$$\text{D}^2u = \left(\frac{d^2u}{dx^2}\right)dx^2 + \left(\frac{d^2u}{dy^2}\right)dy^2 + \left(\frac{d^2u}{dz^2}\right)dz^2 + \ldots\ldots$$
$$+ 2\left(\frac{d^2u}{dy\,dz}\right)dy\,dz + 2\left(\frac{d^2u}{dz\,dx}\right)dz\,dx + 2\left(\frac{d^2u}{dx\,dy}\right)dx\,dy + \ldots \tag{106}$$

. . . .

If $x, y, z, \ldots\ldots$ are not equicrescent, terms will have to be added analogous to those which, by reason of x and y being equicrescent, vanished from equations (100) and (101).

Also, using the notation of partial differentials, the above equations become

$$\left.\begin{aligned} \mathrm{D}u &= d_x u + d_y u + d_z u + \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \\ \mathrm{D}^2 u &= d^2{}_x u + d^2{}_y u + d^2{}_z u + 2 d_y d_z u + 2 d_z d_x u + 2 d_x d_y u + \ldots \end{aligned}\right\} \quad (107)$$

. . . .

Ex. 1. $u = \sin(ax + by + cz)$;

$$\left(\frac{du}{dx}\right) = a\cos(ax+by+cz), \qquad \left(\frac{d^2u}{dx^2}\right) = -a^2\sin(ax+by+cz),$$

$$\left(\frac{du}{dy}\right) = b\cos(ax+by+cz), \qquad \left(\frac{d^2u}{dy^2}\right) = -b^2\sin(ax+by+cz),$$

$$\left(\frac{du}{dz}\right) = c\cos(ax+by+cz), \qquad \left(\frac{d^2u}{dz^2}\right) = -c^2\sin(ax+by+cz),$$

$$\left(\frac{d^2u}{dy\,dz}\right) = -bc\sin(ax+by+cz),$$

. . . .

$$\therefore \quad \mathrm{D}^2 u = -\{a^2 dx^2 + b^2 dy^2 + c^2 dz^2 + 2bc\,dy\,dz + 2ca\,dz\,dx + 2ab\,dx\,dy\}\sin(ax+by+cz).$$

In the preceding inquiry all the subject-variables of the function have been assumed to be independent, and thus the results are general. It however frequently happens that some of them are dependent on others; and although it is unnecessary to consider generally the modifications which the results undergo in such cases, yet it is expedient to give an example, so that the student may perceive the kind of results which such problems present.

Ex. 1. $u = \tan^{-1}\frac{x}{y}, \qquad y^2 + x^2 = a^2$;

$$\left(\frac{du}{dx}\right) = \frac{y}{x^2+y^2}, \qquad \left(\frac{d^2u}{dx^2}\right) = \frac{-2xy}{(x^2+y^2)^2},$$

$$\left(\frac{du}{dy}\right) = -\frac{x}{x^2+y^2}, \qquad \left(\frac{d^2u}{dx\,dy}\right) = \frac{x^2-y^2}{(x^2+y^2)^2},$$

$$\left(\frac{d^2u}{dy^2}\right) = \frac{2xy}{(x^2+y^2)^2}.$$

Therefore from (100), Art. 80,

$$\mathrm{D}^2 u = \frac{1}{(x^2+y^2)^2}\{-2xy\,dx^2 + 2(x^2-y^2)\,dx\,dy + 2xy\,dy^2\} + \frac{y\,d^2x - x\,d^2y}{x^2+y^2}.$$

Let x be equicrescent; so that $d^2x = 0$; and from the relation given between x and y in the preceding equation of condition,

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{a^2}{y^3};$$

so that
$$\mathrm{D}^2u = \frac{x}{y^3}\,dx^2;$$

and if we express it in terms of x only,

$$\frac{d^2u}{dx^2} = \frac{x}{(a^2-x^2)^{\frac{3}{2}}}.$$

82.] As one of the first and most useful applications of the general results of the last Article, let us prove certain properties of homogeneous functions, which are due to Euler, and are generally known by the name of Euler's Theorems of Homogeneous Functions.

DEF. A homogeneous function of many variables is one which has the sum of the indices of the variables in every term the same; and if the sum of the indices in each term $= n$, the function is said to be homogeneous of n dimensions. Thus

$$ax^3 + bx^2y + cz^3 + exyz + gx^2z = 0$$

is a homogeneous function of three dimensions.

Let $u = \mathrm{F}(x, y, z, \ldots)$ be a homogeneous function of n dimensions and r variables; for $x, y, z, \ldots$ let $tx, ty, tz, \ldots$ be written, and suppose the function to become u' when these substitutions are made; then, by the definition of homogeneous functions,

$$u' = \mathrm{F}(tx, ty, tz, \ldots = t^n\,\mathrm{F}(x, y, z, \ldots). \tag{108}$$

Let
$$\left.\begin{array}{ll} tx = x', & \therefore\ \dfrac{dx'}{dt} = x, \\ ty = y', & \ \dfrac{dy'}{dt} = y, \\ tz = z', & \ \dfrac{dz'}{dt} = z, \\ \ldots\ldots & \ldots\ldots \end{array}\right\} \tag{109}$$

$$\therefore\quad u' = \mathrm{F}(x', y', z', \ldots\ldots);$$

and therefore by (46), Art. 49,

$$\frac{\mathrm{D}u'}{dt} = \left(\frac{du'}{dx'}\right)\frac{dx'}{dt} + \left(\frac{du'}{dy'}\right)\frac{dy'}{dt} + \left(\frac{du'}{dz'}\right)\frac{dz'}{dt} + \cdots$$

$$= x\left(\frac{du'}{dx'}\right) + y\left(\frac{du'}{dy'}\right) + z\left(\frac{du'}{dz'}\right) + \dots; \tag{110}$$

now the t-differential of (108) is

$$\frac{du'}{dt} = nt^{n-1}\,\mathrm{F}(x, y, z, \dots); \tag{111}$$

equating this to (110), because they are equal, we have

$$nt^{n-1}\,\mathrm{F}(x, y, z, \dots) = x\left(\frac{du'}{dx'}\right) + y\left(\frac{du'}{dy'}\right) + z\left(\frac{du'}{dz'}\right) + \dots;$$

let $t = 1$, then $x' = x$, $y' = y$, $z' = z$, ... ;

$$\therefore\ \ n\,\mathrm{F}(x,y,z,\dots) = nu = x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) + z\left(\frac{du}{dz}\right) + \dots \tag{112}$$

Again, taking the second total differential of u', and dividing by $(dt)^2$, we have

$$\frac{\mathrm{D}^2 u'}{dt^2} = \left(\frac{d^2u'}{dx'^2}\right)\frac{dx'^2}{dt^2} + 2\left(\frac{d^2u'}{dx'\,dy'}\right)\frac{dx'}{dt}\frac{dy'}{dt} + \left(\frac{d^2u'}{dy'^2}\right)\frac{dy'^2}{dt^2} + \dots,$$

and substituting from (109), we have

$$\frac{\mathrm{D}^2 u'}{dt^2} = \left(\frac{d^2u'}{dx'^2}\right)x^2 + 2\left(\frac{d^2u'}{dx'\,dy'}\right)xy + \left(\frac{d^2u'}{dy'^2}\right)y^2 + \dots \tag{113}$$

also the t-differential of (111) is

$$\frac{d^2u'}{dt^2} = n(n-1)\,t^{n-2}\,\mathrm{F}(x, y, z, \dots); \tag{114}$$

whence, equating (113) and (114), and making $t = 1$, we have

$$n(n-1)\,u = x^2\left(\frac{d^2u}{dx^2}\right) + 2xy\left(\frac{d^2u}{dx\,dy}\right) + y^2\left(\frac{d^2u}{dy^2}\right) + \dots \tag{115}$$

Similarly it may be shewn that

$$n(n-1)(n-2)\,u = x^3\left(\frac{d^3u}{dx^3}\right) + 3x^2y\left(\frac{d^3u}{dx^2\,dy}\right)$$
$$+ 3xy^2\left(\frac{d^3u}{dx\,dy^2}\right) + y^3\left(\frac{d^3u}{dy^3}\right) + \dots; \tag{116}$$

and similar theorems are true for every other order of differentials. One or two examples are subjoined, in which the theorems are shewn to be true.

Ex. 1. $\mathrm{F}(x, y, z) = u = \mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + \mathrm{E}yz + \mathrm{G}zx + \mathrm{H}xy,$ which is an homogeneous function of two dimensions and three variables.

$$\left(\frac{du}{dx}\right) = 2\text{A}x+\text{G}z+\text{H}y, \qquad \left(\frac{d^2u}{dx^2}\right) = 2\text{A}, \qquad \left(\frac{d^2u}{dy\,dz}\right) = \text{E},$$

$$\left(\frac{du}{dy}\right) = 2\text{B}y+\text{E}z+\text{H}x, \qquad \left(\frac{d^2u}{dy^2}\right) = 2\text{B}, \qquad \left(\frac{d^2u}{dz\,dx}\right) = \text{G},$$

$$\left(\frac{du}{dz}\right) = 2\text{C}z+\text{E}y+\text{G}x, \qquad \left(\frac{d^2u}{dz^2}\right) = 2\text{C}, \qquad \left(\frac{d^2u}{dx\,dy}\right) = \text{H};$$

therefore by equations (112) and (115),

$$x\left(\frac{du}{dx}\right)+y\left(\frac{du}{dy}\right)+z\left(\frac{du}{dz}\right) = 2\{\text{A}x^2+\text{B}y^2+\text{C}z^2+\text{E}yz+\text{G}zx+\text{H}xy\}$$
$$= 2u;$$

$$x^2\left(\frac{d^2u}{dx^2}\right)+y^2\left(\frac{d^2u}{dy^2}\right)+z^2\left(\frac{d^2u}{dz^2}\right)+2yz\left(\frac{d^2u}{dy\,dz}\right)$$
$$+2zx\left(\frac{d^2u}{dz\,dx}\right)+2xy\left(\frac{d^2u}{dx\,dy}\right) =$$
$$2\{\text{A}x^2+\text{B}y^2+\text{C}z^2+\text{E}yz+\text{G}zx+\text{H}xy\} = 2u.$$

Ex. 2. $u = \text{F}\left(\frac{y}{x}\right)$, which is a homogeneous function of 0 dimensions.

Of $\text{F}(z)$ let $\text{F}'(z)$ be the derived-function, that is, be that quantity which multiplied by dz is the z-differential of $\text{F}(z)$; then

$$\left(\frac{du}{dx}\right) = -\frac{y}{x^2}\,\text{F}'\left(\frac{y}{x}\right), \qquad \left(\frac{du}{dy}\right) = \frac{1}{x}\,\text{F}'\left(\frac{y}{x}\right);$$

$$\therefore \quad x\left(\frac{du}{dx}\right)+y\left(\frac{du}{dy}\right) = \left(\frac{y}{x}-\frac{y}{x}\right)\text{F}'\left(\frac{y}{x}\right) = 0.$$

Article 53, Ex. 4, is another case in which the preceding theorem is exhibited.

83.] As another application of the results of Article 81, let us investigate the equivalents of the second derived-functions of a function of three variables, when one, say z, is an explicit function of the other two, in terms of the derived-functions, when all are implicitly involved in an equation. That is, if $z = f(x,y)$ and $\phi(x,y,z) = 0$ are equivalent equations, it is required to express $\left(\frac{d^2z}{dx^2}\right)$, $\left(\frac{d^2z}{dx\,dy}\right)$, $\left(\frac{d^2z}{dy^2}\right)$ in terms of $\left(\frac{d\phi}{dx}\right)$, … $\left(\frac{d^2\phi}{dx^2}\right)$, … $\left(\frac{d^2\phi}{dy\,dz}\right)$, …

Since $\phi(x, y, z) = 0$,

$$\left(\frac{d\phi}{dx}\right) dx + \left(\frac{d\phi}{dy}\right) dy + \left(\frac{d\phi}{dz}\right) dz = 0; \tag{117}$$

therefore if we estimate the simultaneous variations of x and z, as in Art. 50,

$$\left(\frac{d\phi}{dx}\right) dx + \left(\frac{d\phi}{dz}\right) dz = 0; \tag{118}$$

let the x- and z-differential of this be taken, and let x be equicrescent; then we have

$$\left(\frac{d^2\phi}{dx^2}\right) dx^2 + 2\left(\frac{d^2\phi}{dx\,dz}\right) dx\,dz + \left(\frac{d^2\phi}{dz^2}\right) dz^2 + \left(\frac{d\phi}{dz}\right) d^2z = 0;$$

therefore dividing through by dx^2, and replacing $\left(\frac{dz}{dx}\right)$ by its value from (118), we have

$$\left(\frac{d^2\phi}{dx^2}\right)\left(\frac{d\phi}{dz}\right)^2 - 2\left(\frac{d^2\phi}{dx\,dz}\right)\left(\frac{d\phi}{dx}\right)\left(\frac{d\phi}{dz}\right) + \left(\frac{d^2\phi}{dz^2}\right)\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dz}\right)^3\left(\frac{d^2z}{dx^2}\right) = 0;$$

whence $\left(\frac{d^2z}{dx^2}\right)$ is expressed in terms of the partial derived of $\phi(x, y, z) = 0$. Similarly for the y- and z-partial differential we have

$$\left(\frac{d^2\phi}{dy^2}\right)\left(\frac{d\phi}{dz}\right)^2 - 2\left(\frac{d^2\phi}{dy\,dz}\right)\left(\frac{d\phi}{dy}\right)\left(\frac{d\phi}{dz}\right) + \left(\frac{d^2\phi}{dz^2}\right)\left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^3\left(\frac{d^2z}{dy^2}\right) = 0.$$

Also taking the y- and z-partial differential of (118), and substituting for $\left(\frac{dz}{dx}\right)$, and $\left(\frac{dz}{dy}\right)$, as they are given in Art. 50, we have

$$\left(\frac{d^2\phi}{dx\,dy}\right)\left(\frac{d\phi}{dz}\right)^2 - \left(\frac{d^2\phi}{dx\,dz}\right)\left(\frac{\phi}{dy}\right)\left(\frac{d\phi}{dz}\right) - \left(\frac{d^2\phi}{dy\,dz}\right)\left(\frac{d\phi}{dz}\right)\left(\frac{d\phi}{dx}\right)$$
$$+ \left(\frac{d^2\phi}{dz^2}\right)\left(\frac{d\phi}{dx}\right)\left(\frac{d\phi}{dy}\right) + \left(\frac{d^2z}{dx\,dy}\right)\left(\frac{d\phi}{dz}\right)^3 = 0;$$

which gives us $\left(\frac{d^2z}{dx\,dy}\right)$ in terms of the partial derived-functions of $\phi(x, y, z) = 0$.

We proceed now to consider the theory of successive derivation of an implicit function in its application to many and important theorems.

SECTION 6.—*Successive differentiation of implicit functions.*

84.] Firstly, let us consider a function of two variables x and y, and of the form,

$$u = \mathrm{F}(x, y) = c,$$

c being a constant or being zero; then

$$\mathrm{D}u = 0 = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy,$$

$$\mathrm{D}^2u = 0 = \left(\frac{d^2u}{dx^2}\right) dx^2 + 2\left(\frac{d^2u}{dx\,dy}\right) dx\,dy + \left(\frac{d^2u}{dy^2}\right) dy^2$$
$$+ \left(\frac{du}{dx}\right) d^2x + \left(\frac{du}{dy}\right) d^2y,$$

$$\mathrm{D}^3u = 0 = \left(\frac{d^3u}{dx^3}\right) dx^3 + \ldots\ldots;$$

then if x is equicrescent, we have

$$\left(\frac{du}{dy}\right)\frac{dy}{dx} + \left(\frac{du}{dx}\right) = 0, \tag{119}$$

$$\left(\frac{du}{dy}\right)\frac{d^2y}{dx^2} + 2\left(\frac{d^2u}{dx\,dy}\right)\frac{dy}{dx} + \left(\frac{d^2u}{dy^2}\right)\frac{dy^2}{dx^2} + \left(\frac{d^2u}{dx^2}\right) = 0; \tag{120}$$

whence we have the following values of $\frac{dy}{dx}$, and of $\frac{d^2y}{dx^2}$,

$$\frac{dy}{dx} = -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)}, \tag{121}$$

$$\frac{d^2y}{dx^2} = \frac{-\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2 + 2\left(\frac{d^2u}{dx\,dy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) - \left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dy}\right)^3}. \tag{122}$$

The latter may be deduced from (121) by differentiating with respect to the equicrescent variable x, as well as from (120), whence we have deduced it.

Similarly from $\mathrm{D}^3u = 0$ may be deduced $\frac{d^3y}{dx^3}$; and from subsequent total differentials the other derived-functions of y may be formed.

85.] Ex. 1. Given $y^2-2yx+a^2=u=0$, to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\left(\frac{du}{dx}\right) = -2y, \qquad \left(\frac{d^2u}{dx^2}\right) = 0, \qquad \left(\frac{d^2u}{dy^2}\right) = 2;$$

$$\left(\frac{du}{dy}\right) = 2y-2x, \qquad \left(\frac{d^2u}{dx\,dy}\right) = -2,$$

therefore by (121), $\quad \frac{dy}{dx} = \frac{y}{y-x}$;

and by (122), $\quad \frac{d^2y}{dx^2} = \frac{y(y-2x)}{(y-x)^3}$.

Instead however of introducing *formally* the general values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ given in equations (121) and (122), it is more convenient to differentiate the given function *immediately* according to the principles contained in Art. 48, and illustrated in the last three examples of that Article; the two following examples illustrate the process.

Ex. 2. If $x^3+3axy+y^3=0$; to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Differentiating, we have

$$(x^2+ay)\,dx+(y^2+ax)\,dy = 0; \tag{123}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2+ay}{y^2+ax};$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{\left(2x+a\frac{dy}{dx}\right)(y^2+ax)-\left(2y\frac{dy}{dx}+a\right)(x^2+ay)}{(y^2+ax)^2}$$

$$= \frac{2a^3xy}{(y^2+ax)^3}.$$

Ex. 3. If $x^2+y^2=a^2$; to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$x\,dx+y\,dy = 0; \qquad \therefore \quad y\frac{dy}{dx}+x = 0,$$

and $\quad \frac{dy}{dx} = -\frac{x}{y}$; $\qquad$ also $\quad y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1 = 0$;

$\therefore \quad y\frac{d^2y}{dx^2}+\frac{x^2+y^2}{y^2} = 0$; $\qquad$ and $\quad \frac{d^2y}{dx^2} = -\frac{a^2}{y^3}$.

86.] Hence also, if an implicit function is given involving x and y, we may calculate the several coefficients of the powers of x in Maclaurin's Theorem, see Art. 57, and thus expand y in a series of terms of ascending powers of x.

Let the given implicit function be $u = \mathrm{F}(x, y) = c$; then, since in the series (8), Art. 57, $f(0)$, $f'(0)$, $f''(0)$, ... are severally the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... when $x = 0$, we have by equations (121) and (122),

$$y = f(x) = [y]_0 + \left[-\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)}\right]_0 \frac{x}{1}$$
$$+ \left[\frac{-\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2 + 2\left(\frac{d^2u}{dx\,dy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) - \left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dy}\right)^3}\right]_0 \frac{x^2}{1.2}$$
$$+ \cdot \cdot \cdot \cdot \cdot \qquad (124)$$

the notation signifying that particular values of the coefficients are taken, viz. when $x = 0$. In applications however of this theorem it is much less laborious to calculate the successive values directly from the given equation than to substitute in the preceding formulæ.

Ex. 1. $ay^3 - xy = a$.

$$\text{Let} \quad x = 0; \qquad \therefore \; f(0) = 1;$$

$$\frac{dy}{dx}(3ay^2 - x) - y = 0, \qquad f'(0) = \frac{1}{3a};$$

$$\frac{d^2y}{dx^2}(3ay^2 - x) + \frac{dy}{dx}\left(6ay\frac{dy}{dx} - 1\right) - \frac{dy}{dx} = 0, \quad f''(0) = 0;$$

$$\frac{d^3y}{dx^3}(3ay^2 - x) + 2\frac{d^2y}{dx^2}\left(6ay\frac{dy}{dx} - 1\right)$$
$$+ 6a\frac{dy}{dx}\left(\frac{dy^2}{dx^2} + y\frac{d^2y}{dx^2}\right) - \frac{d^2y}{dx^2} = 0, \quad f'''(0) = -\frac{2}{27a^3};$$

. . . .

$$\therefore \quad y = 1 + \frac{1}{3a}\frac{x}{1} - \frac{2}{27a^3}\frac{x^3}{1.2.3} + \ldots\ldots;$$

which is one of the values of y in terms of x deduced from the given cubic equation; the other two values may be found by taking the other two impossible cube roots of unity in the value of $f(0)$.

Ex. 2. $y^3-3y+x=0;\qquad \therefore\ f(0)=0, \text{ and } = \pm\sqrt{3};$

$$\frac{dy}{dx}(3y^2-3)+1=0,\qquad f'(0)=\frac{1}{3}, \text{ and } = -\frac{1}{6};$$

$$\frac{d^2y}{dx^2}(y^2-1)+2y\frac{dy^2}{dx^2}=0,\qquad f''(0)=0, \text{ and } = \mp\frac{\sqrt{3}}{36};$$

$$\frac{d^2y}{dx^2}(y^2-1)+6y\frac{dy}{dx}\frac{d^2y}{dx^2}+2\frac{dy^3}{dx^3}=0,$$

$$f'''(0)=\frac{2}{27}, \text{ and } = -\frac{1}{27};$$

. . . .

$$\therefore\quad y=\frac{x}{3}+\frac{2}{27}\frac{x^3}{1.2.3}+\dots,$$

$$\text{and}\quad y=\pm\sqrt{3}-\frac{x}{6}\mp\frac{\sqrt{3}}{36}\frac{x^2}{1.2}-\frac{1}{27}\frac{x^3}{1.2.3}+\dots;$$

and the three series give three different values of y, which are the three roots of the given cubic equation in terms of x.

An equation also may often be put in other algebraic forms, and then expanded by Maclaurin's Theorem in a different form of developement, and sometimes in a series of descending powers of x. As, for instance, consider the last example,

$$y^3-3y+x=0.$$

Divide by x, and we have,

$$\frac{y^3}{x}-\frac{3y}{x^{\frac{1}{3}}}\frac{1}{x^{\frac{2}{3}}}+1=0;$$

for $\frac{y}{x^{\frac{1}{3}}}$ write y', and for $\frac{1}{x^{\frac{2}{3}}}$ write x'; whereby we have,

$$y'^3-3y'x'+1=0.$$

Expanding which, as in the last two examples, and taking only the possible cube root, we have

$$y'=-1-\frac{x'}{1}+\frac{x'^3}{3}-\dots\dots$$

and replacing y' and x' by their values, and multiplying through by $x^{\frac{1}{3}}$, we have

$$y=-x^{\frac{1}{3}}-\frac{1}{x^{\frac{1}{3}}}+\frac{1}{3x^{\frac{5}{3}}}+\dots\dots$$

87.] The above theory is also applicable to the expansion of a function which is of great importance in the calculation of many series; and gives some numerical coefficients known by the name of *Bernoulli's Numbers*.

It is required to expand $y = \frac{x}{e^x - 1} = f(x)$.

We may observe that $f(-x) = \frac{-x}{e^{-x} - 1} = \frac{xe^x}{e^x - 1}$;

$$\therefore \quad f(x) - f(-x) = -x;$$

and as every odd power of x, which entered into the expansion of $f(x)$, would also enter into that of $f(x) - f(-x)$, it follows that no odd power of x enters into $f(x)$ except the first.

Now from above $\qquad y e^x = y + x; \qquad (125)$

$$\therefore \quad e^x \left(\frac{dy}{dx} + y\right) = \frac{dy}{dx} + 1,$$

$$e^x . \left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y\right) = \frac{d^2y}{dx^2},$$

$$e^x . \left(\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y\right) = \frac{d^3y}{dx^3},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$e^x . \left(\frac{d^ny}{dx^n} + n\frac{d^{n-1}y}{dx^{n-1}} + \ldots + n\frac{dy}{dx} + y\right) = \frac{d^ny}{dx^n};$$

the last term being the nth derived-function of (125), and being calculated by the method of Art. 55. Let $x = 0$ in these several equations, and we have

$f(0) = f(0)$, which is an identity;

$f(0) = 1, \qquad \therefore \quad f(0) = 1;$

$2f'(0) + f(0) = 0, \qquad f'(0) = -\frac{1}{2};$

$3f''(0) + 3f'(0) + f(0) = 0, \qquad f''(0) = \frac{1}{6};$

$4f'''(0) + 6f''(0) + 4f'(0) + f(0) = 0, \qquad f'''(0) = 0;$

$5f^{\text{v}}(0) + 10f'''(0) + 10f''(0) + 5f'(0) + f(0) = 0, \; f^{\text{v}}(0) = -\frac{1}{30},$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$nf^{n-1}(0) + \frac{n(n-1)}{1.2} f^{n-2}(0) + \ldots\ldots + nf'(0) + f(0) = 0. \quad (126)$$

The last is the general equation, by means of which $f^{n-1}(0)$ may be determined in terms of the preceding coefficients. But the labour of calculation from this expression may be diminished by observing that, as $f(x)$ involves no odd powers of x except the first,

$$f'''(0) = \ldots\ldots = f^{2r-1}(0) = 0.$$

Hence in (126), if n is of the form $2r+1$, we have

$$(2r+1)f^{2r}(0) + \frac{(2r+1)\,2r\,(2r-1)}{1.2.3}f^{2r-2}(0) + \ldots$$
$$\ldots + (2r+1)f'(0) + f(0) = 0, \quad (127)$$

by means of which any coefficient may be found; and we have

$$\frac{x}{e^x-1} = 1 - \frac{1}{2}\frac{x}{1} + \frac{1}{6}\frac{x^2}{1.2} - \frac{1}{30}\frac{x^4}{1.2.3.4} + \frac{1}{42}\frac{x^6}{1.2.3.4.5.6} + \ldots (128)$$

The values of $f(0)$, $f'(0)$, $f''(0)$, ... are commonly called the numbers of Bernoulli; and though they do not *explicitly* follow any palpably regular law, yet they are implicitly connected with each other by the formula (127). It is convenient to represent them by distinctive symbols; we will therefore substitute as follows:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \ldots\ldots$$

so that we have

$$\frac{x}{e^x-1} = B_0 + B_1\frac{x}{1} + B_2\frac{x^2}{1.2} + B_4\frac{x^4}{1.2.3.4} + \ldots \quad (129)$$

88.] Of this theorem, it is expedient to give some examples.

Ex. 1. It is required to develope $\frac{1}{e^x+1}$ in terms of Bernoulli's numbers.

$$\because \quad \frac{x}{e^x-1} - \frac{x}{e^x+1} = \frac{2x}{e^{2x}-1};$$

$$\therefore \quad \frac{x}{e^x+1} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1}$$
$$= B_0 + B_1 x + B_2\frac{x^2}{1.2} + \ldots - \left\{B_0 + B_1 2x + B_2\frac{4x^2}{1.2} + \ldots\right\};$$

$$\therefore \quad \frac{1}{e^x+1} = -B_1 - (2^2-1)B_2\frac{x^2}{1.2} - (2^4-1)B_4\frac{x^3}{1.2.3.4}$$
$$- (2^6-1)B_6\frac{x^5}{1.2.3.4.5.6}\ldots \quad (130)$$

the law of which is sufficiently obvious.

Ex. 2. It is required to expand $\tan x$ by means of Bernoulli's numbers.

By equation (33), Art. 61,

$$\tan x = \frac{1}{\sqrt{-1}} \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}},$$

$$= \frac{1}{\sqrt{-1}} \frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1},$$

$$= \frac{1}{\sqrt{-1}} \left\{ 1 - \frac{2}{e^{2x\sqrt{-1}} + 1} \right\},$$

the second term of which is of the form expanded in equation (130); so that

$$\tan x = \frac{1}{\sqrt{-1}} \left\{ 1 + 2\mathrm{B}_1 + 2\sqrt{-1}\,(2^2-1)\,\mathrm{B}_2 \frac{2x}{1.2} \right.$$

$$\left. - 2\sqrt{-1}\,(2^4-1)\,\mathrm{B}_4 \frac{2^3 x^3}{1.2.3.4} \cdots \right\}$$

$$= 2\mathrm{B}_2(2^2-1)\frac{2x}{1.2} - 2\mathrm{B}_4(2^4-1)\frac{2^3x^3}{1.2.3.4}$$

$$+ 2\mathrm{B}_6(2^6-1)\frac{2^5x^5}{1.2\ldots5.6} - \ldots \quad (131)$$

which, being written at length, becomes

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots;$$

and the law of the series is sufficiently obvious in equation (131). Also, differentiating the above, we can find the expansion of $(\tan x)^2$.

Ex. 3. By a similar process of development, it may be shewn that

$$\cot x = \frac{1}{x} - \mathrm{B}_2 \frac{2^2 x}{1.2} + \mathrm{B}_4 \frac{2^4 x^3}{1.2.3.4} - \ldots; \quad (132)$$

and hence by differentiation may $(\operatorname{cosec} x)^2$ and $(\cot x)^2$ be found.

Ex. 4. Also since $\operatorname{cosec} x = \cot x + \tan\frac{x}{2}$, from (131) and (132), we have

$$\operatorname{cosec} x = \frac{1}{x} + 2\mathrm{B}_2(2^1-1)\frac{x}{1.2} - 2\mathrm{B}_4(2^3-1)\frac{x^3}{1.2.3.4}$$

$$+ 2\mathrm{B}^6(2^5-1)\frac{x^5}{1.2\ldots5.6} - \ldots \quad (133)$$

Substitute for this investigation that in Chauvenet's Trigonometry page 121 Art. 215.

89.] The sums of the powers of the reciprocals of the natural numbers may also be expressed as follows in terms of Bernoulli's numbers.

Since $\sin z = 0$, when $z = 0, = \pm \pi, = \pm 2\pi, = \pm 3\pi, = \dots;$ therefore $\sin z$ must be composed of factors of the form

$$z(z-\pi)(z+\pi)(z-2\pi)(z+2\pi)\dots;$$

and therefore a relation of the following form must exist;

$$\sin z = kz\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{2^2\pi^2}\right)\left(1-\frac{z^2}{3^2\pi^2}\right)\dots, \tag{134}$$

where k is an undetermined constant. But since by Lemma II, Art. 22, $\sin z = z$, when z is an infinitesimal, therefore $k=1$; and

$$\sin z = z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{2^2\pi^2}\right)\left(1-\frac{z^2}{3^2\pi^2}\right)\dots\dots$$

For z let us substitute πx; then

$$\sin \pi x = \pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\dots\dots \tag{135}$$

Of this equation let us take the logarithmic differential, and we have

$$\pi \cot \pi x = \frac{1}{x} - \frac{2x}{1^2-x^2} - \frac{2x}{2^2-x^2} - \frac{2x}{3^2-x^2} - \dots \tag{136}$$

and developing the several terms in the right-hand member, we have

$$\begin{aligned}
\pi \cot \pi x = \frac{1}{x} &- 2x\left\{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right\} \\
&- 2x^3\left\{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right\} \\
&- 2x^5\left\{\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\dots\right\} \\
&\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&- 2x^{2n-1}\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\right\}.
\end{aligned} \tag{137}$$

In (132) let x be replaced by πx; so that

$$\pi \cot \pi x = \frac{1}{x} - B_2\frac{2^2\pi^2 x}{1.2} + B_4\frac{2^4\pi^4 x^3}{1.2.3.4} - B_6\frac{2^6\pi^6 x^5}{1.2\dots5.6} + \dots \tag{138}$$

and equating coefficients of the same powers of x in (137) and (138),

$$\left.\begin{aligned}
\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots &= \frac{1}{2}\cdot\frac{(2\pi)^2}{1.2}\,\mathrm{B}_2\\
\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots &= -\frac{1}{2}\cdot\frac{(2\pi)^4}{1.2.3.4}\,\mathrm{B}_4\\
\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\dots &= \frac{1}{2}\cdot\frac{(2\pi)^6}{1.2\dots5.6}\,\mathrm{B}_6\\
\dots\dots\dots\dots&\dots\dots\dots\dots\\
\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots &= \pm\frac{1}{2}\cdot\frac{(2\pi)^{2n}}{1.2\dots2n}\,\mathrm{B}_{2n}
\end{aligned}\right\}, \tag{139}$$

the + or − sign being taken according as $2n$ is par impar or par par.

Another development is consequent upon the preceding investigations. In (136) for x substitute $x\sqrt{-1}$, then

$$\pi\cot\pi x\sqrt{-1} = \frac{1}{x\sqrt{-1}} - 2x\sqrt{-1}\left\{\frac{1}{1^2+x^2}+\frac{1}{2^2+x^2}+\dots\right\};$$

but $\cot\pi x\sqrt{-1} = \sqrt{-1}\,\dfrac{e^{-\pi x}+e^{\pi x}}{e^{-\pi x}-e^{\pi x}}$;

$$\therefore\quad \frac{1}{1^2+x^2}+\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}+\dots = \frac{\pi}{2x}\cdot\frac{e^{\pi x}+e^{-\pi x}}{e^{\pi x}-e^{-\pi x}} - \frac{1}{2x^2}. \tag{140}$$

Other theorems may be obtained by a similar process, if we take as the fundamental theorem the equation for the cosine which is analogous to that for the sine given in (135).

90.] It is unnecessary to enter at any length on the *general* subject of implicit functions of more than two variables, as the principles above explained and illustrated are applicable to all such cases. But as particular forms occur, and particular examples have to be solved in the sequel of our work, it is convenient to consider them at this point of the Treatise where they naturally occur; and I proceed to consider Lagrange's Theorem for the development of an implicit function of three variables, of the form $y = z + x\phi(y)$, in ascending powers of x.

Given that $y = z + x\phi(y)$, in which equation y is an implicit function of two variables z and x, which are supposed to have no other relation to each other besides that given by this equation, so that they may vary independently of each other; it is required to determine $f(y)$, another function of y, in ascending powers of x.

Let $u = f(y)$, and therefore u is a function of x; whence, by Maclaurin's Series,

$$u = [u]_0 + \left[\frac{du}{dx}\right]_0 \frac{x}{1} + \left[\frac{d^2u}{dx^2}\right]_0 \frac{x^2}{1.2} + \left[\frac{d^3u}{dx^3}\right]_0 \frac{x^3}{1.2.3} + \ldots; \quad (141)$$

using the notation of Art. 86 to indicate that particular values of the coefficients are to be taken, viz. when $x = 0$; that is, if $u = \mathrm{F}(x)$, $[u]_0 = \mathrm{F}(0)$, $\left[\frac{du}{dx}\right]_0 = \mathrm{F}'(0)$, and so on. Hence we have the following data:

$$u = f(y), \qquad y = z + x\phi(y); \quad (142)$$

therefore when $x = 0$, $y = z$, and $[u]_0 = f(z)$;

now our first object is to determine the values of the quantities within the square brackets.

Calculating the partial derived-functions of (142), by considering y to vary in consequence of changes separately of x and of z, that is, calculating $\left(\frac{dy}{dx}\right)$ and $\left(\frac{dy}{dz}\right)$, we have

$$\left(\frac{dy}{dx}\right) = \phi(y) + x\phi'(y)\left(\frac{dy}{dx}\right), \qquad \therefore \quad \left(\frac{dy}{dx}\right) = \frac{\phi(y)}{1 - x\phi'(y)};$$

$$\left(\frac{dy}{dz}\right) = 1 + x\phi'(y)\left(\frac{dy}{dz}\right), \qquad \therefore \quad \left(\frac{dy}{dz}\right) = \frac{1}{1 - x\phi'(y)};$$

$$\therefore \quad \left(\frac{dy}{dx}\right) = \phi(y)\left(\frac{dy}{dz}\right). \quad (143)$$

$$\text{But} \qquad \left(\frac{du}{dx}\right) = \frac{du}{dy}\frac{dy}{dx};$$

whence, by reason of (143),

$$\frac{du}{dx} = \frac{du}{dy}\phi(y)\left(\frac{dy}{dz}\right).$$

Let $x = 0$, then $y = z$; $\therefore$ $dy = dz$, and $u = [u]_0 = f(z)$;

$$\therefore \quad \left[\frac{du}{dx}\right]_0 = \frac{d.f(z)}{dz}\phi(z). \quad (144)$$

$$\text{Again, as} \qquad \frac{du}{dx} = \frac{du}{dy}\phi(y)\frac{dy}{dz},$$

and as $\frac{du}{dy}$ and $\phi(y)$ are explicitly functions of y only, although they are also functions of x and z implicitly by virtue of equa-

For a more concise proof
see Chauvenet's Astronomy
Vol. I. p. 145.

tion (142), and as $\frac{dy}{dz}$ is explicitly a function of x and z, it is convenient for the purposes of differentiation to consider $\frac{du}{dx}$ the product of *two* functions, viz. of $\frac{du}{dy}\phi(y)$ and of $\frac{dy}{dz}$; whence, differentiating, we have

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left\{\frac{du}{dy}\phi(y)\frac{dy}{dz}\right\}$$

$$= \frac{d\left\{\frac{du}{dy}\phi(y)\right\}}{dy}\frac{dy}{dx}\cdot\frac{dy}{dz} + \frac{du}{dy}\phi(y)\frac{d^2y}{dx\,dz}$$

$$= \frac{d\left\{\frac{du}{dy}\phi(y)\right\}}{dy}\frac{dy}{dz}\cdot\frac{dy}{dx} + \frac{du}{dy}\phi(y)\frac{d^2y}{dz\,dx}$$

$$= \frac{d}{dz}\left\{\frac{du}{dy}\phi(y)\frac{dy}{dx}\right\};$$

and substituting for $\frac{dy}{dx}$ from equation (143),

$$\frac{d^2u}{dx^2} = \frac{d}{dz}\left\{\frac{du}{dy}\{\phi(y)\}^2\frac{dy}{dz}\right\};$$

let $x = 0$, in which case, as before, $y = z$, $dy = dz$, and $u = f(z)$;

$$\therefore \quad \left[\frac{d^2u}{dx^2}\right]_0 = \frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}.$$

Again, considering $\frac{d^2u}{dx^2}$ to involve a product of two functions, viz. $\frac{du}{dy}\{\phi(y)\}^2$ and $\frac{dy}{dz}$, the former of which is explicitly a function of y only, and the latter is an explicit function of both x and z, and differentiating and substituting from (143),

$$\frac{d^3u}{dx^3} = \frac{d^2}{dz^2}\left\{\frac{du}{dy}\{\phi(y)\}^3\frac{dy}{dz}\right\};$$

$$\therefore \quad \left[\frac{d^3u}{dx^3}\right]_0 = \frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}.$$

Let us assume that the form is true for $\frac{d^{n-1}u}{dx^{n-1}}$; viz.

$$\frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-2}}{dz^{n-2}}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dz}\right\};$$

$$\therefore\quad \frac{d^n u}{dx^n} = \frac{d}{dx}\frac{d^{n-2}}{dz^{n-2}}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dz}\right\};$$

and since the order of differentiation may be reversed by virtue of Art. 79,

$$\frac{d^n u}{dx^n} = \frac{d^{n-2}}{dz^{n-2}}\frac{d}{dx}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dz}\right\},$$

$$= \frac{d^{n-2}}{dz^{n-2}}\left\{\frac{d\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\right\}}{dy}\cdot\frac{dy}{dz}\cdot\frac{dy}{dx} + \frac{du}{dy}\{\phi(y)\}^{n-1}\frac{d^2y}{dz\,dx}\right\},$$

$$= \frac{d^{n-2}}{dz^{n-2}}\left\{\frac{d\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\right\}}{dy}\,\frac{dy}{dx}\cdot\frac{dy}{dz} + \frac{du}{dy}\{\phi(y)\}^{n-1}\frac{d^2y}{dx\,dz}\right\},$$

$$= \frac{d^{n-2}}{dz^{n-2}}\frac{d}{dz}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dx}\right\},$$

$$= \frac{d^{n-1}}{dz^{n-1}}\left\{\frac{du}{dy}\{\phi(y)\}^{n}\frac{dy}{dz}\right\},\text{ by virtue of equation (143);}$$

$$\therefore\quad \left[\frac{d^n u}{dx^n}\right]_0 = \frac{d^{n-1}}{dz^{n-1}}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^n\right\}. \qquad (145)$$

If therefore the formulæ are true for $n-1$, they are true for n; they are true when $n=3$, therefore they are true when $n=4$, and therefore are true for all positive integral values of n, which are the only cases in which it is necessary for us to find them. Substituting then, in equation (141), the values above determined, we have

$$f(y) = f(z) + \frac{d.f(z)}{dz}\phi(z)\frac{x}{1} + \frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}\frac{x^2}{1.2}$$
$$+ \frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}\frac{x^3}{1.2.3} + \cdots$$
$$\cdots + \frac{d^{n-1}}{dz^{n-1}}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^n\right\}\frac{x^n}{1.2.3\ldots n} + \cdots \qquad (146)$$

If, having given $y = z + x\phi(y)$, the problem is to determine y, then $f(y) = y$, and $f(z) = z$; and the above formula becomes

$$y = z + \phi(z)\frac{x}{1} + \frac{d}{dz}\{\phi(z)\}^2\frac{x^2}{1.2} + \frac{d^2}{dz^2}\{\phi(z)\}^3\frac{x^3}{1.2.3} + \cdots \qquad (147)$$

In applying the above theorems to particular examples, it is most convenient first to substitute the specific forms of the functions, and subsequently to replace the variables by their specific values.

91.] Ex. 1. Given $a-by+cy^2=0$; to find y.

On comparing the given equation $y=\frac{a}{b}+\frac{c}{b}y^2$ with the typical form $y=z+x\phi(y)$, we have

$$\left.\begin{aligned} f(y) &= y \\ \therefore\ f(z) &= z \end{aligned}\right\}, \qquad \left.\begin{aligned} \phi(y) &= y^2 \\ \phi(z) &= z^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} z &= \frac{a}{b} \\ x &= \frac{c}{b} \end{aligned}\right\};$$

whence we have from

$$f(y) = f(z) + \frac{d.f(z)}{dz}\phi(z)\frac{x}{1} + \frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}\frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}\frac{x^3}{1.2.3} + \dots$$

$$y = z + z^2\frac{x}{1} + \frac{d}{dz}\{z^4\}\frac{x^2}{1.2} + \frac{d^2}{dz^2}\{z^6\}\frac{x^3}{1.2.3} + \dots$$

$$= \text{- - -} + \quad 4z^3\frac{x^2}{1.2} + \quad 6.5z^4\frac{x^3}{1.2.3} + \dots$$

$$= \frac{a}{b} + \frac{a^2}{b^2}\frac{c}{b} + 4\frac{a^3}{b^3}\frac{c^2}{b^2}\frac{1}{1.2} + 6.5\frac{a^4}{b^4}\frac{c^3}{b^3}\frac{1}{1.2.3} + \dots$$

$$= \frac{a}{b}\left\{1 + \frac{ac}{b^2} + \frac{4}{1.2}\frac{a^2c^2}{b^4} + \frac{6.5}{1.2.3}\frac{a^3c^3}{b^6} + \dots\right\};$$

a series which is identical with that arising from the development of $\frac{b}{2c}-\frac{(b^2-4ac)^{\frac{1}{2}}}{2c}$, which is the least of the two roots of the given equation.

Ex. 2. Given $y^3-ay+b=0$; to find y^n.

On comparing the given equation $y=\frac{b}{a}+\frac{1}{a}y^3$ with the typical form, we have

$$\left.\begin{aligned} f(y) &= y^n \\ \therefore\ f(z) &= z^n \end{aligned}\right\}, \qquad \left.\begin{aligned} \phi(y) &= y^3 \\ \phi(z) &= z^3 \end{aligned}\right\}, \qquad \left.\begin{aligned} z &= \frac{b}{a} \\ x &= \frac{1}{a} \end{aligned}\right\};$$

$$\therefore \quad f(y) = f(z) + \frac{d.f(z)}{dz}\phi(z)\frac{x}{1} + \frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}\frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}\frac{x^3}{1.2.3} + \dots$$

$$y^n = z^n + nz^{n-1}z^3\frac{x}{1} + \frac{d}{dz}\{nz^{n-1}z^6\}\frac{x^2}{1.2} + \frac{d^2}{dz^2}\{nz^{n-1}z^9\}\frac{x^3}{1.2.3} + \dots$$

$$= z^n + nz^{n+2}\frac{x}{1} + n(n+5)z^{n+4}\frac{x^2}{1.2} + n(n+8)(n+7)z^{n+6}\frac{x^3}{1.2.3} + \dots$$

$$y^n = \frac{b^n}{a^n}\left\{1 + n\frac{b^2}{a^2}\frac{1}{a} + \frac{n(n+5)}{1.2}\frac{b^4}{a^4}\frac{1}{a^2} + \frac{n(n+8)(n+7)}{1.2.3}\frac{b^6}{a^6}\frac{1}{a^3} + \dots\right\}.$$

Ex. 3. Given $y = a + be^y$; to find $\log_e y$.

$$\left.\begin{array}{l} f(y) = \log_e y \\ f(z) = \log_e z \end{array}\right\}, \quad \left.\begin{array}{l} \phi(y) = e^y \\ \phi(z) = e^z \end{array}\right\}, \quad \left.\begin{array}{l} z = a \\ x = b \end{array}\right\};$$

$$f(y) = f(z) + \frac{d.f(z)}{dz}\phi(z)\frac{x}{1} + \frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}\frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}\frac{x^3}{1.2.3} + \dots$$

$$\log_e y = \log_e z + \frac{e^z}{z}\frac{x}{1} + \frac{d}{dz}\left\{\frac{e^{2z}}{z}\right\}\frac{x^2}{1.2} + \frac{d^2}{dz^2}\left\{\frac{e^{3z}}{z}\right\}\frac{x^3}{1.2.3} + \dots$$

$$= - - - + \frac{2z-1}{z^2}e^{2z}\frac{x^2}{1.2} + \frac{9z^2-6z+2}{z^3}e^{3z}\frac{x^3}{1.2.3} + \dots$$

$$= \log_e a + \frac{e^a}{a}\frac{b}{1} + \frac{2a-1}{a^2}e^{2a}\frac{b^2}{1.2} + \frac{9a^2-6a+2}{a^3}e^{3a}\frac{b^3}{1.2.3} + \dots$$

Ex. 4. Given $y = a + e\sin y$; to find $\cos y$ and $\sin 2y$.

(*a*) To find $\cos y$.

$$\left.\begin{array}{l} f(y) = \cos y \\ f(z) = \cos z \end{array}\right\}, \quad \left.\begin{array}{l} \phi(y) = \sin y \\ \phi(z) = \sin z \end{array}\right\}, \quad \left.\begin{array}{l} x = e \\ z = a \end{array}\right\};$$

$$\cos y = \cos z - (\sin z)^2\frac{x}{1} - \frac{d}{dz}\{(\sin z)^3\}\frac{x^2}{1.2} - \frac{d^2}{dz^2}\{(\sin z)^4\}\frac{x^3}{1.2.3} + \dots$$

$$= - - - 3(\sin z)^2\cos z\frac{x^2}{1.2} + 4\{4(\sin z)^4 - 3(\sin z)^2\}\frac{x^3}{1.2.3} + \dots$$

$$\cos y = \cos a - (\sin a)^2\frac{e}{1} - 3(\sin a)^2\cos a\frac{e^2}{1.2}$$

$$+ 4(\sin a)^2\{4(\sin a)^2 - 3\}\frac{e^3}{1.2.3} + \dots$$

(β) To find $\sin 2y$.

$$\left.\begin{array}{l} f(y) = \sin 2y \\ f(z) = \sin 2z \end{array}\right\}, \qquad \left.\begin{array}{l} \phi(y) = \sin y \\ \phi(z) = \sin z \end{array}\right\}, \qquad \left.\begin{array}{l} x = e \\ z = a \end{array}\right\};$$

$$\begin{aligned} \sin 2y &= \sin 2z + 2\cos 2z \sin z \frac{x}{1} + \frac{d}{dz}\{2\cos 2z\,(\sin z)^2\}\frac{x^2}{1.2} + \ldots \\ &= \text{- - - - - - - - -} + 4\cos 3z \sin z \frac{x^2}{1.2} + \ldots \\ &= \sin 2a + 2\cos 2a \sin a \frac{e}{1} + 4\cos 3a \sin a \frac{e^2}{1.2} + \ldots \end{aligned}$$

92.] By the preceding theorem of Lagrange any function of x, say $f(x)$, may be expanded, when certain conditions are fulfilled, in ascending powers of any other function $\phi(x)$; so that, as by Maclaurin's Theorem a function is expanded in ascending powers of x, by this Theorem it is expanded in ascending powers of another function of x.

Let the form of the required series be

$$f(x) = \text{A}_0 + \text{A}_1 \frac{\phi(x)}{1} + \text{A}_2 \frac{\{\phi(x)\}^2}{1.2} + \text{A}_3 \frac{\{\phi(x)\}^3}{1.2.3} + \ldots; \qquad (148)$$

our object is to determine the coefficients $\text{A}_0, \text{A}_1, \text{A}_2, \ldots$ which are independent of x.

Let $x-a$ be a factor of $\phi(x)$; that is, let a be a value of x which renders $\phi(x) = 0$; and let $\frac{1}{\psi(x)}$ be the product of all the other factors; so that

$$\phi(x) = \frac{x-a}{\psi(x)}, \qquad (149)$$

$$\therefore \quad x = a + \phi(x) \times \psi(x). \qquad (150)$$

It will be convenient to replace $\phi(x)$ by t; and thus (150) becomes

$$x = a + t\,\psi(x).$$

Now this form is clearly the same as (142); and when $t = 0$, $x = a$; and our object is to expand $f(x)$ in ascending powers of t; therefore replacing y by x and z by a in (146), we have

$$\begin{aligned} f(x) = [f(x)] + \left[\frac{d.f(x)}{dx}\psi(x)\right]\frac{t}{1} + \left[\frac{d}{dx}\left\{\frac{d.f(x)}{dx}\{\psi(x)\}^2\right\}\right]\frac{t^2}{1.2} \\ + \left[\frac{d^2}{dx^2}\left\{\frac{d.f(x)}{dx}\{\psi(x)^3\}\right\}\right]\frac{t^3}{1.2.3} + \ldots \end{aligned}$$

wherein the square brackets indicate that particular values of the quantities within them are to be taken, viz. when $x = a$. And replacing t by $\phi(x)$ and $\psi(x)$ by its value given in (149), we have

$$f(x) = [f(x)] + \left[\frac{d.f(x)}{dx}\frac{x-a}{\phi(x)}\right]\frac{\phi(x)}{1} + \left[\frac{d}{dx}\left\{\frac{d.f(x)}{dx}\left(\frac{x-a}{\phi(x)}\right)^2\right\}\right]\frac{\{\phi(x)\}^2}{1.2}$$
$$+ \left[\frac{d^2}{dx^2}\left\{\frac{d.f(x)}{dx}\left(\frac{x-a}{\phi(x)}\right)^3\right\}\right]\frac{\{\phi(x)\}^3}{1.2.3} + \dots \quad (151)$$

where the square brackets indicate particular values of the quantities within them, viz. those which correspond to $x = a$.

If $\phi(x) = x - a = h$, (151) becomes Taylor's series.

As an example of (151), let $f(x) = e^x$, $\phi(x) = (x-1)(x-2)$, and let $a = 2$; then

$$e^x = e^2 + e^2(x-1)(x-2) - e^2\frac{(x-1)^2(x-2)^2}{1.2} + \dots$$

which gives the expansion of e^x in ascending powers of $(x-1)(x-2)$. Also if $a = 1$, we have

$$e^x = e - e(x-1)(x-2) + 3e\frac{(x-1)^2(x-2)^2}{1.2} + \dots,$$

which gives another expansion of e^x in powers of $(x-1)(x-2)$.

93.] A still more general form of expansion than that of Lagrange was discovered by Laplace, and is known by the name of Laplace's Theorem.

Given $y = \mathrm{F}\{z + x\,\phi(y)\}$; it is required to find $f(y)$.

Using the same notation of Maclaurin's Theorem as heretofore, we have

$$u = [u]_0 + \left[\frac{du}{dx}\right]_0\frac{x}{1} + \left[\frac{d^2u}{dx^2}\right]_0\frac{x^2}{1.2} + \left[\frac{d^3u}{dx^3}\right]_0\frac{x^3}{1.2.3} + \dots$$

Let $u = f(y)$; $\therefore$ $[u]_0 = f\{\mathrm{F}(z)\}$.

And by reason of Ex. 1, Art. 53,

$$\left(\frac{dy}{dx}\right) = \phi(y)\left(\frac{dy}{dz}\right);$$

$$\therefore\ \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}\phi(y)\frac{dy}{dz}.$$

Let $x = 0$, then $y = \mathrm{F}(z)$, $u = f\{\mathrm{F}(z)\}$;

$$\therefore\ \left[\frac{du}{dx}\right]_0 = \frac{d.f\{\mathrm{F}(z)\}}{dz}\phi\{\mathrm{F}(z)\}.$$

And by a process similar to that employed in the proof of Lagrange's Theorem, it may be shewn that

$$\left[\frac{d^2u}{dx^2}\right]_0 = \frac{d}{dz}\left\{\frac{d.f\{\mathrm{F}(z)\}}{dz}(\phi\{\mathrm{F}(z)\})^2\right\};$$

and so on for other and for the nth terms; whence

$$f(y) = f\{\mathrm{F}(z)\} + \frac{d.f\{\mathrm{F}(z)\}}{dz}\phi\{\mathrm{F}(z)\}\frac{x}{1} + \frac{d}{dz}\left\{\frac{d.f\{\mathrm{F}(z)\}}{dz}(\phi\{\mathrm{F}(z)\})^2\right\}\frac{x^2}{1.2} + \ldots \quad (152)$$

As an example of this Theorem, let it be required to find e^y, if $y = \log\{z + x\sin y\}$;

$$f(y) = e^y, \qquad \mathrm{F}(z) = \log z, \qquad \phi(y) = \sin y,$$
$$f(z) = e^z; \qquad \therefore\ f.\mathrm{F}(z) = z; \qquad \therefore\ \phi\{\mathrm{F}(z)\} = \sin\log z;$$

whence, substituting in the formula (152), we have

$$e^y = z + \sin\log z\frac{x}{1} + \frac{d}{dz}\{(\sin\log z)^2\}\frac{x^2}{1.2} + \frac{d^2}{dz^2}\{(\sin\log z)^3\}\frac{x^3}{1.2.3} + \ldots$$
$$= - - - - - - + \frac{2\sin\log z\cos\log z}{z}\frac{x^2}{1.2} + \ldots$$
$$= z + \sin\log z\frac{x}{1} + \frac{\sin(2\log z)}{z}\frac{x^2}{1.2} + \ldots$$
$$= z + \sin\log z\frac{x}{1} + \frac{\sin(\log z^2)}{z}\frac{x^2}{1.2} + \ldots$$

Laplace's Theorem, it will be observed, becomes Lagrange's, when $\mathrm{F} = 1$; and Taylor's Series is also a particular case of Laplace's; for as

$$y = \mathrm{F}\{z + x\phi(y)\}; \qquad \text{let } \phi(y) = a, \qquad \text{and } f(y) = y;$$

then $y = \mathrm{F}(z+ax) = \mathrm{F}(z) + \dfrac{d.\mathrm{F}(z)}{dz}\dfrac{ax}{1} + \dfrac{d^2.\mathrm{F}(z)}{dz^2}\dfrac{a^2x^2}{1.2} + \ldots;$

and writing h for ax, we have

$$\mathrm{F}(z+h) = \mathrm{F}(z) + \frac{d.\mathrm{F}(z)}{dz}\frac{h}{1} + \frac{d^2.\mathrm{F}(z)}{dz^2}\frac{h^2}{1.2} + \frac{d^3.\mathrm{F}(z)}{dz^3}\frac{h^3}{1.2.3} + \ldots$$

94.] Another form of function which it is often necessary to expand by Maclaurin's Theorem is that in which a subsidiary variable z is introduced; and where we have two equations of the form

$$y = f(z), \qquad \text{and } z = \phi(x); \qquad (153)$$

and wherein it is required to expand y in ascending powers of x.

Using Lagrange's notation of derived-functions, we have

$$\frac{dy}{dz} = f'(z), \qquad \frac{dz}{dx} = \phi'(x); \tag{154}$$

$$\therefore \quad \frac{dy}{dx} = f'(z)\,\phi'(x). \tag{155}$$

$$\frac{d^2y}{dx^2} = f''(z)\,\{\phi'(x)\}^2 + f'(z)\,\phi''(x), \tag{156}$$

$$\frac{d^3y}{dx^3} = f'''(z)\,\{\phi'(x)\}^3 + 3f''(z)\,\phi'(x)\,\phi''(x) + f'(z)\,\phi'''(x), \tag{157}$$

$$\frac{d^4y}{dx^4} = f^{\text{iv}}(z)\,\{\phi'(x)\}^4 + 6f'''(z)\,\{\phi'(x)\}^2\,\phi''(x) + 3f''(z)\,\{\phi''(x)\}^2 \\ + 4f''(z)\,\phi'(x)\,\phi'''(x) + f'(z)\,\phi^{\text{iv}}(x); \tag{158}$$

and so on. Now substituting these quantities in the several terms of Maclaurin's Series, (13), Art. 58, and putting $x = 0$, and introducing the corresponding value of z, we shall have the required series. To take a simple case, let

$$y = e^z, \text{ where } z = \sin x;$$

$$\frac{dy}{dx} = e^z \cdot \frac{dz}{dx}, \qquad \frac{dz}{dx} = \cos x;$$

$$\therefore \quad \frac{dy}{dx} = e^z \cos x,$$

$$\frac{d^2y}{dx^2} = e^z (\cos x)^2 - e^z \sin x,$$

$$\frac{d^3y}{dx^3} = e^z (\cos x)^3 - 3e^z \cos x \sin x - e^z \cos x,$$

and so on; but when $x = 0$, $z = 0$; therefore using the square brackets in the same signification as heretofore,

$$[y]_0 = 1, \quad \left[\frac{dy}{dx}\right]_0 = 1, \quad \left[\frac{d^2y}{dx^2}\right]_0 = 1, \quad \left[\frac{d^3y}{dx^3}\right]_0 = 0, \quad \left[\frac{d^4y}{dx^4}\right]_0 = -3, \ldots$$

$$\therefore \quad e^{\sin x} = 1 + \frac{x}{1} + \frac{x^2}{1.2} - \frac{3x^4}{1.2.3.4} + \cdots$$

95.] But in the case wherein z is a series of terms in ascending powers of x, the preceding expansion takes a particular form which deserves much attention; and gives rise to a process which has been called Derivation*; and on which Arbogast has constructed his Calcul des Dérivations. It will be convenient to take the exponential series for the base-form of the series, and I shall accordingly assume

---

* This process, though called by the same name, is essentially different from that explained in Art. 18. The title of Arbogast's work is, Du Calcul des Dérivations; it was published at Strasbourg, An VIII. (1800.)

$$z = a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + a_3\frac{x^3}{1.2.3} + \dots; \qquad (159)$$

so that the problem is, the expansion in ascending powers of x of $f\left(a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + a_3\frac{x^3}{1.2.3} + \dots\right)$.

Let $y = f(z)$, (160)

and $z = a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + a_3\frac{x^3}{1.2.3} + \dots$ (161)

$$\therefore \quad \frac{dz}{dx} = a_1 + a_2\frac{x}{1} + a_3\frac{x^2}{1.2} + \dots$$

$$\frac{d^2z}{dx^2} = a_2 + a_3\frac{x}{1} + a_4\frac{x^2}{1.2} + \dots$$

$$\frac{d^3z}{dx^3} = a_3 + a_4\frac{x}{1} + \dots$$

. . . .

In all these equations let $x = 0$; then $z = a_0$, $\frac{dz}{dx} = a_1$, $\frac{d^2z}{dx^2} = a_2$, $\frac{d^3z}{dx^3} = a_3, \dots$; and $y = f(a_0)$; in whatever manner therefore the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$ given in (155), (156), (157), … are composed of $f(z), f'(z), f''(z), \dots$ combined with $\frac{dz}{dx}, \frac{d^2z}{dx^2}, \frac{d^3z}{dx^3}, \dots$; in the same manner will these values, when $x = 0$, be composed of $f(a_0), f'(a_0), f''(a_0), \dots$ combined with $a_1, a_2, a_3, \dots$. From this peculiarity we may deduce the following process; $\left[\frac{dy}{dx}\right]_0$ is the differential of $f(a_0)$ on the supposition that $da_0 = a_1$; $\left[\frac{d^2y}{dx^2}\right]_0$ is the second differential of $f(a_0)$ on the supposition that $da_0 = a_1$, and $da_1 = a_2$; and $\left[\frac{d^ny}{dx^n}\right]_0$ is the nth differential of $f(a_0)$ on the supposition that $da_0 = a_1$, $da_1 = a_2$, $da_2 = a_3$, … $da_{n-1} = a_n$. In this power of substituting new constants for the differentials of other constants does Arbogast's method of derivation consist.

If therefore we replace the successive coefficients of the powers of x in Maclaurin's Series, equation (18), Art. 58, by their values determined as above, we have

$$y = f(a_0) + [df(a_0)]\frac{x}{1} + [d^2f(a_0)]\frac{x^2}{1.2} + \dots \qquad (162)$$

the square brackets in this case indicating that particular values of the functions enclosed in them are to be taken, viz. when we replace da_0 by a_1, da_1 by a_2, ...

And if we perform the several operations of derivation and introduce Lagrange's notation of derived-functions, we have

$$f\left(a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + a_3\frac{x^3}{1.2.3} + \dots + a_n\frac{x^n}{1.2\dots n} + \dots\right)$$

$$= f(a_0) + f'(a_0)\,a_1\frac{x}{1} + \{f''(a_0)\,a_1^2 + f'(a_0)\,a_2\}\frac{x^2}{1.2}$$

$$+ \{f'''(a_0)\,a_1^3 + 3f''(a_0)\,a_1a_2 + f'(a_0)\,a_3\}\frac{x^3}{1.2.3} + \dots \qquad (163)$$

Of this process we propose to give a few examples.

Ex. 1. It is required to expand in ascending powers of x

$$\{a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + \dots\}^{-1}.$$

In this case $f(a_0) = a_0^{-1}$;

$$\begin{aligned} \therefore\quad [df(a_0)] &= -a_0^{-2}a_1, \\ [d^2f(a_0)] &= 2a_0^{-3}a_1^2 - a_0^{-2}a_2, \\ [d^3f(a_0)] &= -6a_0^{-4}a_1^3 + 6a_0^{-3}a_1a_2 - a_0^{-2}a_3, \\ &\dots\dots\dots\dots \end{aligned}$$

$$\therefore\quad \{a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + \dots\}^{-1}$$

$$= \frac{1}{a_0} - \frac{a_1}{a_0^2}\frac{x}{1} + \frac{2a_1^2 - a_0a_2}{a_0^3}\frac{x^2}{1.2} - \frac{6a_1^3 - 6a_0a_1a_2 + a_0^2a_3}{a_0^4}\frac{x^4}{1.2.3.4} + \dots$$

Ex. 2. It is required to develope $\left(a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2}\right)^3$.

$$\left\{a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2}\right\}^3 = a_0^3 + 3a_0^2a_1\frac{x}{1} + 3(2a_0a_1^2 + a_0^2a_2)\frac{x^2}{1.2}$$

$$+ 6(a_1^3 + 3a_0a_1a_2)\frac{x^3}{1.2.3} + 18(2a_1^2a_2 + a_0a_2^2)\frac{x^4}{1.2.3.4}$$

$$+ 90a_1a_2^2\frac{x^5}{1.2\dots5} + 90a_2^3\frac{x^6}{1.2\dots6}.$$

Ex. 3. To develope $\left\{a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+a_3\frac{x^3}{1.2.3}+\dots\right\}^m$.

$$
\begin{aligned}
f(a_0) &= a_0^m; \\
[df(a_0)] &= [ma_0^{m-1}da_0] = ma_0^{m-1}a_1, \\
[d^2f(a_0)] &= [m(m-1)\,a_0^{m-2}a_1da_0+ma_0^{m-1}da_1] \\
&= m(m-1)\,a_0^{m-2}a_1^2+ma_0^{m-1}a_2, \\
[d^3f(a_0)] &= m(m-1)(m-2)\,a_0^{m-3}a_1^3 \\
&\qquad +3m(m-1)\,a_0^{m-2}a_1a_2+ma_0^{m-1}a_3,
\end{aligned}
$$

. . . .

$$
\therefore\quad \left\{a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+\dots\right\}^m
$$

$$
= a_0^m+ma_0^{m-1}a_1\frac{x}{1}+ma_0^{m-2}\{(m-1)\,a_1^2+a_0a_2\}\frac{x^2}{1.2}
$$

$$
+ma_0^{m-3}\{(m-1)(m-2)\,a_1^3+3(m-1)\,a_0a_1a_2+a_0^2a_3\}\frac{x^3}{1.2.3}+\dots;
$$

which is the Multinomial Theorem; and in which it is to be observed, that a_r does not appear in any coefficient before that of x^r.

Ex. 4. To develope e^z, where

$$
z = a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+a_3\frac{x^3}{1.2.3}+\dots.
$$

$$
\begin{aligned}
f(a_0) &= e^{a_0}, \\
[df(a_0)] &= [e^{a_0}da_0] = a_1e^{a_0}, \\
[d^2f(a_0)] &= [e^{a_0}a_1da_0+e^{a_0}da_1] = e^{a_0}a_1^2+e^{a_0}a_2, \\
[d^3f(a_0)] &= e^{a_0}\{a_1^3+3a_1a_2+a_3\},
\end{aligned}
$$

. . . .

$$
\therefore\quad e^{(a_0+a_1x+\dots)} = e^{a_0}\left\{1+a_1x+(a_1^2+a_2)\frac{x^2}{1.2}+\dots\right\}.
$$

Similarly,

$$
\sin\left(a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+a_3\frac{x^3}{1.2.3}+\dots\right)
$$

$$
= \sin a_0+a_1\cos a_0\frac{x}{1}+(a_2\cos a_0-a_1^2\sin a_0)\frac{x^2}{1.2}+\dots.
$$

$$
\tan^{-1}\left(a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+\dots\right)
$$

$$
= \tan^{-1}a_0+\frac{a_1}{1+a_0^2}\frac{x}{1}+\frac{a_2(1+a_0^2)-2a_0a_1^2}{(1+a_0^2)^2}\frac{x^2}{1.2}+\dots.
$$

$$\log\left(a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+a_3\frac{x^3}{1.2.3}+\dots\right)$$

$$=\log a_0+\frac{a_1}{a_0}\frac{x}{1}+\frac{a_2a_0-a_1^2}{a_0^2}\frac{x^2}{1.2}+\frac{a_3a_0^2-3a_0a_1a_2+2a_1^3}{a_0^3}\frac{x^3}{1.2.3}+\dots;$$

and the student is recommended to practise himself in these and other examples, as the process is simple and very useful.

The preceding process is, from another point of view, an extension of Taylor's Series; for whereas by that we are able to expand $f(a+x)$, by this we can develope in ascending powers of x, $f\left(a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+\dots\right)$; and therefore if we put $a_2=a_3=\dots=0$, the preceding formulæ become those of Taylor's Theorem.

96.] Sometimes, instead of finding the actual coefficients, it is convenient to find the law of their dependence; that is, to determine the equation by means of which they are related to each other; such as has been found by implicit differentiation in Art. 87, equation (126).

Thus in Ex. 1 of the preceding Article, let

$$y=\frac{1}{z}=A_0+A_1\frac{x}{1}+A_2\frac{x^2}{1.2}+\dots$$

$$z=a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+\dots$$

then $$zy=1; \qquad (164)$$

and $$zy=a_0A_0+C_1\frac{x}{1}+C_2\frac{x^2}{1.2}+C_3\frac{x^3}{1.2.3}+\dots \qquad (165)$$

where $C_1, C_2, C_3, \dots$ are the coefficients of the powers of x, arising from the product of the developed values of y and z, and which, as determined above, are the several values of $d.A_0a_0$, $d^2.A_0a_0$, $d^3.A_0a_0$, ... on the supposition that $da_0=a_1$, $da_1=a_2$, ... $dA_0=A_1$, $dA_1=A_2$, $dA_2=A_3$, ...; but, on comparing (164) and (165), it appears, that

$$A_0a_0=1; \qquad \therefore\quad d^n.A_0a_0=0.$$

And therefore by Leibnitz's Theorem, Art. 55, equation (4),

$$a_0\,d^nA_0+n\,da_0\,d^{n-1}A_0+\frac{n(n-1)}{1.2}d^2a_0\,d^{n-2}A_0+\dots$$
$$+n\,dA_0\,d^{n-1}a_0+A_0\,d^na_0=0;$$

and therefore

$$A_n a_0 + n A_{n-1} a_1 + \frac{n(n-1)}{1.2} A_{n-2} a_2 + \dots + A_0 a_n = 0; \qquad (166)$$

whence A_1, A_2, ... may be successively calculated.

As an example of this process, consider the problem which was discussed in Art. 87, viz. the expansion of

$$\begin{aligned} y &= \frac{x}{e^x - 1}, \\ &= \frac{1}{1 + \frac{1}{2}\cdot\frac{x}{1} + \frac{1}{3}\cdot\frac{x^2}{1.2} + \frac{1}{4}\cdot\frac{x^3}{1.2.3} + \dots} \end{aligned}$$

Here we have merely to replace in (166) $a_0, a_1, a_2, \dots$ severally by $1, \frac{1}{2}, \frac{1}{3}, \dots$ and we have

$$A_n + \frac{1}{2} n A_{n-1} + \frac{1}{3}\cdot\frac{n(n-1)}{1.2} A_{n-2} + \dots + \frac{1}{n+1} A_0 = 0;$$

by means of which the successive coefficients can easily be calculated.

Thus let $\quad n = 1; \qquad \therefore \quad A_1 + \frac{1}{2} A_0 = 0;$

but $\quad A_0 = \frac{1}{a_0} = 1; \qquad \therefore \quad A_1 = -\frac{1}{2};$

$n = 2, \qquad \therefore \quad A_2 + \frac{1}{2} A_1 + \frac{1}{3} A_0 = 0, \quad$ and $\quad A_2 = \frac{1}{6};$

$n = 3, \qquad A_3 = 0;$

$n = 4, \qquad A_4 = -\frac{1}{30};$

. . . .

and by a similar process the other coefficients may be found; and these are severally the numbers of Bernoulli.

97.] Again, if $u = f(a, \beta, \gamma, \dots)$, where

$$\begin{aligned} a &= a_0 + a_1 x + a_2 \frac{x^2}{1.2} + \dots, \\ \beta &= b_0 + b_1 x + b_2 \frac{x^2}{1.2} + \dots, \\ \gamma &= c_0 + c_1 x + c_2 \frac{x^2}{1.2} + \dots, \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ; \end{aligned}$$

then $\quad u = A_0 + A_1 x + A_2 \frac{x^2}{1.2} + \dots,$

where $A_0 = f(a_0, b_0, c_0, \dots)$, and $A_n = d^n f(a_0, b_0, c_0, \dots)$; the total differentials being taken, and on the supposition, as heretofore, of $da_0 = a_1$, $da_1 = a_2$, ... ; $db_0 = b_1$, $db_1 = b_2$, ... ; $dc_0 = c_1$, $dc_1 = c_2$, ... ; for example,

$$\frac{a_0 + a_1 \frac{x}{1} + a_2 \frac{x^2}{1.2} + \cdots}{b_0 + b_1 \frac{x}{1} + b_2 \frac{x^2}{1.2} + \cdots} = \mathrm{A}_0 + \mathrm{A}_1 \frac{x}{1} + \mathrm{A}_2 \frac{x^2}{1.2} + \cdots,$$

$$\mathrm{A}_0 = \frac{a_0}{b_0},$$

$$\mathrm{A}_1 = \frac{b_0 da_0 - a_0 db_0}{b_0^2} = \frac{b_0 a_1 - a_0 b_1}{b_0^2},$$

$$\mathrm{A}_2 = \frac{b_0 \{b_0 a_2 - a_0 b_2\} - 2 b_1 \{b_0 a_1 - a_0 b_1\}}{b_0^3},$$

. . . .

Also since $a_0 = \mathrm{A}_0 b_0$, we may by successive differentiation, as in the last example, implicitly determine the relation between the successive coefficients.

As a particular example of the last theorem, suppose

$$u = f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots\right),$$

where

$$y = a_0 + a_1 x + a_2 \frac{x^2}{1.2} + \ldots\ldots;$$

$$\frac{dy}{dx} = a_1 + a_2 \frac{x}{1} + \ldots\ldots,$$

$$\frac{d^2y}{dx^2} = a_2 + a_3 \frac{x}{1} + \ldots\ldots,$$

. . . .

$$\therefore \quad u = \mathrm{A}_0 + \mathrm{A}_1 x + \mathrm{A}_2 \frac{x^2}{1.2} + \ldots\ldots,$$

where

$$\mathrm{A}_0 = f(a_0, a_1, a_2, \ldots\ldots),$$

and A_1, A_2, ... are the successive total differentials of A_0 on the supposition that $da_0 = a_1$, $da_1 = a_2$, ...

Thus if

$$u = \frac{\frac{dy}{dx}}{y}, \qquad \mathrm{A}_0 = \frac{a_1}{a_0},$$

$$\mathrm{A}_1 = \frac{a_0 da_1 - a_1 da_0}{a_0^2} = \frac{a_0 a_2 - a_1^2}{a_0^2},$$

. . . .

$$u = \frac{a_1}{a_0} + \frac{a_0 a_2 - a_1^2}{a_0^2} \frac{x}{1} + \ldots\ldots$$

The process which is developed and applied in the preceding Articles was first communicated to me by Mr. W. F. Donkin, the Savilian Professor of Astronomy at Oxford. It is somewhat similar to that which is the subject of Arbogast's Calculus of Derivations. The form of the series which is here taken as the subject of the functional symbol is different to that chosen by Arbogast; and the method by which I have shewn that the differentials of the several constants will in the expansion be replaced by their next following constants is also different. It is however convenient to call the process by the name which he has given to it, although the preceding form of it is due to Professor Donkin.

The student also may consult the Treatise on the Differential and Integral Calculus, by Augustus De Morgan, London, 1842, Art. 214—227; the large treatise on the Differential Calculus, by S. F. Lacroix, Paris, 1810; and papers in the Cambridge and Dublin Mathematical Journal, by Mr. De Morgan, vol. i. p. 238 and vol. vi. p. 35, and by Professor Donkin, vol. vi. p. 141.

To complete the theory of the expansion of functions in ascending powers of the subject-variables, it is necessary to inquire into the form which the general or *n*th term of such an expansion assumes: because on the form of this and its relation to the preceding and the succeeding terms does the convergency or the divergency of the series depend. But this requires conditions which at present we are unable to assign: and therefore the remainder of our investigations on the subject must be reserved until the sixth Chapter.

SECTION 7.—*On the formation of Differential Equations by the elimination of constants and functions by differentiation.*

98.] An equation which contains differentials or derived-functions is called a Differential Equation; and it is called an equation of the first, second, ... *n*th order, according to the order of the differential or derived-function involved in it. It is also called a partial or a total differential equation, according as the differentials which it contains are partial or total. Thus equations (56), (62) of Art. 52 are partial differential equations of the first order; Ex. 1 and 2 of Art. 77 are total differential equations of the second order.

Now the simplest process by which these equations are formed from integral equations, is the elimination of one or more constants, or of a determinate function from the equation. If the function is indeterminate, and of variables combined in a given relation as in Art. 52, the resulting differential equation will be partial: and so also may it be when the function is of three or more variables. Of all these several cases instances will be given in the following Articles; and we shall first take the most simple; that viz. in which the equation arises immediately by differentiation, and the consequent elimination of one or more constants.

Since a constant quantity connected with a variable by the symbols of addition and subtraction disappears in differentiation, it follows that if an equation can be put under the form,

$$y = f(x) + c, \tag{167}$$

its derived will be $\dfrac{dy}{dx} = f'(x)$;

from which the constant has disappeared. And therefore, whatever may have been its value in the first or *primitive* equation, the derived equation contains no trace of it. Similarly, if an equation can be put under the form,

$$\begin{aligned} y &= f(x) + c_1 x + c_0, \\ \frac{dy}{dx} &= f'(x) + c_1, \\ \frac{d^2y}{dx^2} &= f''(x); \end{aligned}$$

and thus in two differentiations two arbitrary constants will have disappeared; and the resulting differential equation is of the second order.

And similarly, if

$$\begin{aligned} y &= f(x) + c_n x^n + c_{n-1} x^{n-1} + \dots + c_2 x^2 + c_1 x + c_0, \\ \frac{dy}{dx} &= f'(x) + n c_n x^{n-1} + \dots + 2 c_2 x + c_1, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{d^{n+1}y}{dx^{n+1}} &= f^{n+1}(x), \end{aligned}$$

and thus all the $n+1$ arbitrary constants have disappeared in the process of $n+1$ differentiations; whenever therefore the functions admit of being put under the above forms, at each differentiation one arbitrary constant will disappear; and the resulting differential equation is of the $(n+1)$th order.

Ex. 1. $y = ax + b$.

$$\frac{dy}{dx} = a, \qquad \frac{d^2y}{dx^2} = 0;$$

or substituting $\frac{dy}{dx}$ for a in the primitive,

$$y = x\frac{dy}{dx} + b, \qquad \therefore \quad \frac{dy}{dx} = \frac{dy}{dx} + x\frac{d^2y}{dx^2},$$

$$\therefore \quad \frac{d^2y}{dx^2} = 0;$$

whence it appears, that by one differentiation either a or b may be eliminated, and the result is a differential equation of the first order; and by two differentiations both may be made to disappear, and the differential equation is of the second order.

99.] Suppose however that the equation of relation between x and y is implicit, involves m arbitrary constants, and is of the form,

$$u = \text{F}(x, y) = 0; \tag{168}$$

for the sake of simplicity consider x to be equicrescent, and differentiate (168) n times in succession; thereby n different equations will be formed, which, ~~added to~~ with (168), give us $n+1$ different equations involving m constants. Let us suppose m to be greater than n; then by means of these, n of the constants may be eliminated, and, theoretically at least, any n of them; whereby the final equation will contain $m-n$ of the constants; and of course there may be as many different final equations as there are combinations of the m constants taken n and n together; that is, there may be

$$\frac{m(m-1)(m-2)\dots(m-n+1)}{1.2.3\dots n} \text{ different equations.}$$

Thus suppose (168) to be differentiated twice; three equations will then be given containing m constants. By means of which two may be eliminated, and the resulting equation will contain $m-2$ constants; and as any two may be eliminated, we may have as many different final equations as there are combinations of the m constants taken 2 and 2 together. Hence it follows, that if we differentiate m times, there will be altogether $m+1$ equations, from which the m arbitrary constants may be entirely eliminated; and that, *generally*, m constants involved in the primitive cannot be eliminated, unless there are formed m derived-functions of the primitive expression.

Ex. 1. $y^2 = m(a^2 - x^2)$; it is required to eliminate m and a.

$$y\frac{dy}{dx} = -mx; \tag{169}$$

also eliminating m between this and the primitive, we have

$$yx + (a^2 - x^2)\frac{dy}{dx} = 0; \tag{170}$$

and differentiating again (169),

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -m;$$

$$\therefore \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0. \tag{171}$$

That is, we have two differential equations of the first order, which contain $\frac{dy}{dx}$ only, and which respectively do not involve a and m; and one differential equation of the second order, viz. (171), from which both the constants have disappeared.

Ex. 2. Eliminate the constant a from

$$(1-x^2)^{\frac{1}{2}} + (1-y^2)^{\frac{1}{2}} = a(x-y).$$

$$\frac{-x\,dx}{(1-x^2)^{\frac{1}{2}}} - \frac{y\,dy}{(1-y^2)^{\frac{1}{2}}} = a(dx-dy);$$

whence, by the common process of elimination, and dividing out the common factors,

$$\frac{dx}{(1-x^2)^{\frac{1}{2}}} = \frac{dy}{(1-y^2)^{\frac{1}{2}}};$$

which is a differential equation of the first order.

Ex. 3. Eliminate c from $x^2 + y^2 = cx$.

$$2(x\,dx + y\,dy) = c\,dx;$$

$$\therefore \quad c = 2\left(y\frac{dy}{dx} + x\right); \qquad \therefore \quad x^2 + y^2 = 2xy\frac{dy}{dx} + 2x^2,$$

$$y^2 - 2xy\frac{dy}{dx} - x^2 = 0;$$

which is a differential equation of the first order.

Ex. 4. Given $(x-a)^2+(y-\beta)^2=\rho^2$; it is required to eliminate a and β by differentiation.

Let neither x nor y be equicrescent; then

$$(x-a)\,dx+(y-\beta)\,dy = 0,$$

$$(x-a)\,d^2x+(y-\beta)\,d^2y+dx^2+dy^2 = 0;$$

$$\therefore \quad x-a = \frac{(dx^2+dy^2)\,dy}{dx\,d^2y-dy\,d^2x}, \qquad y-\beta = -\frac{(dx^2+dy^2)\,dx}{dx\,d^2y-dy\,d^2x};$$

therefore squaring and adding,

$$(x-a)^2+(y-\beta)^2 = \rho^2 = \frac{(dx^2+dy^2)^3}{(dx\,d^2y-dy\,d^2x)^2}.$$

And if x is equicrescent,

$$\rho^2 = \frac{(dx^2+dy^2)^3}{(dx\,d^2y)^2} = \frac{\left(1+\frac{dy^2}{dx^2}\right)^3}{\left(\frac{d^2y}{dx^2}\right)^2};$$

which is a differential expression of the second order.

100.] Also since the differentials of logarithmic and inverse-circular functions are algebraical quantities; the differentials of exponential functions reproduce themselves, and of circular functions are other circular quantities related to them by trigonometrical formulæ; such transcendental functions may be eliminated from given primitive equations by means of differentiation, and differential equations will thereby be formed. The following examples will explain the process, and enable the student to apply it to other similar problems.

Ex. 1. If $y=(a^2+x^2)^{\frac{m}{n}}$; it is required to eliminate the irrational function.

$$y = (a^2+x^2)^{\frac{m}{n}};$$

$$\therefore \quad \log y = \frac{m}{n}\log(a^2+x^2), \qquad \frac{dy}{y} = \frac{m}{n}\,\frac{2x\,dx}{a^2+x^2};$$

$$\therefore \quad \frac{dy}{dx} = \frac{2mxy}{n(a^2+x^2)};$$

a differential equation which is free from radical quantities.

Ex. 2.

$$y = a\sin x+b\cos x,$$

$$\frac{dy}{dx} = a\cos x-b\sin x,$$

$$\frac{d^2y}{dx^2} = -a\sin x-b\cos x;$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0;$$

which is a differential equation of the second order.

Ex. 3. $$y = ce^{\sin^{-1}x}.$$

$$\frac{dy}{dx} = ce^{\sin^{-1}x}\frac{1}{(1-x^2)^{\frac{1}{2}}},$$

$$= \frac{y}{(1-x^2)^{\frac{1}{2}}},$$

$$\therefore \quad (1-x^2)^{\frac{1}{2}}\frac{dy}{dx} = y;$$

which is a differential equation of the first order, and contains an irrational quantity which we may remove by a subsequent differentiation. Thus

$$(1-x^2)^{\frac{1}{2}}\frac{d^2y}{dx^2} - \frac{x}{(1-x^2)^{\frac{1}{2}}}\cdot\frac{dy}{dx} = \frac{dy}{dx};$$

$$\therefore \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = (1-x^2)^{\frac{1}{2}}\frac{dy}{dx},$$

$$= y;$$

a differential equation of the second order which is free from radicals and transcendental functions.

Ex. 4. $$y = be^{ax}\cos(nx+c);$$

$$\frac{dy}{dx} = abe^{ax}\cos(nx+c) - nbe^{ax}\sin(nx+c),$$

$$= ay - nbe^{ax}\sin(nx+c),$$

$$\therefore \quad \frac{d^2y}{dx^2} = a\frac{dy}{dx} - nbae^{ax}\sin(nx+c) - n^2be^{ax}\cos(nx+c),$$

$$= - - - + a\left(\frac{dy}{dx} - ay\right) - n^2y;$$

$$\therefore \quad \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+n^2)y = 0;$$

a differential equation of the second order, and free from exponential and circular functions.

Ex. 5. $$y = \frac{e^x + e^{-x}}{e^x - e^{-x}}; \qquad \therefore \quad y = \frac{e^{2x}+1}{e^{2x}-1};$$

$$e^{2x} = \frac{y+1}{y-1}; \qquad \therefore \quad 2x = \log(y+1) - \log(y-1),$$

$$2\,dx = \frac{dy}{y+1} - \frac{dy}{y-1};$$

$$\therefore \quad dx = \frac{dy}{1-y^2}, \quad \text{or} \quad \frac{dy}{dx} = 1-y^2.$$

101.] Differential equations also sometimes arise in another way: it is often more convenient to express the relation which holds between two given quantities or functions by a differential than by its primitive equation: the differential equation indeed, as will be seen hereafter, can frequently be found, when the primitive is beyond our powers of expression. Here however I shall give only one or two examples wherein differential equations are formed.

Ex. 1. It is required to express in the form of a differential equation the relation between $\cos nx$ and $\cos x$.

Let $\quad u_n = \cos nx, \qquad$ and $\qquad z = \cos x;$

$$\therefore \quad \frac{du_n}{dz} = -n \sin nx \frac{dx}{dz}; \qquad dz = -\sin x\, dx;$$

$$= \frac{n \sin nx}{\sin x};$$

$$\therefore \quad \sin x \frac{du_n}{dz} = n \sin nx,$$

$$\cos x \frac{du_n}{dz} - (\sin x)^2 \frac{d^2 u_n}{dz^2} = n^2 \cos nx;$$

$$\therefore \quad (1-z^2)\frac{d^2u_n}{dz^2} - z\frac{du_n}{dz} + n^2 u_n = 0;$$

which is a differential equation of the second order. If we assume u_n to be expanded in a series of ascending powers of z, the coefficients of which are to be determined, this equation will enable us to determine the coefficients, because it is to be satisfied by them for all values of n.

In this equation it will be observed that $\cos x$ is the equicrescent variable.

Ex. 2. It is required to express in the form of a differential equation the relation which exists between $\cos x$ and the series

$$u_n = \cos nx + \cos (n-1)x + \ldots + \cos 2x + \cos x.$$

The sum of the series is found without difficulty; and we have

$$u_n \sin \frac{x}{2} = \sin \frac{nx}{2} \cos \frac{n+1}{2} x;$$

$$\therefore \quad 2u_n \sin \frac{x}{2} = \sin \left(n + \frac{1}{2}\right) x - \sin \frac{x}{2}.$$

Let $\cos x = z$; $\therefore$ $dz = -\sin x\, dx$; let z or $\cos x$ be equicrescent; then differentiating twice, we have

$$2(\sin x)^2 \sin\frac{x}{2}\frac{d^2u_n}{dz^2} - 2\cos x \sin\frac{x}{2}\frac{du_n}{dz} - \frac{1}{2}\sin\frac{x}{2}u_n$$

$$= -\left(n+\frac{1}{2}\right)^2 \sin\left(n+\frac{1}{2}\right)x + \frac{1}{4}\sin\frac{x}{2}$$

$$= -\left(n+\frac{1}{2}\right)^2\left(\sin\frac{x}{2} + 2u_n\sin\frac{x}{2}\right) + \frac{1}{4}\sin\frac{x}{2};$$

$$\therefore \quad (1-z^2)\frac{d^2u_n}{dz^2} - z\frac{du_n}{dz} + n(n+1)\left(u_n+\frac{1}{2}\right) = 0;$$

which is the required differential equation of the second order.

These two examples are sufficient to illustrate the process.

102.] Thus far the primitive equations have contained only two variables, and the differentials and derived-functions have been total: it remains still to exhibit the analogous processes, when three or more variables are implicitly or explicitly involved, and when the differentials are therefore partial. I shall consider two cases of this process. In the former the variables will be contained in an equation which is free from any undetermined or arbitrary function. In the latter I shall suppose one or more arbitrary functions to be involved, of which the subject variables enter in certain given relations, such as we have supposed and exemplified in Art. 52 and 53; and the derived functions of which will accordingly bear certain relations to each other dependent on the form in which the subject variables are combined, and will thereby admit of elimination.

Ex. 1. It is required to eliminate A, B, C and D from the equation $Ax+By+Cz+D = 0$, and to express the resulting relation in the form of a partial differential equation.

$$Ax+By+Cz+D = 0.$$

Let us assume z to be the dependent variable, and take successively the x- and y-partial differentials; then, observing the power of brackets,

$$A\,dx + C\,dz = 0, \qquad \therefore \quad A + C\left(\frac{dz}{dx}\right) = 0;$$

$$B\,dy + C\,dz = 0, \qquad \therefore \quad B + C\left(\frac{dz}{dy}\right) = 0;$$

$$\therefore \quad \left(\frac{d^2z}{dx^2}\right) = \left(\frac{d^2z}{dx\,dy}\right) = \left(\frac{d^2z}{dy^2}\right) = 0;$$

either one of which differential equations satisfies the equation. Of these results we shall hereafter see the geometrical meaning.

Ex. 2. Determine the partial differential equations which arise in the elimination of a, b, c from the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Taking z to be the dependent variable, we have

$$\frac{x}{a^2} + \frac{z}{c^2}\left(\frac{dz}{dx}\right) = 0, \tag{172}$$

$$\frac{y}{b^2} + \frac{z}{c^2}\left(\frac{dz}{dy}\right) = 0; \tag{173}$$

and of the former of these two, taking the x-partial differential, we have

$$\frac{1}{a^2} + \frac{1}{c^2}\left\{\left(\frac{dz}{dx}\right)^2 + z\left(\frac{d^2z}{dx^2}\right)\right\} = 0; \tag{174}$$

from which and (172) eliminating a^2 and c^2, we have

$$xz\left(\frac{d^2z}{dx^2}\right) + x\left(\frac{dz}{dx}\right)^2 - z\left(\frac{dz}{dx}\right) = 0. \tag{175}$$

Similarly, if we take the y-differential of (173), and eliminate b^2 and c^2, we shall have

$$yz\left(\frac{d^2z}{dy^2}\right) + y\left(\frac{dz}{dy}\right)^2 - z\left(\frac{dz}{dy}\right) = 0. \tag{176}$$

Either (175) or (176) is the required differential equation.

Ex. 3. It is required to express a property of the following equation in terms of partial derived-functions, and independently of a, b and c.

$$\mathrm{v} = \{(x-a)^2+(y-b)^2+(z-c)^2\}^{-\frac{1}{2}};$$

$$\therefore \quad \left(\frac{d\mathrm{v}}{dx}\right) = -(x-a)\,\{(x-a)^2+(y-b)^2+(z-c)^2\}^{-\frac{3}{2}}$$

$$= -(x-a)\,\mathrm{v}^3,$$

$$\left(\frac{d^2\mathrm{v}}{dx^2}\right) = -\mathrm{v}^3 - 3\mathrm{v}^2(x-a)\left(\frac{d\mathrm{v}}{dx}\right)$$

$$= -\mathrm{v}^3 + 3\mathrm{v}^5(x-a)^2.$$

Similarly, $$\left(\frac{d^2\mathrm{v}}{dy^2}\right) = -\mathrm{v}^3 + 3\mathrm{v}^5(y-b)^2,$$

$$\left(\frac{d^2\mathrm{v}}{dz^2}\right) = -\mathrm{v}^3 + 3\mathrm{v}^5(z-c)^2;$$

$$\begin{aligned}\therefore\quad \left(\frac{d^2\text{v}}{dx^2}\right)+\left(\frac{d^2\text{v}}{dy^2}\right)+\left(\frac{d^2\text{v}}{dz^2}\right) &= -3\text{v}^3+3\text{v}^5\{(x-a)^2+(y-b)^2+(z-c)^2\}\\ &= -3\text{v}^3+3\text{v}^3\\ &= 0.\end{aligned}$$

As many examples of the formation of these and similar differential equations will occur in the sequel, it is unnecessary to insert others here.

103.] Again, let us consider the formation of partial differential equations as they arise from the elimination of arbitrary functions, when the subject-variables of these functions are combined in particular relations. And first let us take the most simple case; that viz. of an explicit function of two independent variables, of the form

$$u = \text{F}(x, y); \tag{177}$$

in which the form of F is undetermined, and wherein x and y enter in particular combinations, such as x divided by y, or x multiplied into y, and so on: so that the x- and the y-partial derived-functions of u will contain the same arbitrary functional term. Let us also suppose that (177) contains only one arbitrary function. Then the partial derived-functions having been calculated, we shall have two equations containing the same arbitrary function, which by means of them may be eliminated; and an equation will result in terms of $\left(\frac{d\text{F}}{dx}\right)$, $\left(\frac{d\text{F}}{dy}\right)$, the variables and constants, some or all, of the primitive equation, and it will be independent of the arbitrary function.

If the primitive contains many arbitrary functions of x and y, with their subject variables combined in given relations, successive partial derived-functions may be formed, and thence equations will be obtained independent of the arbitrary functions, and in terms of the partial derived-functions. The process has already been employed in Art. 52 and 53 of Chapter II, and (56), (59), (62), (65), (76), (86) of those Articles are cases of it. The following examples are in further illustration.

Ex. 1. $\quad u = ax+by+cf(mx+ny);$

then if, as explained in Art. 58, the relation between f and f' is such that $d.f(z) = f'(z)\,dz$,

$$\left(\frac{du}{dx}\right) = a + cmf'(mx+ny), \qquad \left(\frac{du}{dy}\right) = b + cnf'(mx+ny);$$

$$\therefore \quad n\left(\frac{du}{dx}\right) - m\left(\frac{du}{dy}\right) = an - bm.$$

Ex. 2. $u = \text{F}(x^2+y^2)$;

let F' be related to F so that $d.\text{F}(z) = \text{F}'(z)\,dz$, then

$$\left(\frac{du}{dx}\right) = 2x\,\text{F}'(x^2+y^2), \qquad \left(\frac{du}{dy}\right) = 2y\,\text{F}'(x^2+y^2);$$

$$\therefore \quad y\left(\frac{du}{dx}\right) - x\left(\frac{du}{dy}\right) = 0.$$

Ex. 3. $u = y^2 + 2f\left(\frac{1}{x} + \log y\right)$;

let f and f' be related so that $d.f(z) = f'(z)\,dz$; then

$$\left(\frac{du}{dx}\right) = -\frac{2}{x^2}f'\left(\frac{1}{x}+\log y\right), \qquad \left(\frac{du}{dy}\right) = 2y + \frac{2}{y}f'\left(\frac{1}{x}+\log y\right);$$

$$\therefore \quad x^2\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = 2y^2.$$

We will now take a case wherein two arbitrary functions are involved, and two successive differentiations are required.

Ex. 4. $u = f(ax+by) + \phi(bx-ay)$;

then using a notation the same as heretofore,

$$\left(\frac{du}{dx}\right) = af'(ax+by) + b\,\phi'(bx-ay),$$

$$\left(\frac{du}{dy}\right) = bf'(ax+by) - a\,\phi'(bx-ay);$$

$$\therefore \quad a\left(\frac{du}{dx}\right) + b\left(\frac{du}{dy}\right) = (a^2+b^2)f'(ax+by).$$

And taking again successively the x- and the y-differentials of this equation,

$$a\left(\frac{d^2u}{dx^2}\right) + b\left(\frac{d^2u}{dx\,dy}\right) = a(a^2+b^2)f''(ax+by),$$

$$a\left(\frac{d^2u}{dy\,dx}\right) + b\left(\frac{d^2u}{dy^2}\right) = b(a^2+b^2)f''(ax+by);$$

$$\therefore \quad ab\left\{\left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right)\right\} + (b^2-a^2)\left(\frac{d^2u}{dx\,dy}\right) = 0.$$

Ex. 5. $$u = xf\left(\frac{y}{x}\right) + \text{F}(xy);$$

and using a similar notation,

$$\left(\frac{du}{dx}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right) + y\,\text{F}'(xy),$$

$$\left(\frac{du}{dy}\right) = f'\left(\frac{y}{x}\right) + x\,\text{F}'(xy);$$

$$\therefore \quad x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = xf\left(\frac{y}{x}\right) + 2xy\,\text{F}'(xy);$$

and substituting for $f\left(\frac{y}{x}\right)$ from the given primitive, we have

$$x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = u - \text{F}(xy) + 2xy\,\text{F}'(xy),$$

and replacing $2xy\,\text{F}'(xy) - \text{F}(xy)$ by $\phi(xy)$, we have

$$x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = u + \phi(xy);$$

$$\therefore \quad x\left(\frac{d^2u}{dx^2}\right) + \left(\frac{du}{dx}\right) + y\left(\frac{d^2u}{dx\,dy}\right) = \left(\frac{du}{dx}\right) + y\,\phi'(xy),$$

$$x\left(\frac{d^2u}{dx\,dy}\right) + \left(\frac{du}{dy}\right) + y\left(\frac{d^2u}{dy^2}\right) = \left(\frac{du}{dy}\right) + x\,\phi'(xy);$$

$$x^2\left(\frac{d^2u}{dx^2}\right) - y^2\left(\frac{d^2u}{dy^2}\right) = 0.$$

104.] Thus far the dependent variable has been explicitly involved: if however all the variables are implicitly involved, a process similar to the preceding may be employed, if we consider one of the variables to be a function of, and thus dependent on the other two, and on this supposition calculate its partial variations due to the variations of the others. Some cases of this kind have already occurred in Art. 52 and 53, and some functions, the subjects of which are in particular combinations, have in them been eliminated.

Ex. 1. Eliminate the function from

$$x^2 + y^2 + z^2 = f(lx + my + nz).$$

Let z be a variable dependent on x and y, which are two independent variables;

$$\therefore \quad 2x + 2z\left(\frac{dz}{dx}\right) = \left\{l + n\left(\frac{dz}{dx}\right)\right\} f'(lx + my + nz),$$

$$2y+2z\left(\frac{dz}{dy}\right) = \left\{m+n\left(\frac{dz}{dy}\right)\right\} f'(lx+my+nz),$$

$$\therefore \quad (mz-ny)\left(\frac{dz}{dx}\right)+(nx-lz)\left(\frac{dz}{dy}\right) = ly-mx.$$

And if the primitive is in the form $\mathrm{F}(x,y,z)=0$, the resulting partial differential equation is

$$(mz-ny)\left(\frac{d\mathrm{F}}{dx}\right)+(nx-lz)\left(\frac{d\mathrm{F}}{dy}\right)+(ly-mx)\left(\frac{d\mathrm{F}}{dz}\right) = 0.$$

Ex. 2. $\quad u = x\,\mathrm{F}\left(\frac{y}{x},\frac{x}{z}\right). \quad = 0$

Since F has two subjects $\frac{y}{x}$, $\frac{x}{z}$, let F_1 and F_2 be so related to F that they are the functional factors in the derived-functions of F, according as the first or second subject varies.

$$\left(\frac{du}{dx}\right) = \mathrm{F}\left(\frac{y}{x},\frac{x}{z}\right)-\frac{y}{x}\mathrm{F}_1\left(\frac{y}{x},\frac{x}{z}\right)+\frac{x}{z}\mathrm{F}_2\left(\frac{y}{x},\frac{x}{z}\right),$$

$$\left(\frac{du}{dy}\right) = \mathrm{F}_1\left(\frac{y}{x},\frac{x}{z}\right),$$

$$\left(\frac{du}{dz}\right) = -\frac{x^2}{z^2}\mathrm{F}_2\left(\frac{y}{x},\frac{x}{z}\right);$$

$$\therefore \quad x\left(\frac{du}{dx}\right)+y\left(\frac{du}{dy}\right)+z\left(\frac{du}{dz}\right) = x\,\mathrm{F}\left(\frac{y}{x},\frac{x}{z}\right), \; = 0$$

$$= u. \; = 0$$

Ex. 3. $\quad u = f(ax^2+by^3+cz^4)+\phi(\cos lx+\cos my+\cos nz)$.

$$\left(\frac{du}{dx}\right) = 2axf'(ax^2+by^3+cz^4)-l\sin lx\,\phi'(\cos lx+\cos my+\cos nz),$$

$$\left(\frac{du}{dy}\right) = 3by^2f'(ax^2+by^3+cz^4)-m\sin my\,\phi'(\cos lx+\cos my+\cos nz),$$

$$\left(\frac{du}{dz}\right) = 4cz^3f'(ax^2+by^3+cz^4)-n\sin nz\,\phi'(\cos lx+\cos my+\cos nz);$$

therefore by Lagrange's method of cross-multiplication, see Preliminary Proposition II, we have

$$\left(\frac{du}{dx}\right)\{4cmz^3\sin my-3bny^2\sin nz\}+\left(\frac{du}{dy}\right)\{2anx\sin nz-4clz^3\sin lx\}$$
$$+\left(\frac{du}{dz}\right)\{3bly^2\sin lx-2amx\sin my\} = 0,$$

an equation independent of the arbitrary functions, and therefore expressive of the properties of such functions, whatever is their specific form.

Ex. 4. $z = f(x+ay)+\phi(x-ay)$.

Let f' and ϕ', f'' and ϕ'' stand in their usual relation to f and ϕ: then

$$\left(\frac{dz}{dx}\right) = f'(x+ay)+\phi'(x-ay),$$

$$\left(\frac{dz}{dy}\right) = af'(x+ay)-a\phi'(x-ay),$$

$$\left(\frac{d^2z}{dx^2}\right) = f''(x+ay)+\phi''(x-ay),$$

$$\left(\frac{d^2z}{dy^2}\right) = a^2f''(x+ay)+a^2\phi''(x-ay);$$

$$\therefore \quad \left(\frac{d^2z}{dy^2}\right)-a^2\left(\frac{d^2z}{dx^2}\right) = 0.$$

105.] In general, for determining to what order of differentiation we must proceed to eliminate any number of arbitrary functions from an expression containing two variables in given combinations, for we will not enter upon the more general case, let the following considerations suffice.

Suppose $u = 0$ to comprise m arbitrary functions of x and y, then it is plain that each successive differentiation introduces m other arbitrary functions, which are the derived of the given functions; so that by proceeding to the nth order of differentiation, we have $(n+1)m$ different arbitrary functions: but as the original equation $u = 0$ gives one relation amongst these functions, so do

$$\left(\frac{du}{dx}\right) = 0, \qquad \left(\frac{du}{dy}\right) = 0,$$

$$\left(\frac{d^2u}{dx^2}\right) = 0, \qquad \left(\frac{d^2u}{dx\,dy}\right) = 0, \qquad \left(\frac{d^2u}{dy^2}\right) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(\frac{d^nu}{dx^n}\right) = 0, \qquad \left(\frac{d^nu}{dx^{n-1}dy}\right) = 0, \ \ldots \ \left(\frac{d^nu}{dy^n}\right) = 0,$$

give us other relations; and therefore by means of n differentiations we have the number of relations equal to

$$1+2+3+\ldots\ldots+(n+1) = \frac{(n+1)(n+2)}{2}.$$

In order that we may be able to eliminate all these, we must evidently have the number of relations greater than the number of unknown quantities, that is,

$$\frac{(n+1)(n+2)}{1.2} > (n+1)m;$$

that is, $n+2 > 2m$, $n > 2m-2$;

that is, n, which expresses the order of differentiation, must $= 2m-1$ at least; and we shall then have a sufficient number of equations to eliminate the arbitrary functions from. Thus, if the original equation involve but one arbitrary function, $m = 1$, and we need differentiate but once; if it involves two arbitrary functions, we must in the general case differentiate thrice, and so on. An example is subjoined in which three differentiations are required:

$$u = f(x+y) + xy\,\phi(x-y).$$

$$\left(\frac{du}{dx}\right) = f'(x+y) + y\,\phi(x-y) + xy\,\phi'(x-y),$$

$$\left(\frac{du}{dy}\right) = f'(x+y) + x\,\phi(x-y) - xy\,\phi'(x-y);$$

$$\left(\frac{du}{dx}\right) - \left(\frac{du}{dy}\right) = (y-x)\,\phi(x-y) + 2xy\,\phi'(x-y).$$

$$\therefore\ \left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dx\,dy}\right) = (y-x)\,\phi'(x-y) - \phi(x-y) + 2y\,\phi'(x-y) + 2xy\,\phi''(x-y),$$

$$\left(\frac{d^2u}{dx\,dy}\right) - \left(\frac{d^2u}{dy^2}\right) = -(y-x)\,\phi'(x-y) + \phi(x-y) + 2x\,\phi'(x-y) - 2xy\,\phi''(x-y);$$

$$\therefore\ \left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right) = 2(x+y)\,\phi'(x-y).$$

And differentiating again,

$$\left(\frac{d^3u}{dx^3}\right) - \left(\frac{d^3u}{dx\,dy^2}\right) = 2\phi'(x-y) + 2(x+y)\,\phi''(x-y),$$

$$\left(\frac{d^3u}{dy\,dx^2}\right) - \left(\frac{d^3u}{dy^3}\right) = 2\phi'(x-y) - 2(x+y)\,\phi''(x-y),$$

$$\therefore\ \left(\frac{d^3u}{dx^3}\right) + \left(\frac{d^3u}{dx^2\,dy}\right) - \left(\frac{d^3u}{dx\,dy^2}\right) - \left(\frac{d^3u}{dy^3}\right) = 4\phi'(x-y);$$

but $$\phi'(x-y) = \frac{1}{2(x+y)}\left\{\left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right)\right\},$$

$$\therefore\ \left(\frac{d^3u}{dx^3}\right) + \left(\frac{d^3u}{dx^2\,dy}\right) - \left(\frac{d^3u}{dx\,dy^2}\right) - \left(\frac{d^3u}{dy^3}\right) = \frac{2}{x+y}\left\{\left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right)\right\}.$$

SECTION 8.—*Transformation of partial differential expressions into their equivalents in terms of new variables.*

106.] In a previous section of the present chapter we have investigated the process by which an equation involving total differential expressions may be changed into its equivalent in terms of new variables, and we explained the modifications which such expressions underwent according as one or another of the variables is equicrescent. In the last section the elimination of arbitrary functions and other processes have given rise to equations containing partial differential quantities; and for purposes of simplification or for other reasons it is often necessary to transform them into their equivalents in terms of new variables. The process of this transformation I proceed to explain; but I shall take only simple cases, as they will be sufficient to unfold the principles; and the more complicated will be discussed where they arise in the second volume of our work.

Let us first take the most simple case: that, viz., of a differential expression involving $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$, dx, dy; in which accordingly the first partial derived-functions and differentials are involved, and wherein are none of a higher order. Now let it be observed that as a relation between $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$, the variables x and y, and certain constants is given by the given equation, so is there of necessity a relation of the form $\text{F}(u, x, y) = 0$, whether such a relation can be found or not.

Let us suppose that the variables x and y are connected with two new variables r and θ by means of equations of the form

$$\left.\begin{aligned} x &= \phi_1(r, \theta) \\ y &= \phi_2(r, \theta) \end{aligned}\right\}; \tag{178}$$

and that it is required to transform the given differential equation into its equivalent in terms of the new variables. If the primitive function is of the form $u = f(x, y)$, then after the substitutions of the preceding values for x and y, we shall have

$$u = \text{F}(r, \theta); \tag{179}$$

of which the total differential is

$$\text{D}u = \left(\frac{d\text{F}}{dr}\right) dr + \left(\frac{d\text{F}}{d\theta}\right) d\theta; \tag{180}$$

whence, dividing through successively by dx and dy, and changing $\frac{\mathrm{D}u}{dx}$ into $\left(\frac{du}{dx}\right)$, and $\frac{\mathrm{D}u}{dy}$ into $\left(\frac{du}{dy}\right)$, since in these cases we have the ratio of the partial variations of u and of x, and of u and of y, and bracketing them to indicate that they are partial, we have

$$\left.\begin{aligned}\left(\frac{du}{dx}\right) &= \left(\frac{d\mathrm{F}}{dr}\right)\frac{dr}{dx} + \left(\frac{d\mathrm{F}}{d\theta}\right)\frac{d\theta}{dx},\\ \left(\frac{du}{dy}\right) &= \left(\frac{d\mathrm{F}}{dr}\right)\frac{dr}{dy} + \left(\frac{d\mathrm{F}}{d\theta}\right)\frac{d\theta}{dy}.\end{aligned}\right\} \tag{181}$$

But since y does not vary when $\left(\frac{du}{dx}\right)$ is calculated, and x does not vary when $\left(\frac{du}{dy}\right)$ is calculated, we must bear in mind that $\frac{dr}{dx}, \frac{d\theta}{dx}$ are to be calculated on the supposition that $dy = 0$; and that $\frac{dr}{dy}, \frac{d\theta}{dy}$ are to be calculated on the supposition that $dx = 0$. Let us introduce these conditions; from (178) we have

$$dx = \left(\frac{d\phi_1}{dr}\right)dr + \left(\frac{d\phi_1}{d\theta}\right)d\theta,$$

$$dy = \left(\frac{d\phi_2}{dr}\right)dr + \left(\frac{d\phi_2}{d\theta}\right)d\theta;$$

therefore to calculate $\frac{dx}{dr}$ and $\frac{dx}{d\theta}$, let $dy = 0$; and eliminating successively $d\theta$ and dr between these two equations on this supposition, we have

$$\frac{dx}{dr} = \frac{\left(\frac{d\phi_1}{dr}\right)\left(\frac{d\phi_2}{d\theta}\right) - \left(\frac{d\phi_2}{dr}\right)\left(\frac{d\phi_1}{d\theta}\right)}{\left(\frac{d\phi_2}{d\theta}\right)}, \tag{182}$$

$$\frac{dx}{d\theta} = \frac{\left(\frac{d\phi_1}{d\theta}\right)\left(\frac{d\phi_2}{dr}\right) - \left(\frac{d\phi_2}{d\theta}\right)\left(\frac{d\phi_1}{dr}\right)}{\left(\frac{d\phi_2}{dr}\right)}; \tag{183}$$

similarly,
$$\frac{dy}{dr} = \frac{\left(\frac{d\phi_1}{d\theta}\right)\left(\frac{d\phi_2}{dr}\right) - \left(\frac{d\phi_2}{d\theta}\right)\left(\frac{d\phi_1}{dr}\right)}{\left(\frac{d\phi_1}{d\theta}\right)}, \tag{184}$$

$$\frac{dy}{d\theta} = \frac{\left(\frac{d\phi_2}{d\theta}\right)\left(\frac{d\phi_1}{dr}\right) - \left(\frac{d\phi_1}{d\theta}\right)\left(\frac{d\phi_2}{dr}\right)}{\left(\frac{d\phi_1}{dr}\right)}; \tag{185}$$

and if we substitute these quantities in the expressions above for $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, the resulting expressions will be the equivalents of $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, when x and y are replaced by their equivalents in terms of r and θ. It is unnecessary to express the values in their full length, because it is more convenient to work each example with its own particular formulæ.

107.] Ex. 1. To transform $\left(\frac{dR}{dx}\right)$ and $\left(\frac{dR}{dy}\right)$ into their equivalents in terms of r and θ, when $x = r\cos\theta$, $y = r\sin\theta$.

In this problem there is implied a function, $R = f(x, y)$, which becomes, when x and y are replaced by their equivalents,

$$R = F(r, \theta);$$

$$\therefore\quad \left.\begin{aligned} \left(\frac{dR}{dx}\right) &= \left(\frac{dR}{dr}\right)\frac{dr}{dx} + \left(\frac{dR}{d\theta}\right)\frac{d\theta}{dx}, \\ \left(\frac{dR}{dy}\right) &= \left(\frac{dR}{dr}\right)\frac{dr}{dy} + \left(\frac{dR}{d\theta}\right)\frac{d\theta}{dy}. \end{aligned}\right\} \tag{186}$$

But
$$\left.\begin{aligned} dx &= dr\cos\theta - r\sin\theta\, d\theta, \\ dy &= dr\sin\theta + r\cos\theta\, d\theta. \end{aligned}\right\} \tag{187}$$

To calculate dx, dy must be equal to 0; whence, eliminating $d\theta$ and dr in turn from (187), we have

$$dx = dr\cos\theta - r\sin\theta\, d\theta, \qquad 0 = dr\sin\theta + r\cos\theta\, d\theta;$$

and therefore

$$\frac{dx}{dr} = \frac{1}{\cos\theta}, \qquad \frac{dx}{d\theta} = -\frac{r}{\sin\theta}. \tag{188}$$

Similarly to calculate dy, $dx = 0$; wherefore, by means of (187),

$$0 = dr\cos\theta - r\sin\theta\, d\theta, \qquad dy = dr\sin\theta + r\cos\theta\, d\theta;$$

whence
$$\frac{dy}{dr} = \frac{1}{\sin\theta}, \qquad \frac{dy}{d\theta} = \frac{r}{\cos\theta}; \tag{189}$$

and substituting these values in (186),

$$\left.\begin{aligned}\left(\frac{d\text{R}}{dx}\right) &= \left(\frac{d\text{R}}{dr}\right)\cos\theta - \left(\frac{d\text{R}}{d\theta}\right)\frac{\sin\theta}{r},\\ \left(\frac{d\text{R}}{dy}\right) &= \left(\frac{d\text{R}}{dr}\right)\sin\theta + \left(\frac{d\text{R}}{d\theta}\right)\frac{\cos\theta}{r}.\end{aligned}\right\} \tag{190}$$

Hence we have two transformations useful in the Planetary Theory, viz.

$$x\left(\frac{d\text{R}}{dy}\right) - y\left(\frac{d\text{R}}{dx}\right) = \left(\frac{d\text{R}}{d\theta}\right),$$

$$x\left(\frac{d\text{R}}{dx}\right) + y\left(\frac{d\text{R}}{dy}\right) = r\left(\frac{d\text{R}}{dr}\right).$$

Ex. 2. It is required to transform into its equivalent in terms of r, θ, and ϕ, $x\left(\frac{d\text{R}}{dx}\right) + y\left(\frac{d\text{R}}{dy}\right) + z\left(\frac{d\text{R}}{dz}\right)$, when $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

Let $r\sin\theta = \rho$; so that

$$\left.\begin{aligned}x &= \rho\cos\phi\\ y &= \rho\sin\phi\end{aligned}\right\}, \qquad \left.\begin{aligned}\rho &= r\sin\theta\\ z &= r\cos\theta\end{aligned}\right\};$$

therefore by the last example,

$$x\left(\frac{d\text{R}}{dx}\right) + y\left(\frac{d\text{R}}{dy}\right) = \rho\left(\frac{d\text{R}}{d\rho}\right),$$

$$\rho\left(\frac{d\text{R}}{d\rho}\right) + z\left(\frac{d\text{R}}{dz}\right) = r\left(\frac{d\text{R}}{dr}\right);$$

$$\therefore\quad x\left(\frac{d\text{R}}{dx}\right) + y\left(\frac{d\text{R}}{dy}\right) + z\left(\frac{d\text{R}}{dz}\right) = r\left(\frac{d\text{R}}{dr}\right).$$

108.] The general formulæ for the transformation of second and higher partial differentials and derived-functions, which are analogous to those of Art. 107, are evidently very long; and as their discovery is easy, it will be sufficient for our purpose to give one or two examples wherein particular and the most common forms of them appear.

Ex. 1. To transform $\left(\frac{d^2\text{v}}{dx^2}\right) + \left(\frac{d^2\text{v}}{dy^2}\right)$ into its equivalent in terms of r and θ, where $x = r\cos\theta$, $y = r\sin\theta$.

By the process of the preceding Article and (188), we have

$$\left(\frac{d\text{v}}{dx}\right) = \left(\frac{d\text{v}}{dr}\right)\cos\theta - \left(\frac{d\text{v}}{d\theta}\right)\frac{\sin\theta}{r}, \tag{191}$$

$$\left(\frac{d\text{v}}{dy}\right) = \left(\frac{d\text{v}}{dr}\right)\sin\theta + \left(\frac{d\text{v}}{d\theta}\right)\frac{\cos\theta}{r}; \tag{192}$$

differentiating (191), and bearing in mind that $\left(\frac{dv}{dr}\right)$ and $\left(\frac{dv}{d\theta}\right)$ are functions of r and θ, we have

$$\left(\frac{d^2v}{dx^2}\right) = \cos\theta\left\{\left(\frac{d^2v}{dr^2}\right)\frac{dr}{dx} + \left(\frac{d^2v}{dr\,d\theta}\right)\frac{d\theta}{dx}\right\}$$

$$-\frac{\sin\theta}{r}\left\{\left(\frac{d^2v}{dr\,d\theta}\right)\frac{dr}{dx} + \left(\frac{d^2v}{d\theta^2}\right)\frac{d\theta}{dx}\right\}$$

$$-\left(\frac{dv}{dr}\right)\sin\theta\frac{d\theta}{dx} - \left(\frac{dv}{d\theta}\right)\frac{r\cos\theta\,d\theta - \sin\theta\,dr}{r^2\,dx};$$

whence, substituting from (188), we have

$$\left(\frac{d^2v}{dx^2}\right) = (\cos\theta)^2\left(\frac{d^2v}{dr^2}\right) - \frac{2\sin\theta\cos\theta}{r}\left(\frac{d^2v}{dr\,d\theta}\right) + \frac{(\sin\theta)^2}{r^2}\left(\frac{d^2v}{d\theta^2}\right)$$

$$+\left(\frac{dv}{dr}\right)\frac{(\sin\theta)^2}{r} + \frac{2\sin\theta\cos\theta}{r^2}\left(\frac{dv}{d\theta}\right). \quad (193)$$

Similarly,

$$\left(\frac{d^2v}{dy^2}\right) = (\sin\theta)^2\left(\frac{d^2v}{dr^2}\right) + \frac{2\sin\theta\cos\theta}{r}\left(\frac{d^2v}{dr\,d\theta}\right) + \frac{(\cos\theta)^2}{r^2}\left(\frac{d^2v}{d\theta^2}\right)$$

$$+\left(\frac{dv}{dr}\right)\frac{(\cos\theta)^2}{r} - \frac{2\sin\theta\cos\theta}{r^2}\left(\frac{dv}{d\theta}\right); \quad (194)$$

$$\therefore \quad \left(\frac{d^2v}{dx^2}\right) + \left(\frac{d^2v}{dy^2}\right) = \left(\frac{d^2v}{dr^2}\right) + \frac{1}{r^2}\left(\frac{d^2v}{d\theta^2}\right) + \frac{1}{r}\left(\frac{dv}{dr}\right). \quad (195)$$

The results (193) and (194) are no other than particular cases of equation (100), Art. 80, when the right substitutions are made, and consistently with the independence and equicrescence of the variables.

Also by a process similar to that by which $\left(\frac{d^2v}{dx^2}\right)$ and $\left(\frac{d^2v}{dy^2}\right)$ have been found, it may be proved that

$$\left(\frac{d^2v}{dx\,dy}\right) = \sin\theta\cos\theta\left(\frac{d^2v}{dr^2}\right) + \frac{\cos 2\theta}{r}\left(\frac{d^2v}{dr\,d\theta}\right) - \frac{\sin\theta\cos\theta}{r^2}\left(\frac{d^2v}{d\theta^2}\right)$$

$$-\frac{\sin\theta\cos\theta}{r}\left(\frac{dV}{dr}\right) - \frac{\cos 2\theta}{r^2}\left(\frac{dv}{d\theta}\right). \quad (196)$$

Ex. 2. To transform $\left(\frac{d^2v}{dx^2}\right) + \left(\frac{d^2v}{dy^2}\right) + \left(\frac{d^2v}{dz^2}\right)$ into its equivalent in terms of r, θ, and ϕ, when $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

It is convenient in this example to introduce a subsidiary quantity ρ, so that

$$\left.\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{aligned}\right\}, \qquad \left.\begin{aligned} z &= r\cos\theta \\ \rho &= r\sin\theta \end{aligned}\right\}. \tag{197}$$

Then by the preceding example,

$$\left(\frac{d^2\mathrm{v}}{dx^2}\right) + \left(\frac{d^2\mathrm{v}}{dy^2}\right) = \left(\frac{d^2\mathrm{v}}{d\rho^2}\right) + \frac{1}{\rho^2}\left(\frac{d^2\mathrm{v}}{d\phi^2}\right) + \frac{1}{\rho}\left(\frac{d\mathrm{v}}{d\rho}\right), \tag{198}$$

$$\left(\frac{d^2\mathrm{v}}{dz^2}\right) + \left(\frac{d^2\mathrm{v}}{d\rho^2}\right) = \left(\frac{d^2\mathrm{v}}{dr^2}\right) + \frac{1}{r^2}\left(\frac{d^2\mathrm{v}}{d\theta^2}\right) + \frac{1}{r}\left(\frac{d\mathrm{v}}{dr}\right); \tag{199}$$

therefore by addition,

$$\left(\frac{d^2\mathrm{v}}{dx^2}\right) + \left(\frac{d^2\mathrm{v}}{dy^2}\right) + \left(\frac{d^2\mathrm{v}}{dz^2}\right)$$

$$= \left(\frac{d^2\mathrm{v}}{dr^2}\right) + \frac{1}{\rho^2}\left(\frac{d^2\mathrm{v}}{d\phi^2}\right) + \frac{1}{r^2}\left(\frac{d^2\mathrm{v}}{d\theta^2}\right) + \frac{1}{\rho}\left(\frac{d\mathrm{v}}{d\rho}\right) + \frac{1}{r}\left(\frac{d\mathrm{v}}{dr}\right); \tag{200}$$

but by (192), $\left(\frac{d\mathrm{v}}{d\rho}\right) = \left(\frac{d\mathrm{v}}{dr}\right)\sin\theta + \left(\frac{d\mathrm{v}}{d\theta}\right)\frac{\cos\theta}{r}$;

therefore substituting for ρ from (197) we have,

$$\left(\frac{d^2\mathrm{v}}{dx^2}\right) + \left(\frac{d^2\mathrm{v}}{dy^2}\right) + \left(\frac{d^2\mathrm{v}}{dz^2}\right) = \left(\frac{d^2\mathrm{v}}{dr^2}\right) + \frac{2}{r}\left(\frac{d\mathrm{v}}{dr}\right)$$

$$+ \frac{1}{r^2(\sin\theta)^2}\left(\frac{d^2\mathrm{v}}{d\phi^2}\right) + \frac{1}{r^2}\left(\frac{d^2\mathrm{v}}{d\theta^2}\right) + \frac{\cot\theta}{r^2}\left(\frac{d\mathrm{v}}{d\theta}\right). \tag{201}$$

To reduce this farther let $\cos\theta = \mu$; therefore $\theta = \cos^{-1}\mu$; and

$$\begin{aligned} \left(\frac{d^2\mathrm{v}}{d\theta^2}\right) + \cot\theta\left(\frac{d\mathrm{v}}{d\theta}\right) &= (1-\mu^2)\left(\frac{d^2\mathrm{v}}{d\mu^2}\right) - 2\mu\left(\frac{d\mathrm{v}}{d\mu}\right) \\ &= \frac{d}{d\mu}\left\{(1-\mu^2)\left(\frac{d\mathrm{v}}{d\mu}\right)\right\}. \end{aligned}$$

Also $$r^2\left(\frac{d^2\mathrm{v}}{dr^2}\right) + 2r\left(\frac{d\mathrm{v}}{dr}\right) = r\left(\frac{d^2.r\mathrm{v}}{dr^2}\right),$$

so that (201) becomes

$$r\left(\frac{d^2.r\mathrm{v}}{dr^2}\right) + \frac{1}{1-\mu^2}\left(\frac{d^2\mathrm{v}}{d\phi^2}\right) + \frac{d}{d\mu}\left\{(1-\mu^2)\left(\frac{d\mathrm{v}}{d\mu}\right)\right\} = 0. \tag{202}$$

I may observe, see Ex. 3, Art. 102, that the given differential equation expresses a property of

$$\mathrm{v} = \{(x-a)^2 + (y-b)^2 + (z-c)^2\}^{-\frac{1}{2}}.$$

If in this we replace x, y, z by their values given in (197), we have

$$v = \{r^2 - 2r(a\sin\theta\cos\phi + b\sin\theta\sin\phi + c\cos\theta) + a^2 + b^2 + c^2\}^{-\frac{1}{2}}; \quad (203)$$

so that (202) is a property of this last expression.

A further reduction of the equation (202) has been made by Mr. Hargreave in a paper in the Philosophical Transactions for 1841; but as it introduces considerations beside our present purpose, I must omit it. Also a discussion of some properties of it by Mr. Boole will be found in the first volume, page 10, of the Cambridge and Dublin Mathematical Journal. These memoirs however for the most part are with the view of inquiring whether (202) is or is not a differential equation derivable from some finite expression more general than (203).

CHAPTER IV.

THE RELATIONS BETWEEN FUNCTIONS AND DERIVED-FUNCTIONS, ON WHICH CERTAIN APPLICATIONS OF THE CALCULUS DEPEND.

109.] In the preceding part of the work, with the single exception of Art. 71, 72, we have considered the changes of the variables to be *infinitesimal,* and have estimated the changes of the functions and their derivatives as they are due to such infinitesimal increments; but many subsequent applications require a knowledge of the properties of functions, when the variables are increased by *finite* augments. One object of this Chapter is therefore to connect such finite changes with differentials and derived-functions; a relation of the kind has been established in equation (81) of Art. 72, the second member of which consists of a finite number of terms; and an accurate equality exists between the sum of them and the left-hand member, except so far as θ is an undetermined proper fraction. But the proof is insufficient, inasmuch as it does not afford answers to such questions as follow: Are *all* functions of the form $f(x+h)$ capable of expansion in a series of the form of the second member of equation (81); and if all are not, what are the characteristics of those which are? and supposing the function to be capable of expansion, can it be so expanded for all values of x and h, or is it possible for some and impossible for others? and what are the characteristics of x and h for which it is possible? and is the series true when continued to any number of terms, or must it cease at a certain term, because the requisite conditions are not satisfied by subsequent terms? Hence arises the necessity of proving certain theorems which establish relations between functions and their derivatives, and which involve certain conditions subject to which they are, and in failure of which they are not, within the general range of the Calculus; and by means of these, the infinitesimal calculus will be extended to changes of functions due to the finite changes of the variables.

110.] THEOREM I.—Given that $y = f(x)$ is a continuous function of x, for a given value of it, viz. $x = x_0$, and for values a little greater and a little less than x_0, that is, for $x_0 + dx$, and $x_0 - dx$: then, if $f'(x_0)$ is positive, x and $f(x)$ are, for that particular value, increasing or decreasing simultaneously; and if $f'(x_0)$ is negative, as x increases and passes through x_0, $f(x)$ is decreasing, or *vice versâ.*

Let Δy and Δx be, as before, the simultaneous and finite changes in the values of y and x; then it is plain, that according as $\frac{\Delta y}{\Delta x}$ is positive or negative when the increments are less than any assignable quantity, so is $\frac{dy}{dx}$ or $f'(x)$ which is its limit.

Since $\qquad y = f(x), \qquad y + \Delta y = f(x + \Delta x),$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x}.$$

On the supposition that $\frac{\Delta y}{\Delta x}$ is positive, the numerator and denominator of the fraction must have the same signs; and therefore, if $x + \Delta x$ is $> x$, that is, if x increases, $f(x + \Delta x)$ is $>$ than $f(x)$: but if $x + \Delta x$ is $< x$, that is, if x decreases, then $f(x + \Delta x)$ is $<$ than $f(x)$; and the same is true of the limiting value when Δx and Δy become dx and dy. So again if $\frac{\Delta y}{\Delta x}$ is negative, the numerator and denominator must have different signs; and therefore, if x increases, $f(x)$ must decrease: and if x decreases, $f(x)$ increases, and the same will be true in the limit. Hence we conclude that, if $\frac{dy}{dx} = f'(x)$ is positive for $x = x_0$, at that particular value, x and $f(x)$ are increasing or decreasing simultaneously; and if $f'(x)$ is negative when $x = x_0$, as x increases $f(x)$ decreases, or as x decreases $f(x)$ increases. Thus, if

$$f(x) = x^3 - 6x^2 + 11x - 6, \qquad f'(x) = 3x^2 - 12x + 11.$$

Now $f'(x)$ is positive when $x = 1$; therefore $f(x)$ and x are simultaneously increasing at that value of x; but $f'(x)$ is negative when $x = 2$; therefore as x increases, $f(x)$ is decreasing at that value.

COROLLARY I.—Hence if $f(x)$ is continuous for every value of x between x_0 and x_n, x_n being greater than x_0, x and $f(x)$ are increasing or decreasing simultaneously through all values for which $f'(x)$ is positive; and through all values for which $f'(x)$ is negative, as x increases, $f(x)$ decreases, and *vice versâ*.

Cor. II.—Hence also if, up to a certain value, $x = a$, as x increases $f(x)$ increases, and after that value $f(x)$ decreases, $f'(x)$ will be positive until $x = a$, and then will become negative; and as the sign of such a quantity changes only by the quantity passing through zero or infinity, according as the factor, to the change of sign of which the function's change of sign is due, is in the numerator or denominator, so, when $x=a$, will $f'(x)$ be equal to either zero or infinity.

In further illustration of this theorem and its corollaries, let us consider the following examples:

In fig. 10, let OBA be a semicircle, the radius of which is equal to a; let O be the origin and OCA the axis of x; then the equation to it is

$$y^2 = 2ax-x^2;$$

$$\therefore \quad f(x) = y = \{2ax-x^2\}^{\frac{1}{2}}, \qquad \frac{dy}{dx} = f'(x) = \frac{a-x}{(2ax-x^2)^{\frac{1}{2}}}.$$

Now for all values of x between 0 and a, $f'(x)$ is positive, and therefore as x increases, $f(x)$ increases also; and for all values of x between a and $2a$, $f'(x)$ is negative, and therefore as x increases, $f(x)$ decreases. And when $x = a$, $f'(x) = 0$, and changes sign from + to −.

Or again, let $y = f(x) = \sin x$;

$$\therefore \quad \frac{dy}{dx} = f'(x) = \cos x.$$

So that for all values of x in the first quadrant, $\cos x$ is positive, and x and $\sin x$ are simultaneously increasing or decreasing. But for all values of x in the second and third quadrants, $f'(x)$ or $\cos x$ is negative, and $\sin x$ decreases as x increases; and similarly for all other values of the arc.

It is also to be observed, that not only by its sign does $\frac{dy}{dx}$ or $f'(x)$ indicate whether an increase of the variable is accompanied contemporaneously by an increase or decrease of the function, but it also by its value denotes the *rate* of such increase or decrease; the greater $f'(x)$ is, if it is positive, the faster does $f(x)$ increase as x increases; and the less $f'(x)$ is, if it is negative, the slower does $f(x)$ decrease as x increases. Thus if $y = x$, the equation to a straight line passing through the origin, $dy = dx$, and the simultaneous increments of x and y are equal; but if $y = 2x$, $dy = 2\,dx$, and the increment of y is always twice that of x.

111.] THEOREM II.—If x_n and x_0 are two definite values of x, x_n being greater than x_0, and $x_n - x_0$ being a finite quantity; and if $\text{F}(x)$ is a function of x, which, as also its first derived-function, is finite and continuous for all values of x between x_n and x_0, then

$$\text{F}(x_n) - \text{F}(x_0) = (x_n - x_0)\,\text{F}'\{x_0 + \theta(x_n - x_0)\},$$

θ being some proper positive fraction.

Let the difference $x_n - x_0$ be divided into n parts, and let x_1, $x_2, \dots x_{n-1}$ be the values of x corresponding to the $n-1$ points of division; and let us moreover suppose n to be so large, that each of the divided elements, $x_1 - x_0$, $x_2 - x_1$, ... $x_n - x_{n-1}$, is an infinitesimal.

From the definition of a derived-function given in (6), Article 18, we have the following series of equations,

$$\left.\begin{aligned} \text{F}(x_1) - \text{F}(x_0) &= (x_1 - x_0)\,\text{F}'(x_0), \\ \text{F}(x_2) - \text{F}(x_1) &= (x_2 - x_1)\,\text{F}'(x_1), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \text{F}(x_n) - \text{F}(x_{n-1}) &= (x_n - x_{n-1})\,\text{F}'(x_{n-1}); \end{aligned}\right\} \qquad (1)$$

whence, adding all the first and second members of the series of equations, the sum of the first is $\text{F}(x_n) - \text{F}(x_0)$; and the sum of the second is, by Preliminary Theorem III, the product, of the sum of the first factors, viz. $x_n - x_0$, and some mean value of the second factors; that is,

$$\text{F}(x_n) - \text{F}(x_0) = (x_n - x_0)\,\text{F}'\{x_0 + \theta(x_n - x_0)\}, \qquad (2)$$

in which θ is some positive proper fraction: and therefore the last factor is a correct symbol of the required quantity; for putting $\theta = 0$, we have only the first term of the series, and which would therefore be correct if all the derived-functions in equations (1) were equal; and if $\theta = 1$, we have $\text{F}'(x_n)$, which is a term just beyond that which the series reaches; hence θ must be greater than zero and less than unity.

Let the finite difference $x_n - x_0$ be represented by h, so that

$x_n - x_0 = h$; and therefore $x_n = x_0 + h$, and h is the finite increment of x_0; then

$$\mathrm{F}(x_0+h) - \mathrm{F}(x_0) = h\,\mathrm{F}'(x_0+\theta h). \tag{3}$$

It is to be observed that, if h is infinitesimal, we must neglect θh when added to the finite quantity x_0; and the result is, the derived-function as originally defined in Art. 18.

112.] THEOREM III.—If x_n and x_0 are two definite values of x, x_n being greater than x_0 and $x_n - x_0$ being a finite quantity; and if $\mathrm{F}(x)$ and $f(x)$ are functions of x, which, as also their first-derived-functions, are finite and continuous for all values of x between x_0 and x_n; and besides, if $f'(x)$ does not change sign within these limits; then

$$\frac{\mathrm{F}(x_n) - \mathrm{F}(x_0)}{f(x_n) - f(x_0)} = \frac{\mathrm{F}'\{x_0 + \theta(x_n - x_0)\}}{f'\{x_0 + \theta(x_n - x_0)\}},$$

where θ represents a fraction mean to 0 and 1.

Let the difference $x_n - x_0$ be divided into n parts, and let x_1, $x_2, \dots x_{n-1}$ be the values of x corresponding to the $n-1$ points of division; and let us moreover suppose n to be infinite, and let each of the divided elements, $x_1 - x_0$, $x_2 - x_1, \dots x_n - x_{n-1}$, be an infinitesimal. Then, since

$$\mathrm{F}(x_1) - \mathrm{F}(x_0) = (x_1 - x_0)\,\mathrm{F}'(x_0),$$

and

$$f(x_1) - f(x_0) = (x_1 - x_0)\,f'(x_0);$$

$$\left.\begin{aligned} \therefore\quad & \frac{\mathrm{F}(x_1) - \mathrm{F}(x_0)}{f(x_1) - f(x_0)} = \frac{\mathrm{F}'(x_0)}{f'(x_0)}. \\ \text{Similarly,}\quad & \frac{\mathrm{F}(x_2) - \mathrm{F}(x_1)}{f(x_2) - f(x_1)} = \frac{\mathrm{F}'(x_1)}{f'(x_1)}, \\ & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ & \frac{\mathrm{F}(x_n) - \mathrm{F}(x_{n-1})}{f(x_n) - f(x_{n-1})} = \frac{\mathrm{F}'(x_{n-1})}{f'(x_{n-1})}. \end{aligned}\right\} \tag{4}$$

Now this being a series of fractions whose denominators are all of the same sign, according to Preliminary Theorem IV, the ratio of the sum of all the numerators of the left-hand members of the equations to the sum of all the denominators is equal to some mean value of the fractions; that is,

$$\frac{\mathrm{F}(x_n) - \mathrm{F}(x_0)}{f(x_n) - f(x_0)}$$

is equal to some mean value of the fractions which are the right-hand members of the above equations; and such a value will be properly represented by

$$\frac{\mathrm{F}'\{x_0+\theta(x_n-x_0)\}}{f'\{x_0+\theta(x_n-x_0)\}},$$

where θ is a proper and positive fraction; therefore

$$\frac{\mathrm{F}(x_n)-\mathrm{F}(x_0)}{f(x_n)-f(x_0)}=\frac{\mathrm{F}'\{x_0+\theta(x_n-x_0)\}}{f'\{x_0+\theta(x_n-x_0)\}}.$$

For the sake of convenience, let $x_n-x_0=h$, so that $x_n=x_0+h$; and therefore h is the finite increment of x_0, and the functions under consideration include those for all values of x between x_0 and x_0+h; then

$$\frac{\mathrm{F}(x_0+h)-\mathrm{F}(x_0)}{f(x_0+h)-f(x_0)}=\frac{\mathrm{F}'(x_0+\theta h)}{f'(x_0+\theta h)}. \tag{5}$$

The further condition to which $f(x)$ is subject, viz. that $f'(x)$ does not change sign between x_0 and x_0+h, is necessary, in order that the sum of the denominators of the right-hand members of (4) may not vanish, for thereby the first member of (5) might be equal to an infinite quantity, and the equation might be useless.

To enable a student the better to grasp the full meaning of the conditions of this important theorem, the graphical illustration of fig. 11 is given.

Of the two curves therein delineated, let the upper one be that whose equation is $y=\mathrm{F}(x)$, and let the lower one represent $y=f(x)$.

Let O be the origin, $\mathrm{OM}_0=x_0$, $\mathrm{OM}_n=x_n$; therefore $\mathrm{M}_n\mathrm{M}_0=h$; $\mathrm{OM}=x$, x referring to any point intermediate to M_0 and M_n; and suppose that $f'(x)$ is positive for all values of x between x_0 and x_n, which property is represented by the curve being such that the ordinate increases as the abscissa increases; now it is immaterial what forms, or branches, or points of discontinuity the curves may have outside the assigned limits; we consider them only within and between the limits, and restrict them to certain conditions within those limits, and which are expressed in the curves of the figure, viz. that they are finite and continuous; also

$$\mathrm{M}_0\mathrm{P}_0=\mathrm{F}(x_0), \qquad \mathrm{M}_0p_0=f(x_0),$$

$$\mathrm{M}_n\mathrm{P}_n=\mathrm{F}(x_n)=\mathrm{F}(x_0+h); \qquad \mathrm{M}_np_n=f(x_n)=f(x_0+h).$$

COR. I.—Suppose in the above equation (5), that $F(x_0) = 0$ and $f(x_0) = 0$, which expresses the condition that both the curves in fig. 11 pass through the point M_0, then

$$\frac{F(x_0+h)}{f(x_0+h)} = \frac{F'(x_0+\theta h)}{f'(x_0+\theta h)}. \qquad (6)$$

COR. II.—Also suppose that x_0, which is the lower limit of x, $= 0$, which expresses the condition that in fig. 11 the origin of coordinates is at M_0, then

$$\frac{F(h)-F(0)}{f(h)-f(0)} = \frac{F'(\theta h)}{f'(\theta h)},$$

and writing x for h, as h is measured from the origin M_0 in this case, we have

$$\frac{F(x)-F(0)}{f(x)-f(0)} = \frac{F'(\theta x)}{f'(\theta x)}; \qquad (7)$$

and if $F(0) = 0$, and $f(0) = 0$, that is, if both the curves pass through the origin,

$$\frac{F(x)}{f(x)} = \frac{F'(\theta x)}{f'(\theta x)}. \qquad (8)$$

113.] By Corollary I. of the last Article and equation (6), it appears that if any two functions of x vanish when $x = x_0$; then, subject to the necessary conditions,

$$\frac{F(x_0+h)}{f(x_0+h)} = \frac{F'(x_0+\theta h)}{f'(x_0+\theta h)} = \frac{F'(x_0+h_1)}{f'(x_0+h_1)}; \qquad (9)$$

replacing θh by h_1 for the sake of convenience, and observing therefore that h_1 is less than h.

Suppose now that $F'(x_0) = 0$, and $f'(x_0) = 0$, and that $F''(x)$ and $f''(x)$, as well as $F'(x)$ and $f'(x)$, are finite and continuous for all values of x between x_0 and $x_0+\theta h$; and that $f''(x)$ does not change sign within these limits; then, by virtue of (9), we have

$$\frac{F'(x_0+h_1)}{f'(x_0+h_1)} = \frac{F''(x_0+\theta h_1)}{f''(x_0+\theta h_1)};$$

and replacing θh_1 by h_2, and observing that h_2 is less than h_1, we have

$$\frac{F'(x_0+h_1)}{f'(x_0+h_1)} = \frac{F''(x_0+h_2)}{f''(x_0+h_2)}. \qquad (10)$$

Similarly again, if $F''(x_0) = 0$ and $f''(x_0) = 0$, and if the derived-functions $F'''(x)$, $f'''(x)$ are finite and continuous for all values of x between x_0 and x_0+h_2; and if, besides, $f''(x)$ always increases or decreases with x between these limits; then

$$\frac{\mathrm{F}''(x_0+h_2)}{f''(x_0+h_2)} = \frac{\mathrm{F}'''(x_0+\theta h_2)}{f'''(x_0+\theta h_2)} = \frac{\mathrm{F}'''(x_0+h_3)}{f'''(x_0+h_3)}; \qquad (11)$$

replacing θh_2 by h_3; whence it appears, that h_3, which is less than h_2, which is also less than h_1, and therefore less than h, is of the form θh.

Suppose now that these several conditions as to the derived-functions, and others similar to them, hold good up to the $(n-1)$th derived-functions inclusivcly, then

$$\frac{\mathrm{F}'(x_0+\theta h)}{f'(x_0+\theta h)} = \frac{\mathrm{F}^n(x_0+\theta h)}{f^n(x_0+\theta h)}.$$

But by (5) the left-hand member of this equation is equal to

$$\frac{\mathrm{F}(x_0+h)-\mathrm{F}(x_0)}{f(x_0+h)-f(x_0)};$$

so that we have the following proposition:

Theorem IV.—If $\mathrm{F}(x)$ and $f(x)$ are two functions of x, which, and also all the derived-functions up to the nth inclusively, are finite and continuous for all values of x between x_0 and x_0+h; and if

$$\mathrm{F}'(x_0) = 0, \quad \mathrm{F}''(x_0) = 0, \ \ldots\ldots \ \mathrm{F}^{n-1}(x_0) = 0,$$
$$f'(x_0) = 0, \quad f''(x_0) = 0, \ \ldots\ldots f^{n-1}(x_0) = 0;$$

and if, besides, $f'(x)$, $f''(x)$, ... $f^n(x)$ severally do not change sign between the limits; then

$$\frac{\mathrm{F}(x_0+h)-\mathrm{F}(x_0)}{f(x_0+h)-f(x_0)} = \frac{\mathrm{F}^n(x_0+\theta h)}{f^n(x_0+\theta h)}. \qquad (12)$$

Cor. I.—Hence, if $\mathrm{F}(x_0)=0$, and $f(x_0)=0$,

$$\frac{\mathrm{F}(x_0+h)}{f(x_0+h)} = \frac{\mathrm{F}^n(x_0+\theta h)}{f^n(x_0+\theta h)}. \qquad (13)$$

Cor. II.—Also, if x_0, which is the inferior value of x, $=0$; then, writing x for h, as h is measured from the origin, we have from (12),

$$\frac{\mathrm{F}(x)-\mathrm{F}(0)}{f(x)-f(0)} = \frac{\mathrm{F}^n(\theta x)}{f^n(\theta x)}. \qquad (14)$$

Cor. III.—And if in addition $\mathrm{F}(0)=0$, and $f(0)=0$, then

$$\frac{\mathrm{F}(x)}{f(x)} = \frac{\mathrm{F}^n(\theta x)}{f^n(\theta x)}. \qquad (15)$$

114.] In equation (12), which is the result including all others of the above Articles, let a specific form be assigned to $f(x)$, which will satisfy the requisite conditions and render the equation applicable to future purposes; as, for instance, let $f(x) = (x-x_0)^n$, wherein n is positive and integral; then

$$f(x_0) = 0, \quad \text{and} \quad f(x_0+h) = h^n;$$

$$\begin{aligned} f'(x) &= n(x-x_0)^{n-1}, & \therefore\ f'(x_0) &= 0, \\ f''(x) &= n(n-1)(x-x_0)^{n-2}, & f''(x_0) &= 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots & & \\ f^{n-1}(x) &= n(n-1)\ldots 3.2\,(x-x_0), & f^{n-1}(x_0) &= 0, \\ f^n(x) &= n(n-1)\ldots 3.2.1, & f^n(x_0) &= n(n-1)\ldots 3.2.1; \end{aligned}$$

and as $f^n(x)$ does not involve x, it has the same value whatever x is, therefore

$$f^n(x_0+\theta h) = n(n-1)\ldots 3.2.1;$$

and substituting these values in the general result, we have

$$\mathrm{F}(x_0+h)-\mathrm{F}(x_0) = \frac{h^n}{1.2.3\ldots(n-1)\,n}\,\mathrm{F}^n(x_0+\theta h); \qquad (16)$$

the conditions to which this equation is subject being, that $\mathrm{F}(x), \mathrm{F}'(x), \mathrm{F}''(x), \ldots \mathrm{F}^n(x)$ are finite and continuous for all values of x between x_0 and x_0+h, and that $\mathrm{F}'(x_0)=0$, $\mathrm{F}''(x_0)=0$, up to $\mathrm{F}^{n-1}(x_0)=0$; and that $\mathrm{F}^n(x_0)$ does not vanish.

115.] Cor. I.—Hence, if $\mathrm{F}(x_0)=0$, we have

$$\mathrm{F}(x_0+h) = \frac{h^n}{1.2.3\ldots n}\,\mathrm{F}^n(x_0+\theta h); \qquad (17)$$

and as a particular case of this, let $x_0=0$; then

$$\mathrm{F}(h) = \frac{h^n}{1.2.3\ldots n}\,\mathrm{F}^n(\theta h); \qquad (18)$$

and writing x for h,

$$\mathrm{F}(x) = \frac{x^n}{1.2.3\ldots n}\,\mathrm{F}^n(\theta x); \qquad (19)$$

which proposition may be enunciated as follows:

Theorem V.—If $\mathrm{F}(x)$ is a function of x, which and also all its derived-functions up to the nth inclusively are finite and continuous for all values of x between 0 and x; and if $\mathrm{F}(x)$, $\mathrm{F}'(x)$, $\ldots\ \mathrm{F}^{n-1}(x)$ severally vanish when $x=0$, then

$$\mathrm{F}(x) = \frac{x^n}{1.2.3\ldots n}\,\mathrm{F}^n(\theta x).$$

Cor. II.—In equation (16) let $x_0 = 0$; then writing x for h,

$$\text{F}(x) - \text{F}(0) = \frac{x^n}{1.2.3\ldots n} \text{F}^n(\theta x); \tag{20}$$

of which result (19) is a particular case.

116.] Let us consider some particular cases of these general theorems which arise from particular values of n.

In equation (16) suppose that $\text{F}'(x_0)$ does not vanish, then $n = 1$, and

$$\text{F}(x_0 + h) - \text{F}(x_0) = h\,\text{F}'(x_0 + \theta h); \tag{21}$$

which gives a symbolical expression for the right-hand member of equation (7), Art. 18, including its residual expression R.

Hence we have the following result. If for *all* values of x, $\text{F}'(x) = 0$, then also $\text{F}'(x_0 + \theta h) = 0$; and therefore by (21) $\text{F}(x_0 + h) = \text{F}(x_0)$. That is, as x_0 and h are arbitrary, the value of $\text{F}(x)$ is the same, whatever is the value of x; and therefore $\text{F}(x)$ is constant. The quantity therefore of which the derived function is zero is constant. Hence also conversely we infer that if a derived-function is zero for *all* values of its subject, then the function of which it is the derived is a constant; and also that a constant is the only continuous quantity of which the derived-function is zero.

Or suppose again that $\text{F}'(x_0) = 0$, and that $\text{F}''(x_0)$ is finite, then $n = 2$, and

$$\text{F}(x_0 + h) - \text{F}(x_0) = \frac{h^2}{1.2} \text{F}''(x_0 + \theta h). \tag{22}$$

Suppose again, in equation (20), that $\text{F}(0) = 0$, and that $\text{F}'(0)$ is finite, then

$$\text{F}(x) = x\,\text{F}'(\theta x); \tag{23}$$

and therefore every function of a variable x, which vanishes when $x = 0$, has x for a factor, unless the first derived-function is equal to ∞ when $x = 0$.

Thus, for example, let $\text{F}(x) = \sin x$, which $= 0$ if $x = 0$; then its derived-function $= \cos x$, which $= 1$ when $x = 0$; therefore x is a factor of $\sin x$.

Similarly, if $\text{F}(x) = \tan x$, which $= 0$ if $x = 0$; then $\text{F}'(x) = (\sec x)^2$, which $= 1$ when $x = 0$; therefore x is a factor of $\tan x$.

But if $\text{F}(x) = e^{-\frac{1}{x^2}}$, which $= 0$ if $x = 0$, $\text{F}'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$, which $= \frac{0}{0}$, when $x = 0$, and is therefore indeterminate; whence we cannot conclude that x is a factor of $e^{-\frac{1}{x^2}}$.

The statement made in most of the ordinary text books on the Theory of Algebraical Expressions, viz. that x is a factor of $F(x)$, if $F(x) = 0$ when $x = 0$; or, which is equivalent, that, if $F(x) = 0$ when $x = a$, $x-a$ is a factor of $F(x)$, is only one of too many *universal* propositions which are unduly assumed; it may be and is true in most cases, but it is an unphilosophical desire of generalization which leads us to conclude that it is true in *all* cases. All possible functions are not known, and cannot be known; and therefore a conscious ignorance ought to guard us from such an error, and be a much more cogent reason against it than the knowledge of a particular instance which disproves the induction; practically however it is found not to be so; let the student be cautious as to such universal assumptions, which rest for proof only on our ignorance.

CHAPTER V.

THE DETERMINATION OF THE ORDERS OF INFINITESIMALS; AND THE EVALUATION OF QUANTITIES WHICH FOR PARTICULAR VALUES OF THE VARIABLES ASSUME THE FORMS

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\times\infty,\quad \infty-\infty,\quad 0^0,\quad \infty^0,\quad 1^\infty,\quad 0^\infty.$$

117.] IN questions such as those proposed for discussion in the present Chapter, it is important to observe what is the exact meaning of the numerical unit; namely, that it is *the ratio of equality*, and that it is independent of the particular magnitude of the quantities which are compared. Hence it follows, that it is immaterial whether the quantities are infinitesimal, finite, or infinite, provided that we can assure ourselves that they are equal. Whenever therefore the same factor is involved in the numerator and denominator of a fraction, be it of any magnitude and kind, it may be divided out, and the value of the fraction will not be changed by the division: by this means expressions on which operations are to be performed can often be simplified; an instance will illustrate the process and its effect. Suppose that it is required to determine the value of

$$\frac{x-a+(2x^2-2ax)^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}},$$

when $x=a$; in which case the fraction assumes the form $\frac{0}{0}$; but why? Because both numerator and denominator involve a factor $(x-a)^{\frac{1}{2}}$, which is equal to 0 when $x=a$; but the factor in the numerator being exactly the same as the factor in the denominator, the ratio of one to the other is unity; by which therefore $\frac{(x-a)^{\frac{1}{2}}}{(x-a)^{\frac{1}{2}}}$ may be replaced, and the fraction becomes

$$\frac{(x-a)^{\frac{1}{2}}+(2x)^{\frac{1}{2}}}{(x+a)^{\frac{1}{2}}},$$

which is equal to 1, when $x=a$.

SECTION 1.—*The determination of orders of infinitesimals.*

118.] It is necessary first to repeat with greater exactness the main points of the account of infinitesimals which was given in rough outline in Art. 8 and 9, Chapter I.

The determination of the Order of Infinitesimals, such as is there assumed, which is a relative term, requires a standard or a base to compare them with. This standard is called the Base of Infinitesimals; and according to the power of the base, which a given infinitesimal expression involves, is its order called.

Thus suppose i to be the base; then, $a, b, c, \ldots\ldots$ being finite numbers, ai, bi^3, ci^n, are respectively infinitesimals of the first, third, nth orders; $i^{\frac{1}{2}}$, $i^{\frac{3}{4}}$ are infinitesimals respectively of the one-half and three-fourths orders. Similarly there may be infinitesimals of negative orders.

Suppose that $\mathrm{F}(i)$ is a function of infinitesimals, and an infinitesimal itself, whose order is to be compared with i as the base. If $\mathrm{F}(i)$ is an infinitesimal of the nth order, i^n is a factor of it, and the only other factor that enters with i^n is a finite quantity; and of all the terms which contain powers of i entering as factors into their composition, that containing i^n is the lowest: higher ones being neglected of necessity by virtue of Theorem VI, Art. 9. Therefore, if $\mathrm{F}(i)$ is divided by i^n, the quotient is finite; if it is divided by i raised to an index lower than n, the quotient is infinitesimal; and if it is divided by i raised to an index higher than n, the quotient is infinite. Hence we are enabled to construct the following definition of the order of an infinitesimal expression:

DEF.—Let $\mathrm{F}(i)$ be the infinitesimal expression whose order is to be determined; then, if $\frac{\mathrm{F}(i)}{i^r}$ is infinitesimal for all values of r less than n, and infinite for all values of r greater than n, $\mathrm{F}(i)$ is an infinitesimal of the nth order.

Hence every finite quantity is an infinitesimal of the order 0; for suppose k to be finite, then $\frac{k}{i^r}$ is infinitesimal for all values of r less than 0, and infinite for all values of r greater than 0.

Hence, if we use 0 for the general symbol of an infinitesimal, the form, which all finite quantities assume from this point of view, is 0^0. This is also evident from the example given in the last Article, where both numerator and denominator $= 0$, when

$x=a$; so that $x-a$ is an infinitesimal which enters into both numerator and denominator, whereby the fraction is of the form $\frac{k\times 0}{k'\times 0}$, k and k' being constants; and as the infinitesimals, being exactly equal, are of the same order, the form of the fraction is $0^0\frac{k}{k'}$; and as 0^0 is equal to 1 in this case, the true value of the fraction is $\frac{k}{k'}$.

119.] The results of the preceding Chapter enable us to construct a rule by which this definition may be applied to any function of infinitesimals; and by which the order may be determined.

If we replace x by i in equation (19), Art. 115, we have

$$\text{F}(i)=\frac{i^n}{1.2.3\ldots(n-1)n}\,\text{F}^n(\theta i). \tag{1}$$

Now observe the conditions to which $\text{F}(i)$ is subject; viz. that

$$\text{F}(0)=0,\quad \text{F}'(0)=0,\ \ldots\ldots\ \text{F}^{n-1}(0)=0,$$

and that $\text{F}^n(i)$ is the derivative, which first does not vanish when $i=0$; and that $\text{F}(i)$ and all its derived-functions up to the nth inclusively are continuous and finite for all values of i between i and 0; and observe also the limits, viz. i and 0; but that, as we have to use the equation only when $i=0$, we may make the difference between the limits to be infinitesimal. Hence

$$\frac{\text{F}(i)}{i^r}=\frac{i^{n-r}}{1.2.3\ldots(n-1)n}\,\text{F}^n(\theta i); \tag{2}$$

which is infinitesimal for all values of r less than n, and infinite for all values of r greater than n; and therefore we have the following theorem for the determination of the order of infinitesimals.

THEOREM.—If $\text{F}(i)$ is a function of an infinitesimal i, such that $\text{F}(0)=0$, and that all its derived-functions up to the nth vanish, when $i=0$; and that $\text{F}^n(i)$ does not vanish; then $\text{F}(i)$ is an infinitesimal of the nth order, if i is taken to be the infinitesimal-base.

If, when $i=0$, $\text{F}(i)$ assumes an indeterminate form, its value is to be found by the methods which are explained in the following sections of this chapter.

120.] In the following examples i is taken as the base of infinitesimals.

Ex. 1. To determine the order of infinitesimal of $\sin i$.

$$F(i) = \sin i = 0, \text{ when } i = 0;$$
$$F'(i) = \cos i = 1, \text{ - - - - }.$$

Therefore $\sin i$ is an infinitesimal of the first order.

Ex. 2. Find the order of infinitesimal of $\tan i - \sin i$.

$$F(i) = \tan i - \sin i = 0, \text{ when } i = 0;$$
$$F'(i) = (\sec i)^2 - \cos i = 0, \text{ - - - - };$$
$$F''(i) = 2(\sec i)^2 \tan i + \sin i = 0, \text{ when } i = 0,$$
$$F'''(i) = 6(\sec i)^4 - 4(\sec i)^2 + \cos i = 3, \text{ when } i = 0.$$

Thus $\tan i - \sin i$ is a function of an infinitesimal i which vanishes when $i = 0$; and so do all its derived-functions up to the third, which does not vanish; therefore $\tan i - \sin i$ is an infinitesimal of the third order, when i is the base.

Ex. 3. To determine the order of $e^i - 2\sin i - e^{-i}$.

$$F(i) = e^i - 2\sin i - e^{-i} = 0, \text{ when } i = 0;$$
$$F'(i) = e^i - 2\cos i + e^{-i} = 0, \text{ - - - - }$$
$$F''(i) = e^i + 2\sin i - e^{-i} = 0, \text{ - - - - }$$
$$F'''(i) = e^i + 2\cos i + e^{-i} = 4, \text{ - - - - - }.$$

Therefore $e^i - 2\sin i - e^{-i}$ is an infinitesimal of the third order.

Ex. 4. To determine the order of infinitesimal of $e^{i^2} - 1$.

$$F(i) = e^{i^2} - 1 = 0, \text{ when } i = 0;$$
$$F'(i) = 2ie^{i^2} = 0, \text{ - - - - }$$
$$F''(i) = 4i^2 e^{i^2} + 2e^{i^2} = 2, \text{ - - - - - }.$$

Therefore $e^{i^2} - 1$ is an infinitesimal of the second order.

Similarly let the student prove that $\tan i$, $\sin^{-1} i$, $e^i - e^{-i}$, are infinitesimals of the first order; that $\operatorname{versin} i$, $\sin i^2$ are infinitesimals of the second order; that $e^{-\frac{1}{i}}$ is an infinitesimal of the order ∞.

Similarly, if $F(x)$ is a function of x which vanishes when $x = a$, we may replace $x - a$ by i; in which case the function becomes $\phi(i)$, and vanishes when $i = 0$; and its order of infinitesimal may be determined in the manner above explained.

SECTION 2.—*Evaluation of quantities of the forms* $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$.

121.] Evaluation of quantities of the form $\frac{0}{0}$.

If a quotient of two functions of a variable or variables, for particular values of them, assumes the form $\frac{0}{0}$, it is plain that such is the case only because certain factors in the numerator and denominator become 0 for these particular values, that is, become infinitesimals. It is clear too from Theorem IV, Art. 9, that, if these infinitesimals are of the same order, the fraction will be a finite quantity; and, if that in the numerator is of a higher order than that in the denominator, the value of the fraction is 0; and, if that in the denominator is of a higher order than that in the numerator, the fraction is ∞: an example will render this plain.

Suppose that we have to evaluate $\frac{(x-a)^m \text{M}}{(x-a)^n \text{N}}$, when $x=a$; M and N being functions of x or not, as the case may be; but not involving any factor of the form $(x-a)$, and not vanishing when $x=a$; the fraction assumes the form $\frac{0}{0}$; but the law of indices authorizes us to put it under the form,

$$(x-a)^{m-n}\frac{\text{M}}{\text{N}};$$

and if $x=a$, it $=0$, if m is greater than n; it $=\frac{\text{M}}{\text{N}}$, if $m=n$; it $=\infty$, if m is less than n. It will be seen from Art. 125 that analogous results are true, if the numerator and denominator are infinities.

122.] Hence the first step towards the evaluation of such quantities is to detect, if possible, the factors common to both the numerator and the denominator, and to divide them out; and then to evaluate the resulting fraction by giving to the variables the assigned values; of this method some examples are subjoined.

Ex. 1. $$\frac{x^3-1}{x^3-2x^2+2x-1}=\frac{0}{0}, \text{ when } x=1;$$

$$=\frac{(x^2+x+1)(x-1)}{(x^2-x+1)(x-1)}=\frac{x^2+x+1}{x^2-x+1}=3, \text{ when } x=1.$$

Ex. 2. $\dfrac{(a^2-x^2)^{\frac{1}{2}}+(a-x)}{(a-x)^{\frac{1}{2}}+(a^3-x^3)^{\frac{1}{2}}} = \dfrac{0}{0}$, if $x = a$;

the common factor is $(a-x)^{\frac{1}{2}}$; divide numerator and denominator by it, and we have

$$\frac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{1+(a^2+ax+x^2)^{\frac{1}{2}}} = \frac{(2a)^{\frac{1}{2}}}{1+a3^{\frac{1}{2}}}, \text{ if } x = a.$$

123.] The preceding method of evaluating indeterminate forms may often be advantageously employed in one or other of the two following ways.

Firstly, expand by the Binomial Theorem, or by Maclaurin's Series, or some other equivalent method, the given functions in terms of the infinitesimal, and then by dividing out the common factors, the function will assume a determinate form.

Ex. 1. Determine the value of $\dfrac{e^x-e^{-x}}{\sin x}$, when $x = 0$.

$$\frac{e^x-e^{-x}}{\sin x} = \frac{1+\frac{x}{1}+\frac{x^2}{1.2}+\ldots\ldots-\left(1-\frac{x}{1}+\frac{x^2}{1.2}-\ldots\ldots\right)}{x-\frac{x^3}{1.2.3}+\ldots\ldots},$$

$$= \frac{2\left(x+\frac{x^3}{1.2.3}+\ldots\ldots\right)}{x-\frac{x^3}{1.2.3}+\ldots\ldots},$$

$$= \frac{2\left(1+\frac{x^2}{1.2.3}+\ldots\ldots\right)}{1-\frac{x^2}{1.2.3}+\ldots\ldots}, \quad = 2, \text{ when } x = 0.$$

Secondly, if the function assumes an indeterminate form when $x = a$, replace $x-a$, or $a-x$, as may be more convenient, by h; that is, for x substitute $a+h$, or $a-h$, in all the terms of the function; and develope them in series of ascending powers of h; and after cancelling, by means of division, the common powers of h, put $h = 0$; and the result is the determinate value of the function.

Ex. 1. Evaluate $\dfrac{x-a+(2ax-2a^2)^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}}$.

For x write $a+h$, and the function becomes

$$\frac{a+h-a+(2a^2+2ah-2a^2)^{\frac{1}{2}}}{\{(a+h)^2-a^2\}^{\frac{1}{2}}},$$

$$= \frac{h+(2ah)^{\frac{1}{2}}}{(2ah+h^2)^{\frac{1}{2}}},$$

$$= \frac{h^{\frac{1}{2}}+(2a)^{\frac{1}{2}}}{(2a+h)^{\frac{1}{2}}} = 1, \text{ when } h = 0.$$

124.] In cases however where it is difficult to detect the common factors, as well as in all cases where the necessary conditions are fulfilled, the theorems of the preceding Chapter enable us to evaluate these quantities.

Suppose $f(x)$ and $\phi(x)$ to be two functions of x, which $= 0$ when $x = x_0$. Suppose also that, when $x = x_0$, their several derived-functions up to the $(n-1)$th inclusively vanish, but that the nth neither vanishes nor becomes infinite: then, if $f(x)$ and $\phi(x)$ are finite and continuous for all values of x between x_0 and x_0+h, by the theorem contained in equation (17), Art. 115,

$$f(x_0+h) = \frac{h^n}{1.2.3\ldots n} f^n(x_0+\theta h);$$

and $$\phi(x_0+h) = \frac{h^n}{1.2.3\ldots n} \phi^n(x_0+\theta h).$$

$$\therefore \frac{f(x_0+h)}{\phi(x_0+h)} = \frac{f^n(x_0+\theta h)}{\phi^n(x_0+\theta h)}.$$

Suppose now that h, the difference between the superior and inferior limits, becomes infinitesimal, since we have to consider the values of the functions only at the particular value x_0 of the subject-variable; then neglecting h when added to x_0, we have

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f^n(x_0)}{\phi^n(x_0)}. \tag{3}$$

That is, the value of the ratio $\frac{f(x_0)}{\phi(x_0)}$, which presents itself under the indeterminate form $\frac{0}{0}$, is the ratio of the values of the derived-functions of the numerator and denominator, which are the first not to vanish, when $x = x_0$.

Of this general proposition the following are particular instances:

Suppose the functions themselves to vanish, and not their first-derived, then

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f'(x_0)}{\phi'(x_0)}; \tag{4}$$

and if the functions themselves and also their first-derived vanish, and not the second derived, then

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f''(x_0)}{\phi''(x_0)}; \tag{5}$$

and so on.

If the derived-functions which first cease to vanish are not of the same order, then the indeterminate form has for its value either 0 or ∞, according as the denominator or numerator has first ceased to vanish. This is plain from the proof we have given; because h will be of different powers in the equivalents of $f(x_0+h)$ and $\phi(x_0+h)$, and therefore will not disappear in the ratio.

Ex. 1. Evaluate $\dfrac{x^3-1}{x^3-2x^2+2x-1}$, when $x = 1$.

$$\frac{f(x)}{\phi(x)} = \frac{x^3-1}{x^3-2x^2+2x-1} = \frac{0}{0}, \text{ when } x = 1;$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{3x^2}{3x^2-4x+2} = 3, \text{ when } x = 1.$$

Ex. 2. Evaluate, $\dfrac{e^x-e^{-x}}{\sin x}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{e^x-e^{-x}}{\sin x} = \frac{0}{0}, \text{ when } x = 0;$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{e^x+e^{-x}}{\cos x} = 2, \text{ when } x = 0.$$

Ex. 3. Evaluate $\dfrac{1-\cos x}{x^2}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{1-\cos x}{x^2} = \frac{0}{0}, \text{ when } x = 0;$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{\sin x}{2x} = \frac{0}{0}, \text{ when } x = 0;$$

$$= \frac{f''(x)}{\phi''(x)} = \frac{\cos x}{2} = \frac{1}{2}, \text{ - - - - - .}$$

Ex. 4. Evaluate $\dfrac{e^x-2\sin x-e^{-x}}{x-\sin x}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{e^x - 2\sin x - e^{-x}}{x - \sin x} = \frac{0}{0}, \text{ when } x = 0;$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{e^x - 2\cos x + e^{-x}}{1 - \cos x} = \frac{0}{0}, \text{ - - - -}$$

$$= \frac{f''(x)}{\phi''(x)} = \frac{e^x + 2\sin x - e^{-x}}{\sin x} = \frac{0}{0}, \text{ - - - -}$$

$$= \frac{f'''(x)}{\phi'''(x)} = \frac{e^x + 2\cos x + e^{-x}}{\cos x} = 4, \text{ - - - - - .}$$

Ex. 5. Evaluate $\frac{(x-a)^n}{e^x - e^a}$, when $x = a$.

$$\frac{f(x)}{\phi(x)} = \frac{(x-a)^n}{e^x - e^a} = \frac{0}{0}, \text{ when } x = a,$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{n(x-a)^{n-1}}{e^x};$$

which, when $x = a$, is either 0, e^{-a}, or ∞, according as n is greater than, equal to, or less than, 1.

On examination of the above examples, it will be seen that the result is infinitesimal, finite or infinite, according as the order of infinitesimals in the numerator, determined by the method of Section 1 of this Chapter, is higher than, the same as, or lower than, the order of the denominator.

125.] Evaluation of indeterminate quantities of the form $\frac{\infty}{\infty}$.

Let $f(x)$ and $\phi(x)$ be two functions of x, which become infinite when $x = x_0$; then, as their reciprocals become infinitesimals, the ratio of the functions may be evaluated by the former method, as follows;

$$\frac{f(x)}{\phi(x)} = \frac{\infty}{\infty}, \text{ when } x = x_0;$$

$$= \frac{\dfrac{1}{\phi(x)}}{\dfrac{1}{f(x)}} = \frac{0}{0}, \text{ when } x = x_0;$$

$$= \frac{\dfrac{\phi'(x)}{\{\phi(x)\}^2}}{\dfrac{f'(x)}{\{f(x)\}^2}} = \frac{\phi'(x)}{f'(x)} \frac{\{f(x)\}^2}{\{\phi(x)\}^2};$$

whence, dividing out common terms, we have, when $x = x_0$,

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f'(x_0)}{\phi'(x_0)};$$

If the first-derived-functions $f'(x_0)$, $\phi'(x_0)$ also become infinite, their ratio must be evaluated in the same way as $\frac{f(x_0)}{\phi(x_0)}$; and we shall have

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f''(x_0)}{\phi''(x_0)};$$

and if the several derived-functions vanish or, if it is possible, become infinite up to the nth, when $x = x_0$, and the nth are finite,

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f'(x_0)}{\phi'(x_0)} = \dots\dots = \frac{f^n(x_0)}{\phi^n(x_0)}.$$

The determinate value therefore of such indeterminate functions is the ratio of those derived-functions of the numerator and denominator which first become finite when $x = x_0$.

Ex. 1. Evaluate $\frac{\log x}{\cot x}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{\log x}{\cot x} = \frac{-\infty}{\infty}, \text{ when } x = 0;$$

$$= \frac{f'(x)}{\phi'(x)} = -\frac{(\sin x)^2}{x} = \frac{0}{0}, \quad - \ - \ - \ -$$

$$= -\frac{2 \sin x \cos x}{1}, \text{ by the last Article,}$$

$$= 0, \text{ when } x = 0;$$

$$\therefore \quad \frac{\log x}{\cot x} = 0, \text{ when } x = 0.$$

Ex. 2. Evaluate $\frac{1 - \log x}{e^{\frac{1}{x}}}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{1 - \log x}{e^{\frac{1}{x}}} = \frac{\infty}{\infty}, \text{ when } x = 0;$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{\frac{1}{x}}{\frac{1}{x^2} e^{\frac{1}{x}}} = \frac{x}{e^{\frac{1}{x}}} = \frac{0}{\infty} = 0, \text{ when } x = 0.$$

Ex. 3. Evaluate $\frac{x^n}{e^x}$, when $x = \infty$.

$$\begin{aligned}\frac{f(x)}{\phi(x)} &= \frac{x^n}{e^x} = \frac{\infty}{\infty}, \text{ when } x = \infty,\\ &= \frac{f'(x)}{\phi'(x)} = \frac{n x^{n-1}}{e^x} = \frac{\infty}{\infty}, \quad - - - -\\ &= \ldots\ldots\ldots\ldots\\ &= \frac{f^n(x)}{\phi^n(x)} = \frac{n(n-1)(n-2)\ldots 3.2.1}{e^x} = 0, \text{ when } x = \infty.\end{aligned}$$

The result of which shews that e^x is, when $x = \infty$, an infinity of a higher order than x^n; also a similar property is true of a^x; a finite quantity therefore raised to an infinite power is an infinity of a higher order than the same infinite quantity raised to a finite power.

Ex. 4. Evaluate $\frac{e^{-\frac{1}{x^2}}}{x^{2n}}$, when $x = 0$.

Replace x^2 by $\frac{1}{z}$; so that $z = \infty$, when $x = 0$; and the function becomes

$$\begin{aligned}\frac{z^n}{e^z} &= \frac{\infty}{\infty}, \text{ when } z = \infty;\\ &= \frac{n z^{n-1}}{e^z} = \frac{\infty}{\infty},\\ &= \frac{n(n-1) z^{n-2}}{e^z} = \frac{\infty}{\infty},\\ &= \ldots\ldots\ldots\\ &= \frac{n(n-1)\ldots 3.2.1}{e^z} = 0, \text{ when } z = \infty;\end{aligned}$$

therefore the function is equal to 0, when $x = 0$; whence we conclude that $e^{-\frac{1}{x^2}}$ is an infinitesimal of a higher order than x^{2n}, however large be the value of n.

This is a result of some importance; insomuch as it shews that although $e^{-\frac{1}{x^2}}$ vanishes, when $x = 0$, whence it might be inferred that x is a factor of it, yet it admits of no factor of the form x^{2n} however large n be. Compare equation (23), Art. 116, and the remarks thereupon.

Ex. 5. Evaluate $\frac{\log x}{x}$, when $x = \infty$.

$$\begin{aligned}\frac{\log x}{x} &= \frac{\infty}{\infty}, \text{ when } x = \infty;\\ &= \frac{1}{x} = 0, \text{ when } x = \infty;\end{aligned}$$

if x therefore becomes infinite, it is an infinity of a higher order than its logarithm.

126.] Evaluation of functions which for particular values assume the indeterminate form $0 \times \infty$.

Let $f(x)$ and $\phi(x)$ be two functions of x which respectively become 0 and ∞, when $x = x_0$; then their product may be put under the form

$$\frac{f(x)}{\frac{1}{\phi(x)}}, \text{ which } = \frac{0}{0}, \text{ when } x = x_0,$$

and may be evaluated according to the method of the last two Articles.

Ex. 1. Evaluate $e^{-x}\log x$, when $x = \infty$.

$$e^{-x}\log x = \frac{\log x}{e^x} = \frac{\infty}{\infty}, \text{ when } x = \infty;$$

$$= \frac{1}{xe^x} = 0, \text{ when } x = \infty;$$

$$\therefore \quad e^{-x}\log x = 0, \text{ when } x = \infty.$$

Ex. 2. Evaluate $x^n \log x$, when $x = 0$.

$$x^n \log x = \frac{\log x}{x^{-n}} = \frac{1}{-nx^{-n}} = \frac{x^n}{-n} = 0, \text{ when } x = 0;$$

$$\therefore \quad x^n \log x = 0, \text{ when } x = 0.$$

127.] Evaluation of functions which for particular values assume the form $\infty - \infty$.

Let $f(x)$ and $\phi(x)$ be two functions of x, which $= 0$, when $x = x_0$; in which case, $\frac{1}{f(x)}$ and $\frac{1}{\phi(x)}$, both become ∞; then

$$\frac{1}{f(x_0)} - \frac{1}{\phi(x_0)} = \frac{\phi(x_0) - f(x_0)}{\phi(x_0)f(x_0)} = \frac{0}{0};$$

and may be evaluated as heretofore.

Ex. 1. Evaluate $\frac{2}{x^2-1} - \frac{1}{x-1}$, when $x = 1$.

$$\frac{2}{x^2-1} - \frac{1}{x-1} = \frac{2-x-1}{x^2-1} = \frac{0}{0}, \text{ when } x = 1,$$

$$= \frac{-1}{2x} = -\frac{1}{2}, \text{ when } x = 1.$$

Ex. 2. Evaluate $\sec x - \tan x$, when $x = \frac{\pi}{2}$.

$$\begin{aligned}\sec x - \tan x &= \infty - \infty, \text{ when } x = \frac{\pi}{2},\\ &= \frac{1-\sin x}{\cos x} = \frac{0}{0}, \text{ - - - -}\\ &= \frac{-\cos x}{-\sin x} = 0, \text{ when } x = \frac{\pi}{2};\end{aligned}$$

therefore, when $x = \frac{\pi}{2}$, $\sec x$ and $\tan x$ are either absolutely equal, or differ by a quantity which must be neglected in their algebraical sum.

Ex. 3. Evaluate $\frac{1}{\log x} - \frac{x}{\log x}$, when $x = 1$.

$$\begin{aligned}\frac{1}{\log x} - \frac{x}{\log x} = \frac{1-x}{\log x} &= \frac{0}{0}, \text{ when } x = 1,\\ &= -x = -1, \text{ when } x = 1.\end{aligned}$$

SECTION 3.—*Evaluation of Explicit Functions of a single variable, which for particular values assume the forms* 0^0, ∞^0, 1^∞, 0^∞.

128.] Let $f(x)$ and $\phi(x)$ be two functions of x which, when $x = x_0$, assume such values that $f(x_0)^{\phi(x_0)}$ is of one or other of the above forms.

Let $$y = f(x)^{\phi(x)};$$

$$\therefore \quad \log y = \phi(x) \log f(x);$$

and as $\log f(x)$ has singular values when $f(x) = 0$, or $= 1$, or $= \infty$, we may express the last equation in the form

$$\log y = \frac{\log f(x)}{\dfrac{1}{\phi(x)}},$$

which, for the particular value of x_0, will be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and may be evaluated according to the methods explained in the last Section.

Ex. 1. Evaluate x^x, when $x = 0$.

Let $\quad y = x^x, \qquad \therefore \quad \log y = x \log x;$

and by Article 126, Ex. 2, $x \log x = 0$, when $x = 0$; therefore $\log y = 0$; and since $\log 1 = 0$; therefore $y = x^x = 1$, when $x = 0$.

Ex. 2. Evaluate $x^{\frac{1}{x}}$, when $x = \infty$.

$$\text{Let} \qquad y = x^{\frac{1}{x}},$$

$$\therefore \quad \log y = \frac{\log x}{x} = \frac{\infty}{\infty}, \text{ when } x = \infty;$$

$$= \frac{1}{x} = 0, \text{ - - - - - .}$$

$$\therefore \quad y = x^{\frac{1}{x}} = 1, \text{ when } x = \infty.$$

Ex. 3. Evaluate $(1 + ax)^{\frac{b}{x}}$, when $x = 0$, and when $x = \infty$.

$$\text{Let} \qquad y = (1 + ax)^{\frac{b}{x}};$$

$$\therefore \quad \log y = \frac{b \log (1 + ax)}{x} = \frac{0}{0}, \text{ when } x = 0,$$

$$= \frac{ab}{1 + ax} = ab, \text{ - - - - - .}$$

$$\therefore \quad y = (1 + ax)^{\frac{b}{x}} = e^{ab}, \text{ when } x = 0.$$

$$\text{Again,} \qquad \log y = \frac{b \log (1 + ax)}{x} = \frac{\infty}{\infty}, \text{ when } x = \infty,$$

$$= \frac{ab}{1 + ax} = 0, \text{ when } x = \infty;$$

$$\therefore \quad (1 + ax)^{\frac{b}{x}} = 1, \text{ when } x = \infty.$$

Ex. 4. Evaluate $\left\{\frac{a_1^{nx} + a_2^{nx} + \dots + a_n^{nx}}{n}\right\}^{\frac{1}{x}}$, when $x = 0$.

$$\text{Let} \quad y = \left\{\frac{a_1^{nx} + a_2^{nx} + \dots + a_n^{nx}}{n}\right\}^{\frac{1}{x}},$$

$$\therefore \quad \log y = \frac{\log (a_1^{nx} + a_2^{nx} + \dots + a_n^{nx}) - \log n}{x} = \frac{0}{0}, \text{ if } x = 0$$

$$= \frac{n \log a_1 . a_1^{nx} + n \log a_2 . a_2^{nx} + \dots + n \log a_n . a_n^{nx}}{a_1^{nx} + a_2^{nx} + \dots + a_n^{nx}}$$

$$= \log (a_1 a_2 \dots a_n), \text{ if } x = 0.$$

$$\text{Therefore } \left\{\frac{a_1^{nx} + a_2^{nx} + \dots + a_n^{nx}}{n}\right\}^{\frac{1}{x}} = a_1 a_2 \dots a_n, \text{ if } x = 0.$$

CHAPTER VI.

ON EXPANSION OF FUNCTIONS.

SECTION 1.—*On functions of one variable.*

129.] THE Theorems of Chapter IV afford rigorous proofs and limits of application of Taylor's and Maclaurin's Theorems, of the general truth of which perhaps little more than a favourable presumption can be raised from what is said in Chap. III.

On referring to Art. 18, equation (6), it will be seen, that if dx is an infinitesimal increment of x, we have by the definition of a derived-function,

$$\mathrm{F}(x+dx)-\mathrm{F}(x) = \mathrm{F}'(x)\,dx; \tag{1}$$

but if Δx or (as we shall, to preserve an uniform notation, say) h, is finite, see Art. 18, (7),

$$\mathrm{F}(x+h)-\mathrm{F}(x) = h\,\mathrm{F}'(x)+\mathrm{R}_1, \tag{2}$$

writing R_1 for $\mathrm{R}h$. R_1 is therefore *a function of* h, which must be neglected when h is infinitesimal, but has a value finite and determinable when h is finite. Let us assume this function of h to be capable of expression in certain powers of h; then the problem which arises out of (2) takes the following form:

Given that $\mathrm{F}(x)$ is continuous and finite for all values of x between x and $x+h$, it is required to expand $\mathrm{F}(x+h)$ in a series of ascending powers of h.

We may also thus arrive at the equation (2) above.

If $\mathrm{F}'(x)$ does not vanish, and remains finite and continuous for all values of x between x and $x+h$, then by equation (21), Art. 116,

$$\mathrm{F}(x+h)-\mathrm{F}(x) = h\,\mathrm{F}'(x+\theta h);$$

as the right-hand member differs from $h\,\mathrm{F}'(x)$ by reason of the θ which enters into $\mathrm{F}'(x+\theta h)$, we may assume that

$$h\,\mathrm{F}'(x+\theta h) = h\,\mathrm{F}'(x)+\mathrm{R}_1;$$

have

$\mathrm{F}($ $h\,\mathrm{F}'(x)+\mathrm{R}_1.$

ed rom (2),

$x)-h\,\mathrm{F}'(x).$

therefore R_1 is a function of h which is equal to 0, when $h = 0$; also

$$\frac{d\mathrm{R}_1}{dh} = \mathrm{F}'(x+h) - \mathrm{F}'(x) = 0, \text{ when } h = 0;$$

$$\frac{d^2\mathrm{R}_1}{dh^2} = \mathrm{F}''(x+h), \tag{3}$$

which does not vanish, when $h = 0$.

Now by Theorem V, Art. 115, it appears that, If $f(h)$ is a function of h, which, as well as all its derived-functions up to the nth inclusively, is finite and continuous for all values of h between 0 and h; and if, in addition, $f(h)$ and its derived-functions up to the $(n-1)$th vanish, when $h = 0$; then

$$f(h) = \frac{h^n}{1.2.3\ldots n} f^n(\theta h).$$

Accordingly
$$\mathrm{R}_1 = \frac{h^2}{1.2}\mathrm{F}''(x+\theta h); \tag{4}$$

and substituting this value in equation (2) above, we have

$$\mathrm{F}(x+h) = \mathrm{F}(x) + h\,\mathrm{F}'(x) + \frac{h^2}{1.2}\mathrm{F}''(x+\theta h); \tag{5}$$

which may be written in the form,

$$\mathrm{F}(x+h) = \mathrm{F}(x) + h\,\mathrm{F}'(x) + \frac{h^2}{1.2}\mathrm{F}''(x) + \mathrm{R}_2.$$

To determine R_2, we have

$$\mathrm{R}_2 = \mathrm{F}(x+h) - \mathrm{F}(x) - h\,\mathrm{F}'(x) - \frac{h^2}{1.2}\mathrm{F}''(x); \tag{6}$$

and therefore R_2 is a function of h, which vanishes when $h = 0$;

its 1st derived, $\frac{d\mathrm{R}_2}{dh} = \mathrm{F}'(x+h) - \mathrm{F}'(x) - h\mathrm{F}''(x) = 0$, when $h = 0$;

its 2d derived, $\frac{d^2\mathrm{R}_2}{dh^2} = \mathrm{F}''(x+h) - \mathrm{F}''(x) = 0$, when $h = 0$;

its 3d derived, $\frac{d^3\mathrm{R}_2}{dh^3} = \mathrm{F}'''(x+h)$;

and which therefore does not vanish when $h = 0$. Hence, in accordance with the Theorem V, Art. 115, cited as above,

$$\mathrm{R}_2 = \frac{h^3}{1.2.3}\mathrm{F}'''(x+\theta h); \tag{7}$$

subject to the condition that $\mathrm{F}'(x)$, $\mathrm{F}''(x)$, $\mathrm{F}'''(x)$ are continuous and finite between the assigned limits. Substituting this value of R_2, we have

$$\mathrm{F}(x+h) - \mathrm{F}(x) - h\,\mathrm{F}'(x) - \frac{h^2}{1.2}\mathrm{F}''(x) = \frac{h^3}{1.2.3}\mathrm{F}'''(x+\theta h);$$

and continuing in the same manner, if all the derived-functions of $F(x)$ are finite and continuous between the assigned limits up to the nth inclusively, we have

$$F(x+h) - F(x) - F'(x)\frac{h}{1} - \ldots - F^{n-1}(x)\frac{h^{n-1}}{1.2.3\ldots(n-1)}$$
$$= \frac{h^n}{1.2.3\ldots n}F^n(x+\theta h);$$

and therefore

$$F(x+h) = F(x) + F'(x)\frac{h}{1} + F''(x)\frac{h^2}{1.2} + F'''(x)\frac{h^3}{1.2.3} + \ldots$$
$$\ldots + F^{n-1}(x)\frac{h^{n-1}}{1.2.3\ldots(n-1)} + \frac{h^n}{1.2.3\ldots n}F^n(x+\theta h). \quad (8)$$

This expression then gives the equivalent of $F(x+h)$ in terms of a series of ascending powers of h, and the conditions under which it has been formed indicate the cases in which the expansion is possible.

As applications of this series have already been given in Article 73, it is unnecessary to add others.

130.] In equation (8) the equality of the two members is perfect, and the development may be considered as completely effected, except so far as some indeterminateness arises from the nature of θ, to which quantity a specific value cannot be assigned. It was however before shewn that it must be some positive and proper fraction; and sometimes, if the series is convergent, when n is very great, the last terms and their sum become infinitesimal, and we must neglect

$$\frac{h^n}{1.2.3\ldots n}F^n(x+\theta h);$$

and we may write

$$F(x+h) = F(x) + F'(x)\frac{h}{1} + F''(x)\frac{h^2}{1.2} + F'''(x)\frac{h^3}{1.2.3} + \ldots \quad (9)$$

in which incomplete form the series was first given by Dr. Taylor, and now generally bears his name.

131] In equation (8) let $x = 0$; that is, let us consider the function for all values of the variable between 0 and h; and let us write x for h, remembering that x is the superior limit; whereby the conditions are, that none of the functions or derived-functions are infinite or discontinuous for any value of x between 0 and x; then

$$\text{F}(x) = \text{F}(0) + \text{F}'(0)\frac{x}{1} + \text{F}''(0)\frac{x^2}{1.2} + \ldots\ldots$$

$$\ldots\ldots + \text{F}^{n-1}(0)\frac{x^{n-1}}{1.2.3\ldots(n-1)} + \frac{x^n}{1.2.3\ldots n}\text{F}^n(\theta x); \quad (10)$$

this is Maclaurin's Theorem, of which an imperfect proof was given in Art. 57; and it accordingly appears that it is only a particular case of Taylor's. Many examples of this series having been given in Chapter III, it is unnecessary to add others. Of the general series (10) however the following are particular instances:

Let $n = 1$, then $\text{F}(x) = \text{F}(0) + x\,\text{F}'(\theta x)$. (11)

$$\text{Thus} \quad e^x = 1 + xe^{\theta x}, \qquad \therefore \quad \frac{e^x - 1}{x} = e^{\theta x}.$$

$$\sin x = x\cos\theta x, \qquad \frac{\sin x}{x} = \cos\theta x.$$

$$\log(1+x) = \frac{x}{1+\theta x}.$$

132.] If the series (10) is such that $\frac{x^n}{1.2.3\ldots n}\text{F}^n(\theta x)$ becomes infinitesimal as n increases without limit, then

$$\text{F}(x) = \text{F}(0) + x\,\text{F}'(0) + \frac{x^2}{1.2}\text{F}''(0) + \frac{x^3}{1.2\,3}\text{F}'''(0) + \ldots\ldots \quad (12)$$

that is, the limit of the sum of the second member of the equation is $\text{F}(x)$.

133.] Again, taking the other form of Taylor's Series, viz. (84) in Art. 74, we have

$$\text{F}(x) = \text{F}(a) + (x-a)\,\text{F}'(a) + \frac{(x-a)^2}{1.2}\text{F}''(a) + \ldots\ldots$$

$$+ \frac{(x-a)^{n-1}}{1.2\ldots(n-1)}\text{F}^{n-1}(a) + \frac{(x-a)^n}{1.2\ldots(n-1)n}\text{F}^n\{a+\theta(x-a)\}, \quad (13)$$

the superior and inferior values of x in this case being respectively x and a; so that it is for all values of x between these limits that the conditions are to be satisfied.

As particular cases of the formula we have

$$\text{F}(x) = \text{F}(a) + (x-a)\,\text{F}'\{a+\theta(x-a)\}, \quad (14)$$

$$\text{F}(x) = \text{F}(a) + (x-a)\,\text{F}'(a) + \frac{(x-a)^2}{1.2}\text{F}''\{a+\theta(x-a)\}. \quad (15)$$

134.] Hence it appears that if we stop at the nth term of

Taylor's Series, and neglect to take account of all terms after it, the error committed by so doing is

$$\frac{h^n}{1.2.3\ldots n}\,\mathrm{F}^n(x+\theta h); \tag{16}$$

and as θ lies between 0 and 1, the error lies between

$$\frac{h^n}{1.2.3\ldots n}\,\{\mathrm{F}^n(x) \text{ and } \mathrm{F}^n(x+h)\}; \tag{17}$$

which is the same value as that before determined in Art. 72.

Similarly, if we stop at the nth term of Maclaurin's Series, and do not take account of the terms after it, the error is

$$\frac{x^n}{1.2.3\ldots n}\,\mathrm{F}^n(\theta x). \tag{18}$$

which is the same value as that before determined in Art. 58.

The expressions (16) and (18) are called respectively the *limits* of Taylor's and Maclaurin's Theorems.

135.] The series given in Article 133 enables us to put the limit of Taylor's Series under another form which is sometimes convenient.

Let us represent the last term of (13) by $\phi(a)$; so that

$$\phi(a) = \frac{(x-a)^n}{1.2.3\ldots n}\,\mathrm{F}^n\{a+\theta(x-a)\}; \tag{19}$$

$$\therefore \quad \mathrm{F}(x) = \mathrm{F}(a) + \frac{x-a}{1}\,\mathrm{F}'(a) + \frac{(x-a)^2}{1.2}\,\mathrm{F}''(a) + \ldots\ldots$$

$$\ldots\ldots + \frac{(x-a)^{n-1}}{1.2.3\ldots(n-1)}\,\mathrm{F}^{n-1}(a) + \phi(a). \tag{20}$$

Let a be variable, and let us differentiate this; and we have

$$\frac{(x-a)^{n-1}}{1.2.3\ldots(n-1)}\,\mathrm{F}^n(a) + \phi'(a) = 0. \tag{21}$$

It is plain from (19) that $\phi(x) = 0$; therefore, writing in (14) ϕ for F, we have

$$\phi(a) + (x-a)\,\phi'\{a+\theta_1(x-a)\} = 0,$$

θ_1 being different from θ in (19);

$$\therefore \quad \phi(a) = -(x-a)\,\phi'\{a+\theta_1(x-a)\}. \tag{22}$$

In (21) for a write $a+\theta_1(x-a)$, and we have

$$\frac{(x-a)^{n-1}(1-\theta_1)^{n-1}}{1.2.3\ldots(n-1)}\,\mathrm{F}^n\{a+\theta_1(x-a)\} + \phi'\{a+\theta_1(x-a)\} = 0; \tag{23}$$

whence, by elimination between (22) and (23),

$$\phi(a) = \frac{(1-\theta_1)^{n-1}(x-a)^n}{1.2.3\ldots(n-1)} \text{F}^n\{a+\theta_1(x-a)\}; \tag{24}$$

and substituting in (13) this particular form of the remainder, we have

$$\text{F}(x) = \text{F}(a) + \frac{x-a}{1}\text{F}'(a) + \frac{(x-a)^2}{1.2}\text{F}''(a) + \ldots\ldots$$
$$+\frac{(x-a)^{n-1}}{1.2\ldots(n-1)}\text{F}^{n-1}(a) + \frac{(x-a)^n(1-\theta)^{n-1}}{1.2\ldots(n-1)}\text{F}^n\{a+\theta(x-a)\}. \tag{25}$$

Substituting in this series $a+h$ for x, and subsequently writing x for a, we have

$$\text{F}(x+h) = \text{F}(x) + h\,\text{F}'(x) + \frac{h^2}{1.2}(\text{F}''x) + \ldots\ldots$$
$$\ldots + \frac{h^{n-1}}{1.2\ldots(n-1)}\text{F}^{n-1}(x) + \frac{h^n(1-\theta)^{n-1}}{1.2\ldots(n-1)}\text{F}^n(x+\theta h). \tag{26}$$

The corresponding expression for the remainder, in Maclaurin's Series, is

$$\frac{(1-\theta)^{n-1}x^n}{1.2.3\ldots(n-1)}\text{F}^n(\theta x). \tag{27}$$

Two examples of the remainders of series in these forms are subjoined.

Ex. 1. $\text{F}(x) = a^x; \qquad \therefore \quad \text{F}^n(x) = a^x(\log a)^n;$

and therefore the sum of all the terms after the nth in the expansion of a^x is

$$\frac{(1-\theta)^{n-1}x^n}{1.2.3\ldots(n-1)}(\log a)^n a^{\theta x}.$$

Ex. 2. $\text{F}(x) = \sin mx; \qquad \therefore \quad \text{F}^n(x) = m^n \sin\left(mx+n\frac{\pi}{2}\right);$

therefore the sum of all the terms after the nth of $\sin m(x+h)$ is

$$\frac{h^n(1-\theta)^{n-1}m^n}{1.2.3\ldots(n-1)}\sin\left\{m(x+\theta h)+n\frac{\pi}{2}\right\}.$$

136.] In these last Articles, as well as in Chapter IV, we have frequently used the expression "finite and continuous for all values of x between" certain limits; now although we cannot tell *all* the cases wherein these conditions are, and are not fulfilled, for a complete knowledge of *all* functions and *all* their properties would be necessary to our doing so, yet we do know certain cases where they are not satisfied; these we propose to discuss briefly. In them Taylor's and Maclaurin's Series are said to *fail;* which is surely an incorrect term, for we are endeavouring to bring all functions within a particular formula, which is true only of *some;* that is, we are trying to make that which is true only when certain conditions are satisfied, comprise all, whether such conditions are fulfilled or not. We shall omit the consideration of discontinuous functions, as such are excluded by their very forms, and confine our attention to those which for particular values of the variable become infinite, and on that account fail to satisfy the requisite conditions.

First let us consider Taylor's Series. To develope $\mathrm{F}(x+h)$ to $n+1$ terms, and so that the last term of the series may comprise the sum of all the terms after the nth, and be

$$\mathrm{F}^n(x+\theta h)\frac{h^n}{1.2.3\ldots n},$$

it is necessary that $\mathrm{F}(x)$, and all its derived-functions up to $\mathrm{F}^n(x)$ inclusively, should not be infinite for any value of x between x and $x+h$. Now let the following effects of differentiation on functions be observed; (1) that all rational and integral functions of x, and all functions of the form $x^n+cx^{-m}+ex^{\frac{p}{q}}$ are lowered one dimension by it; and that therefore, if n is positive and integral, the nth derived-function of x^n is a constant, and the $(n+1)$th is zero; that however many times x^{-m} is differentiated, the result is never zero, but that the negative exponent is increased; that if $\frac{p}{q}$ is a fraction proper or improper, some derived-function of it, sooner or later, has a negative exponent, and therefore some power of x appears in the denominator; (2) that if $e^{f(x)}$ is a factor of the original function, it is also a factor of every one of the successive derived-functions.

Excluding all consideration as to values outside those for which the series is applied, let a be a particular value of x within them; and suppose that when $x=a$, $\mathrm{F}^n(x)$ is finite, but $\mathrm{F}^{n+1}(x)$ is infinite: then it is plain that some factor of the form $x-a$ must, in passing from $\mathrm{F}^n(x)$ to $\mathrm{F}^{n+1}(x)$, have been introduced into the denominator; and therefore, as far as the above remarks go, there must have been in the original function a factor of the form $(x-a)^{m+\frac{p}{q}}$, where m is an integral and positive number, and $\frac{p}{q}$ is a *proper* fraction. And if the primitive and all its derived-functions are infinite, when $x = a$, there must in the original function have been a factor of the form $(x-a)^{-m}$.

Thus in the former case, suppose that we have to expand

$$F(x) = f(x) + (x-a)^{m+\frac{p}{q}}\phi(x),$$

in which neither $f(x)$ nor $\phi(x)$ involves factors of the form $x-a$; then all the derived-functions up to $F^m(x)$ will be finite, when $x = a$, but the subsequent ones will be infinite; the expansion therefore must not be carried beyond the mth term; and the addition of

$$\frac{h^m}{1.2.3\ldots m}F^m(x+\theta h),$$

will make the equation exact.

Suppose, for instance, that it is required to expand $F(a+h)$, having given

$$F(x) = x^4 + (x-a)^{\frac{5}{2}}\sin x;$$

then a is the least value of x for which the function is considered, and $a+h$ is the greatest; and

$$F'(x) = 4x^3 + \frac{5}{2}(x-a)^{\frac{3}{2}}\sin x + (x-a)^{\frac{5}{2}}\cos x,$$

$$F''(x) = 12x^2 + \frac{15}{4}(x-a)^{\frac{1}{2}}\sin x + 5(x-a)^{\frac{3}{2}}\cos x - (x-a)^{\frac{5}{2}}\sin x.$$

But if we form $F'''(x)$, it involves $(x-a)^{-\frac{1}{2}}$, which becomes infinite when $x = a$, and therefore fails to fulfil the conditions under which equation (8) has been determined: therefore, in this case,

$$F(x+h) = F(x) + \frac{h}{1}F'(x) + \frac{h^2}{1.2}F''(x+\theta h);$$

and substituting the specific values above given, and putting $x = a$,

$$F(a+h) = a^4 + 4a^3\frac{h}{1} + \frac{h^2}{1.2}\{12(a+\theta h)^2\}$$

$$+ \frac{15}{4}(\theta h)^{\frac{1}{2}}\sin(a+\theta h) + 5(\theta h)^{\frac{3}{2}}\cos(a+\theta h) - (\theta h)^{\frac{5}{2}}\sin(a+\theta h)\},$$

θ being a positive and proper fraction. On inspection it is plain that the first three terms of the series are correct terms of the expansion of $(a+h)^4$, and that they might have been carried further; and that the last three express $(x-a)^{\frac{5}{2}}\sin x$, when $x = a+h$.

Now observing that the last three terms involve $h^{\frac{1}{2}}$, we have a good illustration of the reason why Taylor's Series "fails;" viz. because we are attempting to express by a formula which involves only integral powers of h, a function which in its development requires fractional powers. Another and perhaps better illustration is the following: Let us consider the case of $(x^2-a^2)^{\frac{1}{2}}$, which is to be developed, when $x = a+h$; here then, when $x = a$, the original function $= 0$, and the first-derived is ∞; but as the function may be written in the form $(x-a)^{\frac{1}{2}}(x+a)^{\frac{1}{2}}$, this becomes, when $x = a+h$, $h^{\frac{1}{2}}(2a+h)^{\frac{1}{2}}$, the second factor of which may be developed in the ordinary way, and thereby the whole development will consist of terms of the powers of h whose exponents are of the form $n+\frac{1}{2}$, n being a whole number.

Similarly, if the original function has factors of the form $(x-a)^{-m}$, that and all its derived-functions are infinite when $x = a$. Thus suppose that we have to develope by Taylor's Series, when $x = a+h$,

$$\text{F}(x) = \frac{\sin x}{x^2-a^2};$$

this and all its derived-functions become infinite when $x = a$; but the function may be written in the form

$$\frac{1}{x-a}\,\frac{\sin x}{x+a},$$

the first factor of which, when $x = a+h$, becomes $\frac{1}{h}$, and the second fulfils the conditions of Taylor's Series and may be expanded in the ordinary way; but the resulting development will have at least one term involving a negative power of h, which indicates the cause of the failure, viz. that the function does not admit of development in ascending integral and positive powers of h, which alone is given by Taylor's Theorem.

Secondly, as to the failure of Maclaurin's Theorem. Two cases correspond to those of Taylor's Series above discussed. In the former, the original function contains factors of the form $x^{m+\frac{p}{q}}$; and therefore, when these are differentiated more than m times, powers of x with negative indices are introduced; and these become infinite when $x = 0$. Although therefore the series will be correct up to the mth term, yet the development must not be carried beyond that term, and the limit will be the symbolical expression for the sum of all the terms after it. In the latter case the original function, when broken into factors, contains one or more of the form x^{-m}; and thus becomes infinite when $x = 0$. Of this kind of failure cosec x and cotan x are instances: see Ex. 3 and 4, Art. 88; for of each of these when expanded x^{-1} is the first term. And therefore their equivalent series cannot be made to correspond with that of Maclaurin.

Another similar case is that of $\log x$; because this, and also all its derived-functions, $= \infty$ when $x = 0$.

Also another case is $e^{-\frac{1}{x^2}}$; because it and all its derived-functions vanish when $x = 0$; and hence we infer that this function does not admit of development into a series of ascending integral and positive powers of x, which alone is given by Maclaurin's Series.

Thus if the function to be expanded by Maclaurin's Theorem is

$$F(x) = f(x) + e^{-\frac{1}{x^2}},$$

the development of $F(x)$ will be that of $f(x)$ alone; and therefore will not be correct.

There may also be many other functions which are not capable of development in the forms of Taylor's and Maclaurin's Series; and therefore the student must be on his guard against attempts at expansion in a form not suited to the function. The above remarks must be considered as general hints rather than as a secure and scientific enumeration of cases of applicability and non-applicability; the latter is beyond our knowledge, and therefore with the former, however incomplete, we must be content.

There is also another reason why we have treated thus at length of the cases where Taylor's and Maclaurin's Theorems fail; some foreign and most English writers have attempted to raise the Differential Calculus on them as its basis. As far as they are applicable, reasoning founded on them may be correct; but since they are not universally so, it is objected, and validly objected, that the basis of the Calculus is confined within limits narrower than are necessary. And no criteria are known for determining whether functions can be expanded in *their* forms or not, before such principles of continuity, as those which we have made fundamental, have been elucidated; on the expansion-principle therefore we are left to grope our way in the dark, being uncertain whether the matter which we are discussing is within its comprehension or not.

SECTION 2.—*On functions of many variables.*

137.] In Chapter IV. certain relations have been investigated between functions and derived-functions of one variable x, and chiefly when many of the derived-functions vanish for particular values of the variables. Similar theorems are true of functions of two and more variables; but I was unwilling in that Chapter to interrupt the discussion of a difficult question, or in the following Chapter to break in upon the immediate application of it, by introducing functions of a more complex order. Now we must return to the subject, and I shall shew that for functions of two and more variables, theorems are true analogous to those which have been proved for functions of one variable. We will take at first a function of two independent variables x and y.

Let $\text{F}(x,y)$ be the function under consideration; and let us suppose the function to be finite and continuous for all values of x and y between x_0 and x_0+h, y_0 and y_0+k; and let us moreover suppose all and each of the several and successive partial derived-functions up to the nth to vanish, when $x = x_0$ and $y = y_0$; but the nth not to vanish nor to become infinite at these values: then the following theorem is true, as I shall just now prove,

$$\frac{1}{1.2\ldots n} \qquad \text{F}(x_0+h, y_0+k)-\text{F}(x_0, y_0) = {}_{\wedge}\Big[\Big(\frac{d^n\text{F}}{dx^n}\Big)h^n + \frac{n}{1}\Big(\frac{d^n\text{F}}{dx^{n-1}\,dy}\Big)h^{n-1}k + \ldots$$

$$\ldots + \frac{n}{1}\Big(\frac{d^n\text{F}}{dx\,dy^{n-1}}\Big)h\,k^{n-1} + \Big(\frac{d^n\text{F}}{dy^n}\Big)k^n\Big]_{\substack{x_0+\theta h\\ y_0+\theta k}};$$

the meaning of the notation at the end of the right-hand member being that x and y are to be respectively replaced by $x_0+\theta h$ and $y_0+\theta k$.

Let us suppose the finite increments of x_0 and y_0 to be ht and kt; so that ultimately they become h and k, when $t = 1$; thus we may consider $\text{F}(x_0+ht, y_0+kt)$ to be a function of t; so that

$$\text{F}(x_0+ht,\ y_0+kt) = f(t); \tag{28}$$

and therefore

$$\text{F}(x_0, y_0) = f(0). \tag{29}$$

Now by (20), Art. 115, if $f(t)$ is finite and continuous for all values of t between $t = t$ and $t = 0$; and if moreover all the derived-functions of $f(t)$ up to the $(n-1)$th inclusive vanish when $t = 0$, but if $f^n(t)$ does not vanish and is not infinite; then

$$f(t) - f(0) = \frac{t^n}{1.2 \ldots n} f^n(\theta t). \tag{30}$$

To apply this to the present case, I must deduce from (28) the several derived-functions of $f(t)$. To simplify the process, let

$$\left.\begin{array}{lll} x_0 + ht = x', & \therefore & \dfrac{dx'}{dt} = h; \\ y_0 + kt = y', & \therefore & \dfrac{dy'}{dt} = k; \end{array}\right\} \tag{31}$$

$$\therefore \quad f(t) = \text{F}(x', y'),$$

and,

$$f'(t) = \left(\frac{d\text{F}}{dx'}\right)\frac{dx'}{dt} + \left(\frac{d\text{F}}{dy'}\right)\frac{dy'}{dt}$$

$$= \left(\frac{d\text{F}}{dx'}\right)h + \left(\frac{d\text{F}}{dy'}\right)k; \tag{32}$$

and observing that when $t = 0$, $x' = x_0$ and $y' = y_0$; we have

$$f'(0) = \left(\frac{d\text{F}}{dx}\right)_0 h + \left(\frac{d\text{F}}{dy}\right)_0 k;$$

the notation in the last line indicating that x and y are replaced by x_0 and y_0 respectively.

If therefore $\left(\frac{d\text{F}}{dx}\right)_0 = 0$, and $\left(\frac{d\text{F}}{dy}\right)_0 = 0$, these circumstances are equivalent to the vanishing of $f'(t)$, when $t = 0$. Again,

$$f''(t) = \frac{d.f'(t)}{dt} = \left(\frac{d^2\text{F}}{dx'^2}\right)\frac{dx'^2}{dt^2} + 2\left(\frac{d^2\text{F}}{dx'\,dy'}\right)\frac{dx'\,dy'}{dt^2} + \left(\frac{d^2\text{F}}{dy'^2}\right)\frac{dy'^2}{dt^2}, \tag{33}$$

bearing in mind that t is equicrescent; otherwise (30) would not be true.

$$\therefore \quad f''(0) = \left(\frac{d^2\text{F}}{dx^2}\right)_0 h^2 + 2\left(\frac{d^2\text{F}}{dx\,dy}\right)_0 hk + \left(\frac{d^2\text{F}}{dy^2}\right)_0 k^2; \tag{34}$$

and therefore if all the second partial derived-functions of $\text{F}(x, y)$ vanish, when $x = x_0$ and $y = y_0$, these circumstances are equivalent to the vanishing of $f''(t)$, when $t = 0$. Similarly

$$f'''(0) = \left(\frac{d^3\text{F}}{dx^3}\right)_0 h^3 + 3\left(\frac{d^3\text{F}}{dx^2\,dy}\right)_0 h^2k + 3\left(\frac{d^3\text{F}}{dx\,dy^2}\right)_0 hk^2 + \left(\frac{d^3\text{F}}{dy^3}\right)_0 k^3, \tag{35}$$

. . . .

$$f^n(t) = \left(\frac{d^n\text{F}}{dx^n}\right)_{,} h^n + \frac{n}{1}\left(\frac{d^n\text{F}}{dx^{n-1}\,dy}\right)_{,} h^{n-1}k + \ldots + \frac{n}{1}\left(\frac{d^n\text{F}}{dx\,dy^{n-1}}\right)_{,} hk^{n-1} + \left(\frac{d^n\text{F}}{dy^n}\right)_{,} k^n; \tag{36}$$

the notation being used to signify that x and y are replaced by x' and y'.

And thus if all the partial derived-functions of $\mathrm{F}(x,y)$ of the $(n-1)$th order vanish when $x=x_0$ and $y=y_0$, these circumstances are equivalent to the vanishing of $f^{n-1}(0)$; and $f^n(t)$ is given by (36). Suppose now all the partial derived-functions of $\mathrm{F}(x,y)$ up to the $(n-1)$th inclusive to vanish, when $x=x_0$ and $y=y_0$: but that all the nth do not vanish; then substituting in (30) we have

$$\mathrm{F}(x_0+ht,\, y_0+kt) - \mathrm{F}(x_0,\, y_0)$$
$$= \frac{t^n}{1.2.3\ldots n}\Big[\Big(\frac{d^n\mathrm{F}}{dx^n}\Big)h^n + \frac{n}{1}\Big(\frac{d^n\mathrm{F}}{dx^{n-1}\,dy}\Big)h^{n-1}k + \ldots$$
$$\ldots + \frac{n}{1}\Big(\frac{d^n\mathrm{F}}{dx\,dy^{n-1}}\Big)hk^{n-1} + \Big(\frac{d^n\mathrm{F}}{dy^n}\Big)k^n\Big]_{\substack{x_0+\theta ht\\ y_0+\theta kt.}} \qquad (37)$$

In this equation let $t=1$; then

$$\mathrm{F}(x_0+h,\, y_0+k) - \mathrm{F}(x_0,\, y_0)$$
$$= \frac{1}{1.2.3\ldots n}\Big[\Big(\frac{d^n\mathrm{F}}{dx^n}\Big)h^n + \frac{n}{1}\Big(\frac{d^n\mathrm{F}}{dx^{n-1}\,dy}\Big)h^{n-1}k + \ldots$$
$$\ldots + \frac{n}{1}\Big(\frac{d^n\mathrm{F}}{dx\,dy^{n-1}}\Big)hk^{n-1} + \Big(\frac{d^n\mathrm{F}}{dy^n}\Big)k^n\Big]_{\substack{x_0+\theta h\\ y_0+\theta k,}} \qquad (38)$$

which is the theorem required to be proved; and is analogous to that contained in equation (16), Art. 114.

If the given function is of more variables than two, and if all its several partial derived-functions up to those of the $(n-1)$th order inclusive vanish for particular values of the variables, a similar theorem is also true. But as the proof is the same as that which has been applied to a function of two variables, it is unnecessary to insert it.

138.] Suppose x_0 and y_0 to be each zero in (38); then, replacing h and k by x and y, we have

$$\mathrm{F}(x,y) - \mathrm{F}(0,0) = \frac{1}{1.2\ldots n}\Big[\Big(\frac{d^n\mathrm{F}}{dx^n}\Big)h^n + \Big(\frac{d^n\mathrm{F}}{dx^{n-1}\,dy}\Big)h^{n-1}k + \ldots$$
$$\ldots + \Big(\frac{d^n\mathrm{F}}{dx\,dy^{n-1}}\Big)hk^{n-1} + \Big(\frac{d^n\mathrm{F}}{dy^n}\Big)k^n\Big]_{\substack{\theta x\\ \theta y;}} \qquad (39)$$

in which all the derived-functions of $\mathrm{F}(x,y)$ up to those of the

$(n-1)$th order inclusive vanish, when $x=0$ and $y=0$. This theorem is analogous to that of equation (19), Art. 115.

139.] If the ratio to each other of two functions of two or more variables is for particular values of the variables indeterminate, and if a function of two or more variables takes for particular values of the variables one or other of the indeterminate forms given in Sect. 3, Chap. V, we are hereby enabled to evaluate them. We will take functions of two variables only, because the expressions are shorter; and the method required for those of more variables is the same in form.

Let $F(x,y)$ and $\Phi(x,y)$ be two functions of x and y which vanish when $x=x_0$ and $y=y_0$; and let us take the most general case, and assume that all the successive partial derived-functions up to those of the $(n-1)$th order inclusive also vanish for these values of the variables; and that all the nth derived-functions neither vanish nor become infinite; then by (38),

$$F(x_0+h, y_0+k) = \frac{1}{1.2.3\ldots n}\Big[\Big(\frac{d^nF}{dx^n}\Big)h^n + \frac{n}{1}\Big(\frac{d^nF}{dx^{n-1}dy}\Big)h^{n-1}k + \ldots$$

$$\ldots + \frac{n}{1}\Big(\frac{d^nF}{dx\,dy^{n-1}}\Big)hk^{n-1} + \Big(\frac{d^nF}{dy^n}\Big)k^n\Big]_{\substack{x_0+\theta h\\ y_0+\theta k,}} \quad (40)$$

$$\Phi(x_0+h, y_0+k) = \frac{1}{1.2.3\ldots n}\Big[\Big(\frac{d^n\Phi}{dx^n}\Big)h^n + \frac{n}{1}\Big(\frac{d^n\Phi}{dx^{n-1}dy}\Big)h^{n-1}k + \ldots$$

$$\ldots + \frac{n}{1}\Big(\frac{d^n\Phi}{dx\,dy^{n-1}}\Big)hk^{n-1} + \Big(\frac{d^n\Phi}{dy^n}\Big)k^n\Big]_{\substack{x_0+\theta h\\ y_0+\theta k.}} \quad (41)$$

Let us suppose h and k to become infinitesimal; to become, that is, dx and dy; then if we neglect those terms which are infinitesimal and are added to finite quantities, and divide (40) by (41), we have

$$\frac{F(x_0, y_0)}{\Phi(x_0, y_0)} =$$

$$\frac{\Big(\frac{d^nF}{dx^n}\Big)_0 dx^n + n\Big(\frac{d^nF}{dx^{n-1}dy}\Big)_0 dx^{n-1}dy + \ldots + n\Big(\frac{d^nF}{dx\,dy^{n-1}}\Big)_0 dx\,dy^{n-1} + \Big(\frac{d^nF}{dy^n}\Big)_0 dy^n}{\Big(\frac{d^n\Phi}{dx^n}\Big)_0 dx^n + n\Big(\frac{d^n\Phi}{dx^{n-1}dy}\Big)_0 dx^{n-1}dy + \ldots + n\Big(\frac{d^n\Phi}{dx\,dy^{n-1}}\Big)_0 dx\,dy^{n-1} + \Big(\frac{d^n\Phi}{dy^n}\Big)_0 dy^n}. \quad (42)$$

To take particular cases, let us suppose the functions to vanish when $x=x_0$ and $y=y_0$, and their first derived-functions not to vanish; then

$$\frac{\text{F}(x_0, y_0)}{\Phi(x_0, y_0)} = \frac{\left(\frac{d\text{F}}{dx}\right)_0 dx + \left(\frac{d\text{F}}{dy}\right)_0 dy}{\left(\frac{d\Phi}{dx}\right)_0 dx + \left(\frac{d\Phi}{dy}\right)_0 dy}. \tag{43}$$

And again, let us suppose the functions, and their first derived, to vanish when $x = x_0$, $y = y_0$, and the second derived not all to vanish, then

$$\frac{\text{F}(x_0, y_0)}{\Phi(x_0, y_0)} = \frac{\left(\frac{d^2\text{F}}{dx^2}\right)_0 dx^2 + 2\left(\frac{d^2\text{F}}{dx\,dy}\right)_0 dx\,dy + \left(\frac{d^2\text{F}}{dy^2}\right)_0 dy^2}{\left(\frac{d^2\Phi}{dx^2}\right)_0 dx^2 + 2\left(\frac{d^2\Phi}{dx\,dy}\right)_0 dx\,dy + \left(\frac{d^2\Phi}{dy^2}\right)_0 dy^2}.$$

It will be observed that I have assumed the dx's and the dy's to be the same in both functions: and the result must necessarily involve the ratio of these to each other. If however $dx = m\,dy$, where m is a constant multiplier, the result will be independent of the infinitesimal variations.

The following is a case in which the preceding process of evaluation is frequently required. Let $\text{F}(x, y) = c$ be an implicit function of two variables; whence after differentiation we have

$$\frac{dy}{dx} = -\frac{\left(\frac{d\text{F}}{dx}\right)}{\left(\frac{d\text{F}}{dy}\right)}; \tag{44}$$

and suppose x_0, y_0 to be values of x and y which simultaneously satisfy the given equation, and are also such that $\left(\frac{d\text{F}}{dx}\right)_0 = 0$, and $\left(\frac{d\text{F}}{dy}\right)_0 = 0$; then $\frac{dy}{dx}$ becomes indeterminate, and is to be evaluated according to the preceding method. Let us take the total differentials of the numerator and the denominator; then (44) becomes

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\left(\frac{d\text{F}}{dx}\right)}{\left(\frac{d\text{F}}{dy}\right)} \\ &= -\frac{\left(\frac{d^2\text{F}}{dx^2}\right) dx + \left(\frac{d^2\text{F}}{dx\,dy}\right) dy}{\left(\frac{d^2\text{F}}{dx\,dy}\right) dx + \left(\frac{d^2\text{F}}{dy^2}\right) dy}; \end{aligned} \tag{45}$$

and let us suppose all these second derived-functions not to vanish nor to become infinite when $x = x_0$ and $y = y_0$; then we have

$$\left(\frac{d^2\text{F}}{dx^2}\right)_0 dx^2 + 2\left(\frac{d^2\text{F}}{dx\,dy}\right)_0 dx\,dy + \left(\frac{d^2\text{F}}{dy^2}\right)_0 dy^2 = 0; \qquad (46)$$

whence the ratio of dy to dx may be determined.

Again, if all the second derived-functions vanish when $x = x_0$ and $y = y_0$, then taking the total differentials of the numerator and the denominator of (45), we have

$$\frac{dy}{dx} = -\frac{\left(\frac{d^3\text{F}}{dx^3}\right)dx^2 + 2\left(\frac{d^3\text{F}}{dx^2\,dy}\right)dx\,dy + \left(\frac{d^3\text{F}}{dx\,dy^2}\right)dy^2}{\left(\frac{d^3\text{F}}{dx^2\,dy}\right)dx^2 + 2\left(\frac{d^3\text{F}}{dx\,dy^2}\right)dx\,dy + \left(\frac{d^3\text{F}}{dy^3}\right)dy^2}; \qquad (47)$$

and let us suppose all these third derived-functions not to vanish nor to become infinite when $x = x_0$ and $y = y_0$; then (47) becomes

$$\left(\frac{d^3\text{F}}{dx^3}\right)_0 dx^3 + 3\left(\frac{d^3\text{F}}{dx^2\,dy}\right)_0 dx^2\,dy + 3\left(\frac{d^3\text{F}}{dx\,dy^2}\right)_0 dx\,dy^2 + \left(\frac{d^3\text{F}}{dy^3}\right)_0 dy^3 = 0; \quad (48)$$

whence $\frac{dy}{dx}$ may be found by the solution of a cubic equation.

It will be observed that in the total differentials of the numerator and denominator in (42) x and y are equicrescent.

Ex. 1. If $ay^2 - x^3 - bx^2 = 0$, $\frac{dy}{dx} = \frac{3x^2+2bx}{2ay} = \frac{0}{0}$, when $x = 0$ and $y = 0$; it is required to determine the values of $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{3x^2+2bx}{2ay} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{6x\,dx + 2b\,dx}{2a\,dy} = \frac{b\,dx}{a\,dy}, \quad - \; - \; - \; - \; ;$$

$$\therefore \quad \frac{dy^2}{dx^2} = \frac{b}{a}; \qquad \frac{dy}{dx} = \pm\left(\frac{b}{a}\right)^{\frac{1}{2}}.$$

Ex. 2. If $x^4 + ay^3 - 2axy^2 - 3ax^2y = 0$,

$$\frac{dy}{dx} = \frac{4x^3 - 6axy - 2ay^2}{3ax^2 + 4axy - 3ay^2} = \frac{0}{0}, \text{ if } x = y = 0;$$

it is required to determine the values of $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{4x^3 - 6axy - 2ay^2}{3ax^2 + 4axy - 3ay^2} = \frac{0}{0}, \text{ if } x = y = 0,$$

$$= \frac{(12x^2 - 6ay)\,dx - (6ax + 4ay)\,dy}{(6ax + 4ay)\,dx + (4ax - 6ay)\,dy} = \frac{0}{0}, \quad - \; - \; - \; - \; - \; .$$

$$= \frac{24x\,dx^3 - 12a\,dx\,dy - 4a\,dy^2}{6a\,dx^2 + 8a\,dx\,dy - 6a\,dy^2}$$

$$= \frac{-12a\,dx\,dy - 4a\,dy^2}{6a\,dx^2 + 8a\,dx\,dy - 6a\,dy^2}, \text{ when } x = y = 0.$$

$$\therefore\ \frac{dy^3}{dx^3} - 2\frac{dy^2}{dx^2} - 3\frac{dy}{dx} = 0;$$

whence we have $\frac{dy}{dx} = 0$, and $= 3$, and $= -1$.

Other examples will occur subsequently in which $\frac{dy}{dx}$ has to be evaluated in a similar way, when $x = y = \infty$.

140.] Again, let it be required to find in ascending powers of h and k the value of $\text{F}(x+h, y+k)$, $\text{F}(x,y)$ and all its partial derived-functions up to those of the nth order inclusive being finite and continuous for all values of x and y between the values x and $x+h$, y and $y+k$; h and k being finite increments of x and y.

Let us make the same substitutions in terms of t as in Art. 137. Then by Maclaurin's Theorem, Art. 58, (13), we have

$$f(t) = f(0) + f'(0)\frac{t}{1} + f''(0)\frac{t^2}{1.2} + \dots\dots$$
$$\dots + f^{n-1}(0)\frac{t^{n-1}}{1.2.3\dots(n-1)} + f^n(\theta t)\frac{t^n}{1.2.3\dots n}; \quad (49)$$

and substituting for $f(t)$, $f(0)$, $f'(0)$, $f''(0)$, $\dots f^n(t)$ the values given in Art. 137, except that we will replace x_0 and y_0 by the general values x and y, we have

$$\begin{aligned}
\text{F}(x+ht, y+kt) = \text{F}(x,y) & \\
&+ \frac{t}{1}\left\{\left(\frac{d\text{F}}{dx}\right)h + \left(\frac{d\text{F}}{dy}\right)k\right\} \\
&+ \frac{t^2}{1.2}\left\{\left(\frac{d^2\text{F}}{dx^2}\right)h^2 + 2\left(\frac{d^2\text{F}}{dx\,dy}\right)hk + \left(\frac{d^2\text{F}}{dy^2}\right)k^2\right\} \\
&+ \frac{t^3}{1.2.3}\left\{\left(\frac{d^3\text{F}}{dx^3}\right)h^3 + 3\left(\frac{d^3\text{F}}{dx^2\,dy}\right)h^2k + \dots + \left(\frac{d^3\text{F}}{dy^3}\right)k^3\right\} \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots \\
&+ \frac{t^n}{1.2.3\dots n}\left[\left(\frac{d^n\text{F}}{dx^n}\right)h^n + n\left(\frac{d^n\text{F}}{dx^{n-1}\,dy}\right)h^{n-1}k + \dots\right. \\
&\qquad\qquad \left.\dots + \left(\frac{d^n\text{F}}{dy^n}\right)k^n\right]_{\substack{x+\theta ht \\ y+\theta kt.}} \quad (50)
\end{aligned}$$

Let $t = 1$; then we have

$$\begin{aligned}\text{F}(x+h,\, y+k) &= \text{F}(x,y)\\ &+ \left(\frac{d\text{F}}{dx}\right) h + \left(\frac{d\text{F}}{dy}\right) k\\ &+ \frac{1}{1.2}\left\{\left(\frac{d^2\text{F}}{dx^2}\right) h^2 + 2\left(\frac{d^2\text{F}}{dx\,dy}\right) hk + \left(\frac{d^2\text{F}}{dy^2}\right) k^2\right\}\\ &+ \frac{1}{1.2.3}\left\{\left(\frac{d^3\text{F}}{dx^3}\right) h^3 + 3\left(\frac{d^3\text{F}}{dx^2\,dy}\right) h^2 k + \ldots + \left(\frac{d^3\text{F}}{dy^3}\right) k^3\right\}\\ &+ \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\\ &+ \frac{1}{1.2.3\ldots n}\Bigg[\left(\frac{d^n\text{F}}{dx^n}\right) h^n + n\left(\frac{d^n\text{F}}{dx^{n-1}\,dy}\right) h^{n-1}k + \ldots\ldots\\ &\qquad\qquad \ldots + \left(\frac{d^n\text{F}}{dy^n}\right) k^n\Bigg]_{\substack{x+\theta h,\\ y+\theta k}}; \qquad (51)\end{aligned}$$

which is the required development. The following are examples in which the expansion is applied.

141.] Ex. 1. Given that $\text{F}(x,y) = x^2(a+y)^3$; it is required to find $(x+h)^2(a+y+k)^3$.

$$\left(\frac{d\text{F}}{dx}\right) = 2x(a+y)^3, \quad \left(\frac{d^2\text{F}}{dx^2}\right) = 2(a+y)^3, \quad \left(\frac{d^3\text{F}}{dx^3}\right) = 0,$$

$$\left(\frac{d\text{F}}{dy}\right) = 3x^2(a+y)^2, \quad \left(\frac{d^2\text{F}}{dx\,dy}\right) = 6x(a+y)^2, \quad \left(\frac{d^3\text{F}}{dx^2\,dy}\right) = 6(a+y)^2,$$

$$\left(\frac{d^2\text{F}}{dy^2}\right) = 6x^2(a+y), \quad \left(\frac{d^3\text{F}}{dx\,dy^2}\right) = 12x(a+y),$$

$$\left(\frac{d^3\text{F}}{dy^3}\right) = 6x^2;$$

and omitting to write down the subsequent partial derived-functions which vanish, we have $\left(\frac{d^4\text{F}}{dx^2\,dy^2}\right) = 12(a+y)$, $\left(\frac{d^4\text{F}}{dx\,dy^3}\right) = 12x$, $\left(\frac{d^5\text{F}}{dx^2\,dy^3}\right) = 12$; whence, substituting in equation (51),

$$\begin{aligned}(x+h)^2(a+y+k)^3 = x^2(a+y)^3 &+ 2x(a+y)^3 h + 3x^2(a+y)^2 k\\ &+ (a+y)^3 h^2 + 6x(a+y)^2 hk + 3x^2(a+y)\, k^2\\ &+ 3(a+y)^2 h^2 k + 6x(a+y)\, hk^2 + x^2 k^3\\ &+ 3(a+y)\, h^2 k^2 + 2x\, hk^3 + h^2 k^3.\end{aligned}$$

Ex. 2. The equation to a conic is

$$\text{F}(x,y) = \text{A}x^2+\text{B}xy+\text{C}y^2+\text{E}x+\text{G}y+\text{K} = 0;$$

it is required to prove that $\left(\frac{d\text{F}}{dx}\right) = 0$, $\left(\frac{d\text{F}}{dy}\right) = 0$, are the equations to two diameters which are conjugate to diameters parallel to the axes of x and y respectively.

In the first place, it is manifest that $\left(\frac{d\text{F}}{dx}\right) = 0$, and $\left(\frac{d\text{F}}{dy}\right) = 0$ are two equations of the first degree in x and y, and therefore represent straight lines. Let the origin of coordinates be moved to a point (h, k) by replacing x and y by $x+h$ and $y+k$ respectively. And let it be observed that the given equation is only of two dimensions in terms of x and y; and that therefore all the partial derived-functions of it after the second vanish. In $\text{F}(x,y)$ let x and y be replaced by $x+h$ and $y+k$; and let us expand in powers of x and y; and we have

$$\text{F}(x+h, y+k) = 0 = \text{F}(h,k) + \left(\frac{d\text{F}}{dh}\right)x + \left(\frac{d\text{F}}{dk}\right)y$$
$$+\frac{1}{1.2}\left\{\left(\frac{d^2\text{F}}{dh^2}\right)x^2+2\left(\frac{d^2\text{F}}{dh\,dk}\right)xy+\left(\frac{d^2\text{F}}{dk^2}\right)y^2\right\}. \quad (52)$$

See MS.

Wrong

Now if h and k are such that $\left(\frac{d\text{F}}{dh}\right) = 0$, (52) has no term involving the first power of x, and therefore is not changed when $-x$ is substituted for $+x$. And $\left(\frac{d\text{F}}{dh}\right) = 2\text{A}h+\text{B}k+\text{E}$, which is an expression of one dimension in terms of h and k, and therefore represents a straight line, of which h and k are the current coordinates: this line therefore bisects all chords of the conic which are parallel to the axis of x, and is therefore a diameter conjugate to the diameter which is parallel to the axis of x. Similarly, if $\left(\frac{d\text{F}}{dk}\right) = 0$, (52) has no term involving the first power of y; and therefore $\left(\frac{d\text{F}}{dk}\right) = 0$, which is equal to $2\text{C}k+\text{B}h+\text{G} = 0$, and represents a straight line, is the equation to a diameter of the conic which bisects all chords parallel to the axis of y; and is therefore conjugate to the diameter which is parallel to the axis of y. And if we replace h and k, as we evidently may, by x and y, $\left(\frac{dF}{dx}\right) = 0$, and $\left(\frac{dF}{dy}\right) = 0$, or

$$\left.\begin{aligned} 2Ax + By + E &= 0, \\ 2Cy + Bx + G &= 0, \end{aligned}\right\} \tag{53}$$

are the equations to the two diameters of the conic $F(x, y) = 0$, which bisect all chords parallel to the axes of x and y respectively.

If $\left(\frac{dF}{dh}\right)$ and $\left(\frac{dF}{dk}\right)$ simultaneously vanish, (52) contains no term involving the first powers of x and y; and the conic is referred to the centre as origin. The coordinates to the centre are from (53),

$$x = \frac{2CE - BG}{B^2 - 4AC}, \qquad y = \frac{2AG - BF}{B^2 - 4AC}. \tag{54}$$

We shall have a further illustration of this theorem hereafter, when we come to the consideration of tangents and polars of plane curves.

If in the preceding general formula (51) we make $x = 0$, $y = 0$, and then change h and k into x and y, we have

$$\begin{aligned} F(x, y) = F(0, 0) &+ \left[\left(\frac{dF}{dx}\right) x + \left(\frac{dF}{dy}\right) y\right]_0 \\ &+ \frac{1}{1.2}\left[\left(\frac{d^2F}{dx^2}\right) x^2 + 2\left(\frac{d^2F}{dx\,dy}\right) xy + \left(\frac{d^2F}{dy^2}\right) y^2\right]_0 \\ &+ \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \frac{1}{1.2.3\ldots n}\left[\left(\frac{d^nF}{dx^n}\right) x^n + n\left(\frac{d^nF}{dx^{n-1}\,dy}\right) x^{n-1}y + \ldots\right. \\ &\qquad\qquad \left.\ldots + \left(\frac{d^nF}{dy^n}\right) y^n\right]_{\theta x,\, \theta y,} \end{aligned} \tag{55}$$

where we have to replace x and y by the value 0 in all the partial derived-functions, except in those of the last term, where they are to be replaced by θx and θy; and if this last term decreases without limit, as n increases without limit, then the remainder may be neglected, and the series may be written without it.

142.] If it is required to expand $F(x+h,\ y+k,\ z+l, \ldots\ldots)$, then, by a similar process, we have

$$
\begin{aligned}
\mathrm{F}(x+h, y+k, z+l, \ldots\ldots) &= \mathrm{F}(x, y, z, \ldots\ldots) \\
&\quad + \left\{\left(\frac{d\mathrm{F}}{dx}\right) h + \left(\frac{d\mathrm{F}}{dy}\right) k + \left(\frac{d\mathrm{F}}{dz}\right) l + \ldots\ldots\right\} \\
&\quad + \frac{1}{1.2}\left\{\left(\frac{d^2\mathrm{F}}{dx^2}\right) h^2 + \left(\frac{d^2\mathrm{F}}{dy^2}\right) k^2 + \ldots\ldots \right. \\
&\qquad\qquad \left. + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) hk + 2\left(\frac{d^2\mathrm{F}}{dx\,dz}\right) hl + \ldots\ldots\right\} \\
&\quad + \frac{1}{1.2.3}\left\{\left(\frac{d^3\mathrm{F}}{dx^3}\right) h^3 + \ldots + 3\left(\frac{d^3\mathrm{F}}{dx^2\,dy}\right) h^2 k + \ldots \right. \\
&\qquad\qquad \left. \ldots + 3\left(\frac{d^3\mathrm{F}}{dx\,dy^2}\right) hk^2 + \ldots\ldots\right\} \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + \frac{1}{1.2.3\ldots n}\left[\left(\frac{d^n\mathrm{F}}{dx^n}\right) h^n + \left(\frac{d^n\mathrm{F}}{dy^n}\right) k^n + \ldots + n\left(\frac{d^n\mathrm{F}}{dx^{n-1}\,dy}\right) h^{n-1}k \right. \\
&\qquad\qquad \left. + \ldots\ldots\right]_{\substack{x+\theta h, \\ y+\theta k, \\ z+\theta l, \\ \ldots \\ \ldots}} \qquad (56)
\end{aligned}
$$

replacing $x, y, z, \ldots$ in the last term by $x+\theta h$, $y+\theta k$, $z+\theta l, \ldots$

As the equations (51) (55) and (56) stand at present, each side is exactly equal to the other; but if we can assure ourselves that, as n increases without limit, each term, as well as the sum of all the terms, of the part omitted decreases without limit; then the remainders may be neglected, and the equations will be modified accordingly.

Ex. 1. $\mathrm{F}(x, y, z) = \mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\mathrm{E}yz + 2\mathrm{G}zx + 2\mathrm{H}xy$; it is required to expand $\mathrm{F}(x+h, y+k, z+l)$.

$$
\begin{aligned}
\left(\frac{d\mathrm{F}}{dx}\right) &= 2\mathrm{A}x + 2\mathrm{G}z + 2\mathrm{H}y, & \left(\frac{d^2\mathrm{F}}{dx^2}\right) &= 2\mathrm{A}, & \left(\frac{d^2\mathrm{F}}{dy\,dz}\right) &= 2\mathrm{E}, \\
\left(\frac{d\mathrm{F}}{dy}\right) &= 2\mathrm{B}y + 2\mathrm{E}z + 2\mathrm{H}x, & \left(\frac{d^2\mathrm{F}}{dy^2}\right) &= 2\mathrm{B}, & \left(\frac{d^2\mathrm{F}}{dz\,dx}\right) &= 2\mathrm{G}, \\
\left(\frac{d\mathrm{F}}{dz}\right) &= 2\mathrm{C}z + 2\mathrm{E}y + 2\mathrm{G}x, & \left(\frac{d^2\mathrm{F}}{dz^2}\right) &= 2\mathrm{C}, & \left(\frac{d^2\mathrm{F}}{dx\,dy}\right) &= 2\mathrm{H}.
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \mathrm{F}(x+h, y+k, z+l) &= \mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\mathrm{E}yz + 2\mathrm{G}zx + 2\mathrm{H}xy \\
&\quad + 2\{(\mathrm{A}x + \mathrm{G}z + \mathrm{H}y)h + (\mathrm{B}y + \mathrm{E}z + \mathrm{H}x)k + (\mathrm{C}z + \mathrm{E}y + \mathrm{G}x)l\} \\
&\quad + \mathrm{A}h^2 + \mathrm{B}k^2 + \mathrm{C}l^2 + 2\mathrm{E}kl + 2\mathrm{G}lh + 2\mathrm{H}hk.
\end{aligned}
$$

143.] And the result of this last Article may be extended by that method of Derivation, the principles of which have been explained in Art. 95—97, and which has been in those Articles applied to functions of one variable x. I propose now to apply the process to functions of two variables, although it is evidently capable of application to any number of variables. And I will first take the following case. It is required to expand in ascending powers of x and y the function

$$f\left(a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+\ldots,\ b_0+b_1\frac{y}{1}+b_2\frac{y^2}{1.2}+\ldots.\right). \qquad (56)$$

The first term of the expansion is $f(a_0, b_0)$; partial derived-functions of this are to be taken; in the a_0-partial derived-functions da_0 is to be replaced by a_1; and in the b_0-partial derived-functions db_0 is to be replaced by b_1; so that, if f stands for $f(a_0, b_0)$, the second term of the expansion is

$$\left(\frac{df}{da_0}\right)a_1x+\left(\frac{df}{db_0}\right)b_1y; \qquad (57)$$

the third term is

$$\frac{1}{1.2}\Bigg[\left\{\left(\frac{d^2f}{da_0^2}\right)a_1^2+\left(\frac{df}{da_0}\right)a_2\right\}x^2$$

$$+2\left(\frac{d^2f}{da_0\,db_0}\right)a_1b_1xy+\left\{\left(\frac{d^2f}{db_0^2}\right)b_1^2+\left(\frac{df}{db_0}\right)b_2\right\}y^2\Bigg]; \qquad (58)$$

the fourth term is

$$\frac{1}{1.2.3}\Bigg[\left\{\left(\frac{d^3f}{da_0^3}\right)a_1^3+3\left(\frac{d^2f}{da_0^2}\right)a_1a_2+\left(\frac{df}{da_0}\right)a_3\right\}x^3$$

$$+3\left\{\left(\frac{d^3f}{da_0^2\,db_0}\right)a_1^2b_1+\left(\frac{d^2f}{da_0\,db_0}\right)a_2b_1\right\}x^2y$$

$$+3\left\{\left(\frac{d^3f}{da_0\,db_0^2}\right)a_1b_1^2+\left(\frac{d^2f}{da_0\,db_0}\right)a_1b_2\right\}xy^2$$

$$+\left\{\left(\frac{df}{db_0}\right)b_3+3\left(\frac{d^2f}{db_0^2}\right)b_1b_2+\left(\frac{d^3f}{db_0^3}\right)b_1^3\right\}y^3\Bigg]; \qquad (59)$$

and by a similar process other terms may be found; but it is unnecessary to write them at length.

Ex. 1. By the process above explained the product of

$$a_0+a_1\frac{x}{1}+a_2\frac{x^2}{1.2}+\ldots \text{ and } b_0+b_1\frac{y}{1}+b_2\frac{y^2}{1.2}+\ldots \text{ is}$$

$$
\begin{aligned}
&a_0 b_0 + a_1 b_0 x + a_0 b_1 y \\
&\quad + \frac{1}{1.2}\{a_2 b_0 x^2 + 2a_1 b_1 xy + a_0 b_2 y^2\} \\
&\quad + \frac{1}{1.2.3}\{a_3 b_0 x^3 + 3a_2 b_1 x^2 y + 3a_1 b_2 xy^2 + a_0 b_3 y^3\} \\
&\quad + \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot
\end{aligned}
$$

Ex. 2. $\sin\{(a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + \ldots)(b_0 + b_1\frac{y}{1} + b_2\frac{y^2}{1.2} + \ldots)\}$

$$
\begin{aligned}
&= \sin(a_0 b_0) + (a_1 b_0 x + a_0 b_1 y)\cos(a_0 b_0) \\
&- \frac{1}{1.2}(a_1 b_0 x + a_0 b_1 y)^2 \sin(a_0 b_0) \\
&\qquad + \frac{1}{1.2}\cos(a_0 b_0)\{a_2 b_0 x^2 + 2a_1 b_1 xy + a_0 b_2 y^2\} \\
&- \frac{1}{1.2.3}\{(a_1 b_0 x + a_0 b_1 y)^3 - a_3 b_0 x^2 \\
&\qquad - 3a_2 b_1 x^2 y - 3a_1 b_2 xy^2 - a_0 b_3 y^3\}\cos(a_0 b_0) \\
&- \frac{1}{1.2.3}\{3(a_1 b_0 x + a_0 b_1 y)(a_2 b_0 x^2 + 2a_1 b_1 xy + a_0 b_2 y^2)\}\sin(a_0 b_0) \\
&+ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot
\end{aligned}
$$

Another form is that wherein it is required to expand in ascending powers of the variables a function of variables of the form

$$f\{a_{00} + a_{10}x + a_{01}y + \frac{1}{1.2}(a_{20}x^2 + 2a_{11}xy + a_{02}y^2) + \ldots\}.$$

The first term of the expansion is evidently $f(a_{00})$: let the successive derived-functions of this be symbolized according to Lagrange's notation. As however f is a function of two variables, da_{00} will be a_{10} or a_{01} according as the x- or y-partial differential is taken: and thus we have the preceding function equal to

$$
\begin{aligned}
&f(a_{00}) + f'(a_{00})\{a_{10}x + a_{01}y\} \\
&\qquad + \frac{f''(a_{00})}{1.2}\{a_{10}x + a_{01}y\}^2 + \frac{f'(a_{00})}{1.2}\{a_{20}x^2 + 2a_{11}xy + a_{02}y^2\} \\
&+ \frac{1}{1.2.3}[f'''(a_{00})(a_{10}x + a_{01}y)^3 \\
&\qquad + 3f''(a_{00})(a_{10}x + a_{01}y)(a_{20}x^2 + 2a_{11}xy + a_{02}y^2) \\
&\qquad\quad + f'(a_{00})(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3)] + \ldots . \quad (60)
\end{aligned}
$$

CHAPTER VII.

ON THE DETERMINATION OF MAXIMA AND MINIMA VALUES OF FUNCTIONS.

144.] Let us consider a function of a single variable x; and, to fix our thoughts, let the variable be continuously increased; then the corresponding variation of the function need not always be one of increase or of decrease, but it may increase up to a certain value and afterwards decrease, or *vice versâ*. In the former of these two cases, at the value of the variable when the function ceases to increase, it has attained a *greatest* value, or what is technically called a *maximum* state; and in the latter it reaches a least or a *minimum* state; such *singular* conditions of a function the principles of Chapter IV enable us to determine. And we have the following definition:

A particular value of a function, which is greater than *all* its values in the immediate neighbourhood, that is, when the variables are infinitesimally increased or decreased, is said to be *a maximum*. And the particular value which is less than all its immediately adjacent ones, is called *a minimum*.

Maxima and *minima* are therefore terms used not absolutely, but in reference to the values of the functions immediately adjacent to those to which the names are applied.

As a simple illustration, let us consider $\sin x$; and let the radius of the circle be unity; then, when the arc $= 0$, the sine $= 0$; but as the arc increases up to $\frac{\pi}{2}$, the sine increases and at last becomes 1, which is its maximum, for as the arc becomes larger, the sine becomes smaller and continually decreases, passing through 0 when the arc $= \pi$, until, when the arc $= \frac{3\pi}{2}$, the sine $= -1$; after which it continually increases until, when the arc $= \frac{5\pi}{2}$, the sine $= +1$, which is a maximum, and so on. Thus as the arc increases, the sine periodically attains to maxima and minima values.

SECTION 1.—*On maxima and minima of explicit functions of one variable.*

145.] Let $y = F(x)$ be the function of which the maxima and minima are to be investigated.

From the definition it is plain that if, as x increases up to a certain value x_0, $F(x)$ increases; and afterwards as x increases, $F(x)$ decreases; then $F(x)$ has attained a maximum value at $x = x_0$. And if, as x increases up to a certain value x_0, $F(x)$ decreases, and afterwards increases as x increases, then $F(x_0)$ is a minimum value.

Now Theorem I, Chapter IV, is immediately applicable to the determination of these conditions: if x and $F(x)$ are simultaneously increasing, $F'(x)$ is positive; if $F(x)$ decreases as x increases, $F'(x)$ is negative.

If therefore at any point $x = x_0$, $F'(x)$ changes its sign from $+$ to $-$, we have a maximum value; and if $F'(x)$ changes its sign from $-$ to $+$, we have a minimum value; and as changes of sign can take place only when the quantity passes through 0 or ∞, we have the following rule to determine Maxima and Minima:

Suppose $\frac{dy}{dx} = \phi(x).\psi(x)^2$ Then, if we examine $\frac{dy}{dx}$ to see if the roots of $\psi(x) = 0$ give max. or min. vals. we have, $\frac{d^2y}{dx^2} = \phi(x)\,\psi'(x)$ since ex hypoth $\psi(x) = 0$. etc.

Find every value of x which renders $F'(x)$ equal to 0 and to ∞; if such a value makes $F'(x)$ change its sign, the corresponding value of $F(x)$ is a maximum or minimum; being a maximum if $F'(x)$ changes sign from $+$ to $-$, and a minimum if the change is from $-$ to $+$; but if there is no change of sign, there is no such singular value.

Ex. 1.
$$y = F(x) = x^2 - 2ax;$$
$$\therefore \quad F'(x) = 2x - 2a = 0, \text{ if } x = a;$$
and as $F'(x)$ is negative when x is less than a, and positive when x is greater than a, $F'(x)$ changes sign from $-$ to $+$ as x passes through a; and therefore $F(x)$ has a minimum value, viz. $-a^2$.

Ex. 2.
$$y = F(x) = \sin x;$$
$$F'(x) = \cos x = 0, \text{ if } x = \frac{\pi}{2};$$
and as $\cos x$ is positive when x is less than $\frac{\pi}{2}$, and negative when x is greater than $\frac{\pi}{2}$, $F'(x)$ changes sign from $+$ to $-$; and accordingly $F(x)$ has a maximum value, viz. 1.

Also since $F'(x) = \cos x = 0$, when $x = \frac{3\pi}{2}$, and changes sign from $-$ to $+$ as x passes through this value, $\sin x$ has at it a minimum value, viz. -1. Similarly also $x = \frac{5\pi}{2}$ gives to $\cos x$ a change of sign from $+$ to $-$, and therefore gives to $\sin x$ a maximum, viz. 1; and thus may other values be determined.

Ex. 3. $$y = x(a-x)^2 = F(x);$$

$$\therefore \quad F'(x) = (a-x)(a-3x) = 0, \text{ if}$$

$x = a$, and changes sign from $-$ to $+$, $\therefore$ a minimum;

$x = \frac{a}{3}$, - - - - - from $+$ to $-$, $\therefore$ a maximum;

therefore 0 is a minimum value of $x(a-x)^2$, viz. when $x = a$; and when $x = \frac{a}{3}$, $F(x) = \frac{4a^3}{27}$, which is a maximum value.

It is convenient to have a distinctive name for that value of a variable which makes a function of it to vanish, and therefore I propose to call it the *critical* value; thus a and $\frac{a}{3}$ are critical values of x in $F'(x)$ in the last example; 0 and a are the critical values of x in $x(a-x)^2$. Of an algebraical expression, the roots are the critical values. It is plain that critical values do not necessarily cause a function to change sign, although a function cannot change sign except at a critical value: at least such is the case so far as we know. Similarly the factor which vanishes is called the *critical* factor.

146.] When, as in Ex. 3 above, $F'(x)$ is an algebraical function, and has many factors which, when equated to 0, cause it to vanish, it is easy to perceive their forms so that $F'(x)$ may change its sign. Corresponding to every factor of uneven dimensions, that is, of the form $(x-x_0)^{2m+1}$, as x passes through x_0, there is a change of sign; but to factors of even dimensions, viz. of the form $(x-x_0)^{2m}$, there is no change of sign as x passes through x_0, and therefore there is no maximum or minimum value.

To determine the change of sign corresponding to a factor of uneven dimensions, the best method is first to determine the signs of all the factors, short of the critical factor, corresponding to the critical value, and then to investigate the change of sign of the critical factor; the following example will explain the process. Suppose

$$F'(x) = x^3 (x-1)^2 (x-2)^3 (x-4)^4;$$

which is equal to 0, if

$x = 0$, and gives a change of sign from + to −, ∴ a maximum;
$x = 1$, and gives no change of sign, ∴ no max. or minimum;
$x = 2$, and gives a change of sign from − to +, ∴ a minimum;
$x = 4$, and gives no change of sign, ∴ no max. or minimum;

that is, if $x = 0$, the critical factor is x^3; but when $x = 0$, the other factors severally are $+1$, -8, $+16$, the product of which is −; and as x^3, when $x = 0$, changes sign from − to +, it follows that $F'(x)$, when $x = 0$, changes sign from + to −, and accordingly $F(x)$ has a corresponding maximum value; by a similar method the changes of sign due to the other critical factors are determined.

147.] Geometrical illustrations of the several conditions of maxima and minima are given in figs. 12, 13, 14, 15.

Suppose $y = F(x)$ to represent a curve such as those drawn in the figures.

Let $OM_0 = x_0$, and let $M_0P_0 = y_0$ the corresponding ordinate. Then in fig. 12, as x increases up to x_0, $y = F(x)$ increases, and therefore $F'(x)$ is positive; but so soon as x passes the value x_0, y begins to decrease, and $F'(x)$ is negative, and the ordinate y or $F(x)$ has manifestly attained a maximum value at x_0.

In fig. 13, the reverse is the case; as x increases up to x_0, $y = F(x)$ decreases, but so soon as x is greater than x_0, $F(x)$ increases, and thus the sign of $F'(x)$ changes from − to + at x_0, and the corresponding value of $F(x)$ is a minimum.

Fig. 14 illustrates the case of $F'(x)$ being positive up to x_0; and although $F'(x) = 0$, yet it does not change its sign, but continues positive afterwards, and therefore we have no maximum value.

In the curve drawn in fig. 15, $F'(x)$ is negative throughout; at P_0 it is equal to 0, but as it does not change its sign, there is no minimum value.

148.] Examples of maxima and minima.

Ex. 1. To determine the maxima and minima values of y, having given

$$y = F(x) = x^4 - 8ax^3 + 22a^2x^2 - 24a^3x + 12a^4.$$

$$\frac{dy}{dx} = \text{F}'(x) = 4x^3-24ax^2+44a^2x-24a^3,$$
$$= 4(x-a)(x-2a)(x-3a) = 0;$$

if $x = a$, and changes sign from $-$ to $+$, $\therefore$ a minimum;
$x = 2a$, - - - - - - - - $+$ to $-$, $\therefore$ a maximum;
$x = 3a$, - - - - - - - - $-$ to $+$, $\therefore$ a minimum;

whence we have, if $x = a$, $\text{F}(x) = 3a^4$, a minimum;
$x = 2a$, $\text{F}(x) = 4a^4$, a maximum;
$x = 3a$, $\text{F}(x) = 3a^4$, a minimum.

Ex. 2. To determine the maxima and minima values of

$$\text{F}(x) = (x-1)^4(x+2)^3;$$
$$\text{F}'(x) = (x-1)^3(x+2)^2(7x+5) = 0; \text{ if}$$

$x = 1$, and changes sign from $-$ to $+$, $\therefore$ a minimum;
$x = -2$, but does not change sign, $\therefore$ no max. or minimum;
$x = -\frac{5}{7}$, and changes sign from $+$ to $-$, $\therefore$ a maximum;

hence, when $x = 1$, $\text{F}(x) = 0$, a minimum value;
and when $x = -\frac{5}{7}$, $\text{F}(x) = \frac{12^4\, 9^3}{7^7}$, a maximum.

Ex. 3. To determine the maximum and minimum values of $\text{F}(x)$, having given

$$\text{F}(x) = \frac{(x+2)^3}{(x-3)^2};$$
$$\therefore \quad \text{F}'(x) = \frac{(x+2)^2(x-13)}{(x-3)^3};$$

$= 0$, if $x = -2$, and does not change sign, $\therefore$ no max. or min.;
$= 0$, if $x = 13$, and changes sign from $-$ to $+$, $\therefore$ a minimum;
$= \infty$, if $x = 3$, - - - - - - - - $+$ to $-$, $\therefore$ a maximum;

hence, when $x = 13$, $\text{F}(x) = \frac{135}{4}$, a minimum value;
and when $x = 3$, $\text{F}(x) = \infty$, a maximum value.

Ex. 4. To determine the minimum value of x^x.

$$\text{F}(x) = x^x;$$
$$\text{F}'(x) = x^x(1+\log x) = 0;$$

if $\log x = -1$, that is, if $x = e^{-1}$; and as $\text{F}'(x)$ changes sign from $-$ to $+$, the corresponding value of x^x, viz. $(e)^{-\frac{1}{e}}$, is a minimum.

Ex. 5. It is required to find the value of x, when $\sin x + \cos x$ is a maximum.

$$\text{F}(x) = \sin x + \cos x;$$

$$\text{F}'(x) = \cos x - \sin x = 0;$$

if $\tan x = 1$; that is, when

$x = \frac{\pi}{4}$, and changes sign from + to −, ∴ a maximum;

$x = \frac{5\pi}{4}$, - - - - - - - - − to +, ∴ a minimum;

. . . .

Hence the maximum value of the function is $\sqrt{2}$, and the minimum is $-\sqrt{2}$; which values recur whenever x is increased by π.

149.] The change of sign of $\text{F}'(x)$ may often be conveniently determined from the following considerations.

Let us suppose that $\text{F}''(x)$ does not vanish or become infinite when $\text{F}'(x) = 0$; then since $\text{F}''(x) = \frac{d^2y}{dx^2} = \frac{d.\left(\frac{dy}{dx}\right)}{dx} = \frac{d.\text{F}'(x)}{dx}$, it is manifest that if x increases, $\text{F}''(x)$ is positive or negative according as $\text{F}'(x)$ simultaneously increases or decreases; but if, as x increases, $\text{F}'(x)$ changes sign from − to +, it is increasing, and if it changes sign from + to −, it is decreasing; hence for a minimum value $\text{F}''(x)$ is positive, and for a maximum value $\text{F}''(x)$ is negative. Accordingly $\text{F}(x)$ is a maximum or minimum value, according as the value of x, determined from the equation $\text{F}'(x) = 0$, renders $\text{F}''(x)$ negative or positive.

Ex. 1. Let $\text{F}(x) = \frac{x}{1+x^2}$;

$$\text{F}'(x) = \frac{1-x^2}{(1+x^2)^2}; \qquad \text{F}''(x) = \frac{2x^3-6x}{(1+x^2)^3};$$

and $\text{F}'(x) = 0$, if $x = +1$, and if $x = -1$;

and if $x = +1$, $\text{F}''(x) = -\frac{1}{2}$; ∴ $\text{F}(x) = \frac{1}{2}$, a maximum;

if $x = -1$, $\text{F}''(x) = +\frac{1}{2}$; ∴ $\text{F}(x) = -\frac{1}{2}$, a minimum.

Ex. 2.
$$\begin{aligned} F(x) &= \sin x \cos(a-x); \\ F'(x) &= \cos x \cos(a-x) + \sin x \sin(a-x), \\ &= \cos(a-2x); \\ F''(x) &= 2\sin(a-2x); \end{aligned}$$

$$\therefore\ F'(x) = 0, \text{ if } a-2x = -\frac{\pi}{2}; \quad \therefore\ x = \frac{a}{2}+\frac{\pi}{4};$$

$$\text{- - } = 0, \text{ if } a-2x = \frac{\pi}{2}; \quad \therefore\ x = \frac{a}{2}-\frac{\pi}{4}.$$

In the former of which cases $F''(x)$ is negative, and the corresponding value of $F(x)$ is a maximum; and in the latter $F''(x)$ is positive, and the corresponding value of $F(x)$ is a minimum.

150.] This method however of determining such singular values of functions is not applicable whenever the value of x which makes $F'(x) = 0$, also makes $F''(x) = 0$; in which case, as well as in all others, the following method may be employed.

Let $F(x)$ be the function of which the maximum and minimum values are to be determined; then, by (21), Art. 116,

$$F(x+h) - F(x) = hF'(x+\theta h);$$

and if h is infinitesimal, $F'(x+\theta h)$ becomes $F'(x)$.

Suppose now that x_0 is a value of x, such that $F(x_0)$ is a maximum or a minimum; then if we consider the values of $F(x)$ which are immediately adjacent to $F(x_0)$ on both sides of it, we shall be able to detect a criterion of such singular values. For this purpose let us consider $F(x_0+h)$ and $F(x_0-h)$ when h is infinitesimal: then if $F(x_0)$ is a maximum, $F(x_0+h) - F(x_0)$ and $F(x_0-h) - F(x_0)$ are both negative: and if $F(x_0)$ is a minimum, these differences are both positive. Now if $F'(x_0)$ does not vanish,

$$F(x_0+h) - F(x_0) = hF'(x_0),$$

and
$$F(x_0-h) - F(x_0) = -hF'(x_0);$$

and thus these differences have different signs. But this is inconsistent with $F(x_0)$ being a maximum or minimum; and therefore we conclude that $F(x_0)$ is not a maximum or a minimum if $F'(x_0)$ does not vanish.

Suppose however that $F'(x_0) = 0$, and that $F''(x_0)$ does not vanish; then, by equation (22), Art. 116,

$$F(x_0+h) - F(x_0) = \frac{h^2}{1.2} F''(x_0+\theta h);$$

whence, when h is infinitesimal,

$$\text{F}(x_0+h) - \text{F}(x_0) = \frac{h^2}{1.2}\,\text{F}''(x_0);$$

and this does not change sign with h; if therefore $\text{F}''(x_0)$ is positive, $\text{F}(x_0)$ is less than both $\text{F}(x_0+h)$ and $\text{F}(x_0-h)$, that is, $\text{F}(x_0)$ is a minimum; and if $\text{F}''(x_0)$ is negative, $\text{F}(x_0)$ is greater than both $\text{F}(x_0+h)$ and $\text{F}(x_0-h)$, that is, $\text{F}(x_0)$ is a maximum; whence we conclude,

If $\text{F}'(x_0) = 0$, and $\text{F}''(x_0)$ does not vanish, $\text{F}(x_0)$ is a singular value; and if $\text{F}''(x_0)$ is negative, $\text{F}(x_0)$ is a maximum; and if $\text{F}''(x_0)$ is positive, $\text{F}(x_0)$ is a minimum.

Again, if $\text{F}''(x_0) = 0$, and $\text{F}'''(x_0)$ does not vanish, and if h is infinitesimal,

$$\text{F}(x_0+h) - \text{F}(x_0) = \frac{h^3}{1.2.3}\,\text{F}'''(x_0);$$

in which case, as h^3 changes its sign with h, it is plain that there is no maximum or minimum value; but if $\text{F}'''(x_0) = 0$, and $\text{F}^{\text{iv}}(x_0)$ does not vanish, then

$$\text{F}(x_0+h) - \text{F}(x_0) = \frac{h^4}{1.2.3.4}\,\text{F}^{\text{iv}}(x_0);$$

in which case, as before, $\text{F}(x_0)$ will be a maximum or minimum value of $\text{F}(x)$, according as $\text{F}^{\text{iv}}(x_0)$ is negative or positive.

And thus generally if the value x_0, which makes $\text{F}'(x) = 0$, so affects $\text{F}''(x)$, $\text{F}'''(x)$, ... up to $\text{F}^{n-1}(x)$, that all vanish, but that $\text{F}^n(x_0)$ does not vanish; then we have

$$\text{F}(x_0+h) - \text{F}(x_0) = \frac{h^n}{1.2.3\ldots n}\,\text{F}^n(x_0);$$

and if n is an odd number, there is no maximum or minimum value; but if n is an even number, $\text{F}(x_0)$ is a maximum if $\text{F}^n(x_0)$ is negative, and a minimum if $\text{F}^n(x_0)$ is positive.

In the application of this theory to questions of geometrical maxima and minima, it will subsequently appear that figs. 12 and 13 correspond to the analytical conditions of every derived-function vanishing, when $x = x_0$, up to one of an odd order inclusively, and of the next derived-function of an even order remaining finite; and figs. 14 and 15 correspond to the condition that the derived-function, which is the first not to vanish, is of an odd order.

Ex. 1.
$$\begin{aligned}
\mathrm{F}(x) &= e^{x}+2\cos x+e^{-x}, \\
\mathrm{F}'(x) &= e^{x}-2\sin x-e^{-x} = 0, \text{ if } x=0, \\
\mathrm{F}''(x) &= e^{x}-2\cos x+e^{-x} = 0, \text{ - - - }, \\
\mathrm{F}'''(x) &= e^{x}+2\sin x-e^{-x} = 0, \text{ - - - }, \\
\mathrm{F}^{\text{iv}}(x) &= e^{x}+2\cos x+e^{-x} = 4, \text{ - - - };
\end{aligned}$$

therefore if $x=0$, $\mathrm{F}(x)=4$, which is a minimum value, because the fourth derived-function, which is the first not to vanish, is positive.

151.] We subjoin some examples of problems on maxima and minima, the study of which will be sufficient to enable the reader to apply the methods to other similar ones.

Ex. 1. To divide a given number a into two parts such that the product of the nth power of one, and the mth power of the other, may be a maximum.

Let a be the given number, of which let x be one part, and therefore $a-x$ is the other; and we have

$$\mathrm{F}(x) = x^{n}(a-x)^{m};$$

$$\begin{aligned}
\therefore\quad \mathrm{F}'(x) &= x^{n-1}(a-x)^{m-1}(na-nx-mx), \\
&= (n+m)\,x^{n-1}(a-x)^{m-1}\left(\frac{na}{m+n}-x\right),
\end{aligned}$$

$= 0$, if $x=0$, and, if n is an even number, changes sign from $-$ to $+$, which indicates a minimum; but if n is odd, $\mathrm{F}'(x)$ does not change sign, and there is no corresponding maximum or minimum;

$= 0$, if $x=a$, and, if m is an even number, changes sign from $-$ to $+$, which indicates a minimum; but if m is odd, $\mathrm{F}'(x)$ does not change sign, and there is no corresponding maximum or minimum;

$= 0$, if $x = \dfrac{na}{m+n}$, and changes sign from $+$ to $-$, which indicates a maximum value; viz. $\mathrm{F}(x) = m^{m}n^{n}\left(\dfrac{a}{m+n}\right)^{m+n}$.

Ex. 2. To find the number the ratio of which to its logarithm is a minimum.

Let x be the number; then

$$\mathrm{F}(x) = \frac{x}{\log x}; \text{ which is to be a minimum;}$$

$$\therefore \quad F'(x) = \frac{\log x - 1}{(\log x)^2},$$

$= 0$, when $x = e$, and changes sign from $-$ to $+$, which indicates a minimum; viz. $F(x) = e$.

Ex. 3. To find the number of equal parts into which a given number a is to be divided, so that their continued product may be a maximum.

Let x be the number of parts; and thus each part $= \frac{a}{x}$; and therefore, if $F(x) =$ the product of them,

$$F(x) = \left(\frac{a}{x}\right)^x;$$

$$\therefore \quad \log F(x) = x\{\log a - \log x\};$$

$$\begin{aligned} \therefore \quad \frac{F'(x)}{F(x)} &= \log a - \log x - 1, \\ &= \log a - \log x - \log e, \\ &= \log\left(\frac{a}{e}\right) - \log x; \end{aligned}$$

therefore $F'(x) = 0$, if $x = \frac{a}{e}$, and changes sign from $+$ to $-$; which indicates a maximum; therefore each part $= e$, and the product of all $= (e)^{\frac{a}{e}}$.

Ex. 4. To inscribe in a given circle the greatest isosceles triangle.

In fig. 16, let the vertex of the triangle be at the extremity A of the diameter of the circle; let the radius of the circle $= a$; and let P and P′ be the other angular points of the triangle; therefore $MP = MP'$; let $CM = x$, $MP = y$; then the equation to the circle is

$$y^2 + x^2 = a^2;$$

the area of the triangle $= F(x) = AM \times MP$,

$$= (a+x)(a^2-x^2)^{\frac{1}{2}};$$

$$\therefore \quad F'(x) = \frac{a^2 - ax - 2x^2}{(a^2-x^2)^{\frac{1}{2}}} = \frac{(a+x)^{\frac{1}{2}}(a-2x)}{(a-x)^{\frac{1}{2}}},$$

$= 0$, if $x = \frac{a}{2}$, and the sign changes from $+$ to $-$, which indicates a maximum, of which the value is $\frac{3^{\frac{3}{2}}}{4}a^2$.

$F'(x)$ also vanishes when $x = a$, and when $x = -a$; but as the sign of it in both cases passes from $\pm\sqrt{-}$ to $\pm$, and as the geometrical meaning of the values is plain enough, we must reserve the consideration of such critical values to a future part of the work.

Ex. 5. To inscribe the greatest rectangle in a semi-ellipse.

In fig. 17 let A′P′BPA be the given semi-ellipse, and MPP′M′ the rectangle inscribed in it, which is to be a maximum; let $\text{CM} = x$, $\text{MP} = y$; and let the equation to the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1;$$

$$\begin{aligned}\therefore\quad \text{the rectangle} = \text{F}(x) &= \text{M}'\text{M}\times\text{MP},\\ &= 2\text{CM}\times\text{MP},\\ &= 2\frac{b}{a}x(a^2-x^2)^{\frac{1}{2}};\\ \therefore\quad \text{F}'(x) &= 2\frac{b}{a}\frac{a^2-2x^2}{(a^2-x^2)^{\frac{1}{2}}},\end{aligned}$$

$= 0$, if $x = +\dfrac{a}{\sqrt{2}}$, and changes sign from $+$ to $-$, which indicates a maximum, and $\text{F}(x)$ becomes ab; and therefore the greatest rectangle is one-half of that contained by AA and CB.

Ex. 6. The whole surface of a cylinder being given, it is required to find its form when the content is a maximum.

Let $x =$ the radius of the base, $y =$ the height of the cylinder;

$$\therefore\quad \pi x^2 y = \text{the content},$$

$$\text{and}\quad 2\pi x^2+2\pi xy = \text{the whole surface}.$$

Let the given surface be $2\pi a^2$, and let the content be $\text{F}(x)$;

$$\therefore\quad 2\pi x^2+2\pi xy = 2\pi a^2,\qquad \text{and}\quad y = \frac{a^2-x^2}{x};$$

$$\text{F}(x) = \pi x^2 y, = \pi x(a^2-x^2);$$

$$\therefore\quad \text{F}'(x) = \pi(a^2-3x^2),$$

$= 0$, if $x = \dfrac{a}{\sqrt{3}}$, and changes sign from $+$ to $-$; which indicates a maximum; in which case $y =$ the height $= \dfrac{2a}{\sqrt{3}}$, and the content $= \dfrac{2\pi a^3}{3^{\frac{3}{2}}}$.

Ex. 7. Given the content of a cone; it is required to find its dimensions when the whole surface is a minimum.

Let x = the radius of the base, y = the altitude of the cone; therefore by Ex. 6, Art. 24, the content $= \frac{\pi x^2 y}{3}$;

and the whole surface $= \pi x^2 + \pi x(x^2+y^2)^{\frac{1}{2}}$.

Let the content $= \frac{\pi a^3}{3}$, and the surface be $F(x)$;

$$\therefore \quad \pi x^2 y = \pi a^3, \quad \text{and} \quad y = \frac{a^3}{x^2};$$

$$F(x) = \pi x^2 + \pi x(x^2+y^2)^{\frac{1}{2}},$$

$$= \pi x^2 + \pi \frac{(a^6+x^6)^{\frac{1}{2}}}{x};$$

$$F'(x) = 2\pi x - \pi \frac{a^6 - 2x^6}{x^2(a^6+x^6)^{\frac{1}{2}}},$$

$= 0$, if $x = \frac{a}{2^{\frac{1}{6}}}$, and changes sign from $-$ to $+$; which indicates a minimum; and $y = 2a$; in which case the semi-vertical angle $= \cot^{-1}\sqrt{8}$, and the surface $= 2\pi a^2$.

Ex. 8. To describe the least cone about a given sphere.

In fig. 18, let the circle APBP′, and the triangle EFG, represent a plane section of the sphere and circumscribed cone, made by the paper passing through the sphere's centre; let the radius of the sphere $= a$; and $CM = x$, $MP = y$, so that

$$x^2 + y^2 = a^2.$$

Then, by properties of the tangent of a circle,

$$CE = \frac{a^2}{x}, \qquad AF = \frac{a(a+x)}{y} = a\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}.$$

Let the content of the cone $= F(x)$;

$$\therefore \quad F(x) = \frac{\pi}{3} AF^2 \times AE = \frac{\pi}{3} a^3 \frac{(a+x)^2}{ax-x^2};$$

$$\therefore \quad F'(x) = \frac{\pi a^4}{3} \frac{(a+x)(3x-a)}{(ax-x^2)^2},$$

$= 0$, if $x = \frac{a}{3}$, and changes sign from $-$ to $+$, which indicates a minimum value, viz. $\frac{8}{3}\pi a^3$; and which is therefore twice the volume of the sphere. See Ex. 7, Art. 24.

Ex. 9. To cut the greatest parabola from a given cone.

In fig. 19, let the radius of the base of the cone $= b$, let the altitude of the cone $= a$; let $\text{OM} = x$, $\text{MP} = y$; and let the area of the parabola $= \text{F}(x)$.

Now $$\text{MQ}^2 = \text{OM} \times \text{MB} = x(2b-x) = 2bx-x^2,$$

$$\text{MP} = \text{MB}\frac{\text{CO}}{\text{OB}} = \frac{(a^2+b^2)^{\frac{1}{2}}}{2b}(2b-x).$$

By Ex. 8, Art. 24, the area of the parabola $= \frac{2}{3}\text{QQ}' \times \text{MP}$;

$$\therefore \quad \text{F}(x) = \frac{2}{3} 2\text{MQ} \times \text{MP},$$

$$= \frac{4}{3}(2bx-x^2)^{\frac{1}{2}} \frac{(a^2+b^2)^{\frac{1}{2}}}{2b}(2b-x),$$

$$= \frac{2(a^2+b^2)^{\frac{1}{2}}}{3b} x^{\frac{1}{2}}(2b-x)^{\frac{3}{2}},$$

$$\therefore \quad \text{F}'(x) = \frac{2(a^2+b^2)^{\frac{1}{2}}}{3b} \frac{(2b-x)^{\frac{1}{2}}(b-2x)}{x^{\frac{1}{2}}},$$

$= 0$, and changes sign from $+$ to $-$, if $x = \frac{b}{2}$, which indicates a maximum; therefore $\text{OM} = \frac{\text{OA}}{2}$, and the area of the greatest parabola $= \frac{3^{\frac{1}{2}}}{2}(a^2+b^2)^{\frac{1}{2}} b$.

152.] In the preceding Articles we have determined certain properties of maxima and minima, and have investigated methods by which particular cases of such singular values may be found. The general theory however, to be complete, requires a criterion whereby the number of maxima and the number of minima of a given function may be determined; and also means for determining the greatest and the least, or, as I shall call them, the absolute maximum and the absolute minimum, of all the singular values of the function.

Let $\text{F}(x)$ be the function whose singular values are the subject of consideration; and let x_0 be the general symbol of the critical value of x to which such singular values correspond; and let y_0 be the corresponding value of $\text{F}(x)$; and let β be the value which the second derived-function of $\text{F}(x)$ takes, when $x = x_0$. Then we have the following equations, when $x = x_0$:

$$\text{F}(x) - y_0 = 0, \tag{1}$$

$$\text{F}'(x) = 0, \tag{2}$$

$$\text{F}''(x) - \beta = 0. \tag{3}$$

Now as the last two equations coexist when $x = x_0$, a relation exists between them which can, theoretically at least, be expressed in terms of β and the coefficients of the original function, and which will be independent of x and x_0; or, in other words, x may be eliminated from (2) and (3). If this relation is found, and is arranged in terms of β, we shall have an equation in terms of β, the signs of the roots of which will enable us to distinguish maxima and minima; because there will be as many maxima values as there are negative values of β, and as many minima values as there are positive values. In many cases however this process will not enable us to determine the *number* of the singular values, because there may be many of the same value. Thus if $\text{F}(x) = \sin x$, $\text{F}'(x) = \cos x$, $\text{F}''(x) = -\sin x$; and we have to eliminate x between $\cos x = 0$, and $-\sin x - \beta = 0$; whence we have $\beta^2 - 1 = 0$, and $\beta = 1$, and $\beta = -1$; hence $\sin x$ has one minimum value and one maximum value, but it has an infinite number of maxima and minima, corresponding to the infinite number of the roots of the equation $f'(x) = 0$, or $\cos x = 0$.

The determination of the required equation may be performed in any way that is practicable, although it is in most cases beyond our power; if however $\text{F}(x)$ is a rational and integral function of x, then, because (2) and (3) are satisfied when $x = x_0$, they have a common measure of the form $(x - x_0)^m$, where m is either unity or some positive integer greater than unity; and if we proceed by the ordinary algebraical process of finding this common measure, we shall finally come to a remainder independent of x and in terms of β, which must be equal to zero; and if the terms of it are arranged in powers of β, we shall have an equation the signs of the roots of which will fix the numbers of the maxima and the minima of the original function. The following examples illustrate the process.

Ex. 1. Let
$$\begin{aligned} \text{F}(x) &= x^3 - 2x^2 + x; \\ \therefore \quad \text{F}'(x) &= 3x^2 - 4x + 1, \\ \text{F}''(x) &= 6x - 4. \end{aligned}$$

If we proceed by the ordinary method to find the greatest common measure between $3x^2 - 4x + 1$ and $3x - 2 - \beta$, we shall after two divisions come to the remainder $\beta^2 - 1$, which must be equal to zero; whence we have $\beta = \pm 1$; and as the values of β are one positive and one negative, so the singular values of $\mathrm{F}(x)$ are respectively a minimum and a maximum.

Ex. 2.
$$\begin{aligned}\mathrm{F}(x) &= x^3+ax^2-2a^2x,\\ \mathrm{F}'(x) &= 3x^2+2ax-2a^2,\\ \mathrm{F}''(x) &= 6x+2a.\end{aligned}$$

If we proceed as in the last example we shall find $\beta^2-7a^2=0$; so that $\beta = \pm 7^{\frac{1}{2}} a$; and therefore $\mathrm{F}(x)$ has two singular values which are respectively a maximum and a minimum.

If one of the values of β, deduced from the result of the elimination between (2) and (3), is zero, the criterion fails for a singular value, or for the particular kind of it, corresponding to that value of the variable which makes $\beta = 0$. Of this circumstance the following is an example:

$$\begin{aligned}\mathrm{F}(x) &= 3x^4-16x^3+24x^2-11,\\ \mathrm{F}'(x) &= 12(x^3-4x^2+4x),\\ \mathrm{F}''(x) &= 12(3x^2-8x+4).\end{aligned}$$

Now in pursuit of the process by which, as in the preceding examples, the criteria of maxima and minima have been determined, let us endeavour to find the common measure of x^3-4x^2+4x and of $3x^2-8x+4-\beta$; then the first remainder which does not involve x is $3\beta^2-32\beta$; and of this when equated to zero the roots are 0 and a positive quantity; so that corresponding to the factors which make $\mathrm{F}'(x)$ to vanish, for one $\mathrm{F}(x)$ may have no singular value; and the other gives to $\mathrm{F}(x)$ a minimum value, viz. -11.

153.] Let thus much suffice for the number of maxima and the number of minima of a given function $\mathrm{F}(x)$; and let us now investigate a method for determining the greatest and least absolutely of all the singular values which $\mathrm{F}(x)$ has. Since (1) and (2) of the last Article are both satisfied when $x = x_0$, they have a common measure. When $\mathrm{F}(x)$ is transcendental, the determination of this relation is generally beyond our powers; but when $\mathrm{F}(x)$ is a rational and integral algebraical function, it can be found. For if we operate on (1) and (2) in the ordinary process of finding the greatest common measure, we shall finally arrive at a remainder which is independent of x, but which involves y_0 and the coefficients of the original function; this remainder must be equal to zero; and thus we shall have an equation in terms of y_0, the roots of which will be the singular values of $F(x)$; and thus the greatest root will be that value of $F(x)$ which is absolutely the greatest, and the least root will be the absolute minimum. The process is illustrated by the following examples.

Ex. 1.
$$F(x) = x^3-2x^2+x,$$
$$F'(x) = 3x^2-4x+1.$$

If, as in Art. 152, y_0 is the singular value of $F(x)$, and if we pursue the common process for the determination of the common measure of $x^3-2x^2+x-y_0$ and of $3x^2-4x+1$, the first remainder which is independent of x is $27y_0^2-4y_0$; and this is to be equal to zero; so that the two singular values of y_0 are 0 and $\frac{4}{27}$, of which the former is the absolute minimum, and the latter is the absolute maximum of all the relative maxima and minima.

Ex. 2.
$$F(x) = x^4-4ax^3-8a^2x^2+11a^4,$$
$$F'(x) = 4x^3-12ax^2-16a^2x.$$

If we proceed to find the greatest common measure of $F(x)-y_0$, and of $F'(x)$, the first remainder which is independent of x is $y_0^3+109a^4y_0^2-936a^8y_0$, of which, when equated to zero, the roots are 0, $8a^4$, $-117a^4$: whence we know the relative maxima and minima, and therefore the absolute values also.

SECTION 2.—*On maxima and minima of implicit functions of two variables.*

154.] Suppose that it is required to find the values of x which make y a maximum or minimum, when the equation connecting y and x is of the form, $u = F(x, y) = c$; then, by Art. 48,

$$\frac{dy}{dx} = -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)}, \tag{4}$$

and as the necessary condition of such a singular value of y is, that $\frac{dy}{dx}$ changes sign, and as a change of sign can take place only by $\frac{dy}{dx}$ being equal to 0 or to ∞, it follows that either

$\left(\frac{du}{dx}\right) = 0$ or ∞, or $\left(\frac{du}{dy}\right) = 0$ or ∞. Let us however assume that $\frac{dy}{dx}$ is not indeterminate in form, and thus that $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ are not simultaneously equal either to 0 or to ∞.

Suppose that $\left(\frac{du}{dx}\right) = 0$, at a particular value for which $\left(\frac{du}{dy}\right)$ does not vanish or become infinite, then the singular value depends on the sign of $\frac{d^2y}{dx^2}$, see Art. 150; being a maximum or minimum according as $\frac{d^2y}{dx^2}$ is negative or positive; and by Art. 84, equation (122), if $\left(\frac{du}{dx}\right) = 0$,

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{d^2u}{dx^2}\right)}{\left(\frac{du}{dy}\right)}; \tag{5}$$

hence if this expression is positive, there is a minimum, and if it is negative, there is a maximum value.

This method however of determining maxima and minima is very incomplete, as it does not discuss the cases where $\left(\frac{du}{dy}\right)$, or any other of the partial derived-functions, becomes infinite; and it is therefore to be taken as a suggestion of the manner in which such problems are to be solved: the best plan is to determine the special maxima and minima values for each problem separately, as follows.

Ex. 1. It is required to find the maxima and minima values of y, having given, $y^3+x^3-3axy = 0$.

$$\frac{dy}{dx} = -\frac{x^2-ay}{y^2-ax};$$

which $= 0$, if $x^2 = ay$, that is, if $y = \frac{x^2}{a}$, whereby we have from the equation

$$x^6 = 2a^3x^3;$$

$$\therefore \quad x = 0, \quad y = 0; \qquad \text{and} \quad x = 2^{\frac{1}{3}}a, \quad y = 2^{\frac{2}{3}}a.$$

The latter values render the denominator of $\frac{dy}{dx}$ positive; and therefore there is a change of sign of $\frac{dy}{dx}$ from + to −, and thus these values correspond to a maximum.

If $x = 0$, $y = 0$, $\frac{dy}{dx} = \frac{0}{0}$, which must be evaluated as in Art. 139;

$$\frac{dy}{dx} = -\frac{2x - a\dfrac{dy}{dx}}{2y\dfrac{dy}{dx} - a} = -\frac{dy}{dx}, \text{ if } x = 0,\ y = 0;$$

$$\therefore \quad \frac{dy}{dx} = 0, \text{ or } = \infty.$$

If $\frac{dy}{dx} = 0$, it changes sign from − to +, which indicates a minimum.

SECTION 3.—*Maxima and minima of an explicit function of two independent variables.*

155.] Let $u = \text{F}(x, y)$ be the explicit function of two independent variables x and y, of which the maxima and minima are to be determined.

Now observing the definition of maxima and minima, given in Art. 144, viz. that a maximum is greater, and a minimum less, than any and every value of the function in its immediate neighbourhood, it follows that if x_0 and y_0 are specific values of x and y, which give such a singular value to $\text{F}(x, y)$; then $\text{F}(x_0, y_0)$ is greater or less, as the case may be, than any value corresponding to the variables, whether x infinitesimally varies while y does not vary, or whether y infinitesimally varies while x does not vary, or whether x and y are simultaneously and infinitesimally increased, or whether as x is infinitesimally increased y is similarly decreased, or *vice versâ*. This property may be thus expressed: If $\text{F}(x_0, y_0)$ is a maximum or a minimum, $\text{F}(x_0 \pm h, y_0 \pm k)$ is less or greater than $\text{F}(x_0, y_0)$ whatever are the signs of h and k, which are infinitesimal increments, and in whatever manner the signs are combined; and also whether $k = 0$, when x is increased or diminished by h; and whether $h = 0$, when y is increased or diminished by k.

156.] This being the definition of such singular values, I propose to extend to functions of two variables the theory of criteria which, in Art. 150, has been applied to those of one variable.

Let us in the first place assume that neither $\left(\frac{d\mathrm{F}}{dx}\right)$ nor $\left(\frac{d\mathrm{F}}{dy}\right)$ vanishes for the values x_0, y_0; then, by Art. 140, (51),

$$\mathrm{F}(x_0+h, y_0+k) - \mathrm{F}(x_0, y_0) = \left[\left(\frac{d\mathrm{F}}{dx}\right) h + \left(\frac{d\mathrm{F}}{dy}\right) k\right]_{\substack{x_0+\theta h \\ y_0+\theta k}}; \tag{6}$$

the meaning of the notation in the right-hand member of this equation being that x_0 and y_0 are to be respectively replaced by $x_0 + \theta h$ and $y_0 + \theta k$, where θ is the general symbol of some positive and proper fraction. Now if $\mathrm{F}(x_0, y_0)$ is a maximum or a minimum, the left-hand member of (6) will have the same sign, h and k being infinitesimal, whether h and k are positive or negative, and when either one or the other of them is absolutely zero; and of course its equivalent must be subject to the same condition. But these circumstances cannot exist in the right-hand member of (6); that is, so long as the equivalent of the left-hand member takes the form given in (6). It is necessary therefore that it should have another form; and such it will take only when, for $x = x_0$ and $y = y_0$,

$$\left(\frac{d\mathrm{F}}{dx}\right) = 0, \qquad \left(\frac{d\mathrm{F}}{dy}\right) = 0; \tag{7}$$

in which case, by reason of (51), Art. 140,

$$\mathrm{F}(x_0+h, y_0+k) - \mathrm{F}(x_0, y_0)$$
$$= \frac{1}{1.2}\left[\left(\frac{d^2\mathrm{F}}{dx^2}\right) h^2 + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) hk + \left(\frac{d^2\mathrm{F}}{dy^2}\right) k^2\right]_{\substack{x_0+\theta h \\ y_0+\theta k}}. \tag{8}$$

If the left-hand member satisfies the second condition above stated for a maximum or a minimum, viz. that the sign of it is the same whether either h or k, but not both, is zero, it is necessary that $\left(\frac{d^2\mathrm{F}}{dx^2}\right)$ and $\left(\frac{d^2\mathrm{F}}{dy^2}\right)$ should be of the same sign. Also of the quadratic expression in the right-hand member the first and third terms are the same whether h and k are positive or negative: but the sign of the second term may be either positive or negative, according as h and k have the same or different signs; it is necessary therefore that the relation between the three terms should be such that the sign of the whole expression within the brackets should not be altered by changes of sign of h and k. This condition will be obtained, if the roots of the expression are imaginary; because in that case it will be the arithmetical sum of two squares. The roots of a quadratic are imaginary, if four times the product of the first and last terms is greater than the square of the middle term: that is, in this case, if

$$\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) \text{ is greater than } \left(\frac{d^2\text{F}}{dx\,dy}\right)^2. \tag{9}$$

If this condition is satisfied, the sign of the right-hand member of (8) is the same as that of $\left(\frac{d^2\text{F}}{dx^2}\right)$ or $\left(\frac{d^2\text{F}}{dy^2}\right)$; and therefore $\text{F}(x_0, y_0)$ is a maximum or a minimum according as $\left(\frac{d^2\text{F}}{dx^2}\right)$ and $\left(\frac{d^2\text{F}}{dy^2}\right)$ are negative or positive. Hence, if $\text{F}(x_0, y_0)$ is a maximum or a minimum, when $x = x_0$ and $y = y_0$,

$$\left.\begin{array}{l} \left(\frac{d\text{F}}{dx}\right) = 0, \qquad \left(\frac{d\text{F}}{dy}\right) = 0, \\ \left(\frac{d^2\text{F}}{dx^2}\right) \text{ and } \left(\frac{d^2\text{F}}{dy^2}\right) \text{ are of the same sign,} \\ \left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) \text{ is greater than } \left(\frac{d^2\text{F}}{dx\,dy}\right)^2; \end{array}\right\} \tag{10}$$

and $\text{F}(x_0, y_0)$ is a maximum or a minimum according as $\left(\frac{d^2\text{F}}{dx^2}\right)$ and $\left(\frac{d^2\text{F}}{dy^2}\right)$ are negative or positive. The inequality (9) having been first determined by Lagrange, is known by the name of Lagrange's condition.

Examples of illustration are given in a subsequent Article.

157.] If when $x = x_0$, and $y = y_0$, $\left(\frac{d\text{F}}{dx}\right) = \left(\frac{d\text{F}}{dy}\right) = 0$, and also $\left(\frac{d^2\text{F}}{dx^2}\right) = \left(\frac{d^2\text{F}}{dx\,dy}\right) = \left(\frac{d^2\text{F}}{dy^2}\right) = 0$, then by (51), Art. 140,

$$\text{F}(x_0+h, y_0+k) - \text{F}(x_0, y_0) =$$

$$\frac{1}{1.2.3}\left[\left(\frac{d^3\text{F}}{dx^3}\right)h^3 + 3\left(\frac{d^3\text{F}}{dx^2dy}\right)h^2k + 3\left(\frac{d^3\text{F}}{dx\,dy^2}\right)hk^2 + \left(\frac{d^3\text{F}}{dy^3}\right)k^3\right]_{\substack{x_0+\theta h \\ y_0+\theta k.}} \tag{11}$$

But it is clear that the right-hand member, being of uneven dimensions in terms of h and k, will change sign with h and k; and therefore if the equivalent of the left-hand member takes the form given in (11), $F(x_0, y_0)$ cannot be a maximum or a minimum. If therefore all the first and the second derived-functions vanish when $x = x_0$ and $y = y_0$, $F(x_0, y_0)$ cannot be a maximum or a minimum unless also all the third derived-functions vanish; in which case the equivalent of the left-hand member of (6) involves fourth derived-functions, and the sum of the powers of h and k is four in every term. This circumstance is consistent with a singular value of $F(x_0, y_0)$, if the biquadratic expression in terms of h and k is subject to conditions analogous to that given in (9), so that it should not change sign with h and k. The conditions however are such that it is unnecessary to state them. And the singular value will be a maximum or a minimum according as $\left(\frac{d^4 F}{dx^4}\right)$ and $\left(\frac{d^4 F}{dy^4}\right)$ are negative or positive.

And generally if $F(x_0, y_0)$ is a singular value, all the partial derived-functions vanish for the particular values of the variables up to those of an odd order inclusive; and those of the even order, which are the first not to vanish, must be so related that the sign of the expression involving h and k should be the same, whatever are the signs of h and k.

158.] Let us however consider the subject of maxima and minima of functions of two variables from another point of view. If $F(x, y)$ has such a singular value, it is necessary that it should exist for infinitesimal variations of x when y does not vary; and for infinitesimal variations of y when x does not vary. If the function fulfils the former of these conditions, we shall call it an x-partial maximum or minimum; and if it fulfils the latter a y-partial maximum or minimum. And when these partial maxima or minima are combined, we have what may be called a total maximum or minimum. The criteria of such a total maximum or minimum are now to be discussed. If however a partial maximum or minimum is combined with a partial minimum or maximum, the value does not fulfil the conditions of the singular values which are given in Art. 144, viz. that a maximum value is greater, and a minimum value is less, than each and every value of the function in its immediate neighbourhood.

Let us express these conditions and circumstances mathematically. If $F(x, y)$ is that function of which the x-partial singular value is to be found, it is necessary that $\left(\frac{dF}{dx}\right)$ should change its sign at the corresponding values of x and y: and thus it is necessary that $\left(\frac{d\text{F}}{dx}\right)$ should be equal to zero or to infinity: the case where $\left(\frac{d\text{F}}{dx}\right) = \infty$ we will at present omit: and according as $\left(\frac{d\text{F}}{dx}\right)$ at the critical value changes its sign from + to − or from − to +, so is the corresponding singular value a partial maximum or a partial minimum.

Similarly for the y-partial singular value it is necessary that $\left(\frac{d\text{F}}{dy}\right)$ should change its sign at the critical value, and therefore that, omitting the infinite value, $\left(\frac{d\text{F}}{dy}\right) = 0$; and according as the change of sign of $\left(\frac{d\text{F}}{dy}\right)$ is from + to −, or from − to +, so is the corresponding singular value a partial maximum or a partial minimum. To render our notions as free as possible from confusion, I will consider first the subject of maxima, and subsequently that of minima.

Now as the total maximum arises from the combination of two partial maxima, it is necessary that the conditions which they require should be simultaneously satisfied. And therefore at such a total maximum we must have

$$\left(\frac{d\text{F}}{dx}\right) = 0, \qquad \left(\frac{d\text{F}}{dy}\right) = 0. \tag{12}$$

And as no relation is given between dx and dy, and as these equations are simultaneously true, the quality of $\text{F}(x, y)$, which each of them separately represents, is also true for all infinitesimal variations of the function at its singular value; and consequently whatever is the direction along which that variation takes place.

The sufficiency of (12) for *all* directions of variation of the function at the singular value may thus be shewn. Let x and y be expressed in terms of new variables ξ and η by means of the following equations, wherein the a's and b's are constants,

$$\left.\begin{aligned} x &= a_1 \xi + a_2 \eta \\ y &= b_1 \xi + b_2 \eta \end{aligned}\right\}; \tag{13}$$

whence we have, as in Art. 106,

$$\left.\begin{array}{ll}\left(\dfrac{d\xi}{dx}\right) = \dfrac{b_2}{a_1 b_2 - a_2 b_1}, & \left(\dfrac{d\eta}{dx}\right) = \dfrac{-b_1}{a_1 b_2 - a_2 b_1} \\ \left(\dfrac{d\xi}{dy}\right) = \dfrac{-a_2}{a_1 b_2 - a_2 b_1}, & \left(\dfrac{d\eta}{dy}\right) = \dfrac{a_1}{a_1 b_2 - a_2 b_1}\end{array}\right\}; \qquad (14)$$

$$\left.\begin{array}{l}\therefore \quad (a_1 b_2 - a_2 b_1)\left(\dfrac{d\text{F}}{dx}\right) = b_2\left(\dfrac{d\text{F}}{d\xi}\right) - b_1\left(\dfrac{d\text{F}}{d\eta}\right) \\ \quad\;\; (a_1 b_2 - a_2 b_1)\left(\dfrac{d\text{F}}{dy}\right) = -a_2\left(\dfrac{d\text{F}}{d\xi}\right) + a_1\left(\dfrac{d\text{F}}{d\eta}\right)\end{array}\right\}. \qquad (15)$$

And thus if $\left(\frac{d\text{F}}{dx}\right)$ and $\left(\frac{d\text{F}}{dy}\right)$ vanish, $\left(\frac{d\text{F}}{d\xi}\right)$ and $\left(\frac{d\text{F}}{d\eta}\right)$ also vanish. And therefore, inasmuch as the constants in (13) are undetermined, the first derived-functions vanish, whatever is the direction of the infinitesimal variation of the function: and therefore thus far the conditions of a total maximum are satisfied.

Let the critical values of x and y which satisfy (12) be x_0 and y_0; we have to examine the change of sign which $\left(\frac{d\text{F}}{dx}\right)$ and $\left(\frac{d\text{F}}{dy}\right)$ undergo at these values. In certain cases this change may be conveniently determined by an examination of the critical factors themselves, according to the method explained in Articles 145 and 146. In the general case however the following process gives the required criteria.

Let us assume that for the critical values of x and y all the second derived-functions do not vanish, and let it be remembered that we are investigating the conditions for a total maximum. Then by Art. 149, if for the critical values $\left(\frac{d\text{F}}{dx}\right)$ and $\left(\frac{d\text{F}}{dy}\right)$ vanish, and change sign from + to −, the following ratios are both negative, viz.

$$\frac{\text{D}\left(\dfrac{d\text{F}}{dx}\right)}{dx} \quad \text{and} \quad \frac{\text{D}\left(\dfrac{d\text{F}}{dy}\right)}{dy}; \qquad (16)$$

the total differentials being taken in both numerators, so that we may investigate the change of sign of the first partial derived-functions for changes of not only the corresponding variable, but also of both the variables simultaneously; otherwise we might get from the critical values singular values in certain directions and not in all directions. Now

$$\frac{\text{D}}{dx}\left(\frac{d\text{F}}{dx}\right) = \left(\frac{d^2\text{F}}{dx^2}\right) + \left(\frac{d^2\text{F}}{dx\,dy}\right)\frac{dy}{dx}, \qquad (17)$$

$$\frac{\mathrm{D}}{dy}\left(\frac{d\mathrm{F}}{dy}\right) = \left(\frac{d^2\mathrm{F}}{dx\,dy}\right)\frac{dx}{dy} + \left(\frac{d^2\mathrm{F}}{dy^2}\right). \tag{18}$$

In (17) and (18) dx and dy are arbitrary; in fact, so far as we have proceeded, dx and dy in (17) need not be the same as the dx and dy in (18). Let us however suppose them to be so: and for convenience of expression, let us substitute as follows:

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right) = \mathrm{A}, \quad \left(\frac{d^2\mathrm{F}}{dx\,dy}\right) = \mathrm{B}, \quad \left(\frac{d^2\mathrm{F}}{dy^2}\right) = \mathrm{C}. \tag{19}$$

See MS. * To examine the signs of (17) and (18), let us assume θ to represent some quantity to which both may be equated; so that we have

$$\mathrm{A}\,dx + \mathrm{B}\,dy = \theta\,dx,$$

$$\mathrm{B}\,dx + \mathrm{C}\,dy = \theta\,dy;$$

which may be written

$$(\mathrm{A}-\theta)\,dx + \mathrm{B}\,dy = 0, \tag{20}$$

$$\mathrm{B}\,dx + (\mathrm{C}-\theta)\,dy = 0; \tag{21}$$

whence, eliminating dx and dy, we have the following quadratic in θ,

$$(\mathrm{A}-\theta)(\mathrm{C}-\theta) - \mathrm{B}^2 = 0, \tag{22}$$

$$\theta^2 - (\mathrm{A}+\mathrm{C})\,\theta + \mathrm{AC} - \mathrm{B}^2 = 0; \tag{23}$$

of which the two roots are to be negative, since (17) and (18) must be negative, so that $\mathrm{F}(x_0, y_0)$ may be a maximum; therefore all the coefficients must be positive, that is,

A and C must both be negative, (24)

and $\mathrm{AC}-\mathrm{B}^2$ must be positive. (25)

Hence the necessary conditions that x_0, y_0 should render $\mathrm{F}(x, y)$ a maximum are the following:

$$\left.\begin{array}{ll} (\alpha) & \left(\frac{d\mathrm{F}}{dx}\right) = 0, \quad \left(\frac{d\mathrm{F}}{dy}\right) = 0, \\ (\beta) & \left(\frac{d^2\mathrm{F}}{dx^2}\right) \text{ and } \left(\frac{d^2\mathrm{F}}{dy^2}\right) \text{ must be negative,} \\ (\gamma) & \left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d^2\mathrm{F}}{dy^2}\right) - \left(\frac{d^2\mathrm{F}}{dx\,dy}\right)^2 \text{ must be positive.} \end{array}\right\} \tag{26}$$

159.] Similarly to determine a total minimum, it is requisite that it should be a partial minimum of a partial minimum, and therefore, supposing that $\left(\frac{d^2\mathrm{F}}{dx^2}\right)$, $\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)$, $\left(\frac{d^2\mathrm{F}}{dy^2}\right)$ do not all vanish at the critical values, that $\left(\frac{d\text{F}}{dx}\right) = 0$, and $\left(\frac{d\text{F}}{dy}\right) = 0$, and that $\frac{\text{D}}{dx}\left(\frac{d\text{F}}{dx}\right)$ and $\frac{\text{D}}{dy}\left(\frac{d\text{F}}{dy}\right)$ should be positive; accordingly, following the process of the last Article, the two values of θ, and therefore the roots of (23), must be positive; whence it follows, that A and C must be both positive, and that $\text{AC} - \text{B}^2$ must be positive; and therefore the conditions that x_0, y_0 should render $\text{F}(x, y)$ a minimum are, that

$$\left.\begin{array}{ll} (\alpha) & \left(\frac{d\text{F}}{dx}\right) = 0, \quad \left(\frac{d\text{F}}{dy}\right) = 0, \\[2ex] (\beta) & \left(\frac{d^2\text{F}}{dx^2}\right) \text{ and } \left(\frac{d^2\text{F}}{dy^2}\right) \text{ must be positive,} \\[2ex] (\gamma) & \left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) - \left(\frac{d^2\text{F}}{dx\,dy}\right)^2 \text{ must be positive.} \end{array}\right\} \quad (27)$$

160.] As (23) has two roots, and thus gives two determinate values to θ, so by reason of (20) and (21) dx and dy will bear a determinate ratio to each other; it may perhaps hence be inferred that the criteria of maxima and minima, which are thus determined, are applicable for variations of the original function in only two certain directions. This however is not the case: for the criteria are true and sufficient whatever is the direction of variation of the original function which is due to the combined variations of the variables.

Let the variables x and y be replaced by new variables ξ and η, with which they are related by the equations (13), Art. 158; then from (15) we have

$$(a_1 b_2 - a_2 b_1)^2 \left(\frac{d^2\text{F}}{dx^2}\right) = b_2^2 \left(\frac{d^2\text{F}}{d\xi^2}\right) - 2 b_1 b_2 \left(\frac{d^2\text{F}}{d\xi\, d\eta}\right) + b_1^2 \left(\frac{d^2\text{F}}{d\eta^2}\right), \quad (28)$$

$$(a_1 b_2 - a_2 b_1)^2 \left(\frac{d^2\text{F}}{dx\, dy}\right) = -a_2 b_2 \left(\frac{d^2\text{F}}{d\xi^2}\right) + (a_1 b_2 + a_2 b_1)\left(\frac{d^2\text{F}}{d\xi\, d\eta}\right) - a_1 b_1 \left(\frac{d^2\text{F}}{d\eta^2}\right), \quad (29)$$

$$(a_1 b_2 - a_2 b_1)^2 \left(\frac{d^2\text{F}}{dy^2}\right) = a_2^2 \left(\frac{d^2\text{F}}{d\xi^2}\right) - 2 a_1 a_2 \left(\frac{d^2\text{F}}{d\xi\, d\eta}\right) + a_1^2 \left(\frac{d^2\text{F}}{d\eta^2}\right); \quad (30)$$

whence we have

$$\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) - \left(\frac{d^2\text{F}}{dx\, dy}\right)^2 = \left(\frac{d^2\text{F}}{d\xi^2}\right)\left(\frac{d^2\text{F}}{d\eta^2}\right) - \left(\frac{d^2\text{F}}{d\xi\, d\eta}\right)^2; \quad (31)$$

and therefore whatever is the sign of the left-hand member, the same also is that of the right-hand member. If therefore the left-hand member is positive, and thus fulfils the condition necessary for a singular value, the right-hand member is also positive, and similarly fulfils the required condition. Now in the equations (13), which connect x, y, ξ, and η, the directions of ξ and η relatively to those of x and y, or, to speak geometrically, the angles through which the coordinate axes are turned, depend on a_1, a_2, b_1, b_2; and the equivalence (31) is independent of these constants: hence if Lagrange's condition is satisfied for any system of two variables, it is also satisfied for every other system which is connected with it by two linear equations of the form (13), Art. 158; and therefore if Lagrange's condition is satisfied for the infinitesimal variations of the variables in any one direction, it is satisfied for their variations in all directions.

It is also to be observed that if the right-hand member of (31) is positive, then from (28) and (30) $\left(\frac{d^2\mathrm{F}}{d\xi^2}\right)$ and $\left(\frac{d^2\mathrm{F}}{d\eta^2}\right)$ have the same sign as $\left(\frac{d^2\mathrm{F}}{dx^2}\right)$ and $\left(\frac{d^2\mathrm{F}}{dy^2}\right)$; and therefore if all the conditions of a total maximum or minimum which are given in (26) and (27) are satisfied for one direction of the variation of the variables, they are also satisfied for all directions and for all the circumstances of such total singular values as exist at the critical values of the variables.

If all the second derived-functions of the original function vanish at the critical values, the preceding results come to nought: as however I have alluded to this case in Art. 157, it is unnecessary to say more.

Some examples of total maxima and minima of functions of two variables are subjoined.

161.] Ex. 1. To determine whether any, and what, values of x and y render x^2y+xy^2-axy a maximum or minimum.

$$\mathrm{F}(x,y) = x^2y+xy^2-axy;$$

$$\left(\frac{d\mathrm{F}}{dx}\right) = 2xy+y^2-ay = y(2x+y-a);$$

$$\left(\frac{d\mathrm{F}}{dy}\right) = 2xy+x^2-ax = x(2y+x-a);$$

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right) = 2y, \quad \left(\frac{d^2\mathrm{F}}{dx\,dy}\right) = 2x+2y-a, \quad \left(\frac{d^2\mathrm{F}}{dy^2}\right) = 2x;$$

therefore putting $\left(\frac{d\mathrm{F}}{dx}\right)=0$, and $\left(\frac{d\mathrm{F}}{dy}\right)=0$, we have the following systems of simultaneous values:

$$\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\}; \quad \left.\begin{array}{l} x=a \\ y=0 \end{array}\right\}; \quad \left.\begin{array}{l} x=0 \\ y=a \end{array}\right\}; \quad \left.\begin{array}{l} x=\frac{a}{3} \\ y=\frac{a}{3} \end{array}\right\};$$

neither of the first three of which satisfy Lagrange's condition; and from the last we have $\left(\frac{d^2\text{F}}{dx^2}\right)$ and $\left(\frac{d^2\text{F}}{dy^2}\right)$ positive, and $\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right)-\left(\frac{d^2\text{F}}{dx\,dy}\right)^2=\frac{a^2}{3}$, which is a positive quantity;

$\therefore$ x^2y+xy^2-axy is a minimum, viz. $-\frac{a^3}{27}$, when $x=y=\frac{a}{3}$.

Ex. 2. To find a point within a triangle for which, if lines are drawn to the angles, the sum of their squares is a minimum.

Take A, one of the angular points, see fig. 20, of the triangle, for the origin; and let the base $\text{AB}=a$, and the coordinates to C be h, k; and let the coordinates of P be x, y, and the sum of the squares $=\text{F}(x,y)$.

$$\begin{aligned} \therefore \quad \text{F}(x,y) &= x^2+y^2+(a-x)^2+y^2+(h-x)^2+(k-y)^2, \\ &= 3x^2+3y^2-2(a+h)x-2ky+a^2+h^2+k^2; \end{aligned}$$

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) = 6x-2(a+h) = 0, \text{ if } x=\frac{a+h}{3};$$

$$\left(\frac{d\text{F}}{dy}\right) = 6y-2k = 0, \text{ if } y=\frac{k}{3};$$

and as in both cases the change of sign is from $-$ to $+$, we have a partial minimum of a partial minimum, and therefore the necessary conditions of a total minimum.

Ex. 3. Inscribe the greatest triangle in a circle.

Fig. 21. Let one of the angular points of the triangle be at A, the extremity of the diameter ACB; and let P and Q be the other angles; let $\text{AC}=\text{CB}=a$.

$\text{PAB}=\theta$, $\text{QAB}=\phi$; therefore by a property of the circle,

$$\text{PA}=2a\cos\theta, \qquad \text{QA}=2a\cos\phi;$$

$$\begin{aligned} \therefore \quad \text{area of triangle} = \text{F}(\theta,\phi) &= \frac{1}{2}\,\text{PA}\times\text{QA}\times\sin(\theta+\phi), \\ &= 2a^2\cos\theta\cos\phi\sin(\theta+\phi); \end{aligned}$$

$$\begin{aligned} \therefore \quad \left(\frac{d\text{F}}{d\theta}\right) &= 2a^2\cos\phi\,\{\cos\theta\cos(\theta+\phi)-\sin\theta\sin(\theta+\phi)\}, \\ &= 2a^2\cos\phi\cos(2\theta+\phi); \end{aligned}$$

$$\left(\frac{d\text{F}}{d\phi}\right) = 2a^2\cos\theta\cos(2\phi+\theta);$$

$$\therefore \quad \left(\frac{d^2\text{F}}{d\theta^2}\right) = -4a^2\cos\phi\sin(2\theta+\phi);$$

$$\left(\frac{d^2\text{F}}{d\theta\, d\phi}\right) = -2a^2\sin 2(\theta+\phi);$$

$$\left(\frac{d^2\text{F}}{d\phi^2}\right) = -4a^2\cos\theta\sin(2\phi+\theta).$$

But if $\left(\frac{d\text{F}}{d\theta}\right) = 0$, and $\left(\frac{d\text{F}}{d\phi}\right) = 0$, $2\theta+\phi = \frac{\pi}{2}$, and $2\phi+\theta = \frac{\pi}{2}$; therefore $\theta = \phi = \frac{\pi}{6}$; in which case $\left(\frac{d^2\text{F}}{d\theta^2}\right)$ and $\left(\frac{d^2\text{F}}{d\phi^2}\right)$ are both negative, and $\left(\frac{d^2\text{F}}{d\theta^2}\right)\left(\frac{d^2\text{F}}{d\phi^2}\right) - \left(\frac{d^2\text{F}}{d\theta\, d\phi}\right)^2 = 9a^4$; therefore the above critical values of θ and ϕ give a maximum: and as $\theta = \phi = \frac{\pi}{6}$, the triangle is equilateral. Hence the greatest triangle that can be inscribed in a circle is the equilateral one.

162.] In the case in which $\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) - \left(\frac{d^2\text{F}}{dx\, dy}\right)^2$ is negative, the last term of the quadratic in θ of equation (23) is negative, and therefore the two values of θ are of different signs; whence, by means of (17) and (18), it follows that one of the partial singular values is a maximum, and the other is a minimum: and therefore the conditions requisite for a total maximum or minimum are not fulfilled.

And if $\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) = \left(\frac{d^2\text{F}}{dx\, dy}\right)^2$, then the last term of equation (23) = 0, and therefore one value of θ is zero; and therefore either (17) or (18) = 0, and therefore either $\left(\frac{d\text{F}}{dx}\right)$ or $\left(\frac{d\text{F}}{dy}\right)$ undergoes no variation; whereas then there is a partial maximum or minimum with respect to one of the variables, the other is such that corresponding to its variations the function is constant; hence we have a *locus* of such partial maxima or minima.

These several conditions will be more clearly understood from the geometrical analogues of them in the theory of curved surfaces; which however it would be premature to explain in this place, and therefore we reserve them until they naturally arise in the course of the treatise.

SECTION 4.—*Maxima and minima of functions of three and more independent variables.*

163.] Firstly, let us consider a function of three independent variables, x, y, z, of the form

$$u = \text{F}(x, y, z).$$

Extending the principles of Art. 158 and 159 to this more general case, it appears that a total maximum or minimum of a function of three variables must arise from the combination of three several partial maxima or minima with respect to the several variables. And also, as any two of the three variables may vary, while the remaining one does not vary, it appears that the conditions of such a combination of two partial maxima or minima must be fulfilled. Which conditions are,

$$\left(\frac{d\text{F}}{dx}\right) = 0, \qquad \left(\frac{d\text{F}}{dy}\right) = 0, \qquad \left(\frac{d\text{F}}{dz}\right) = 0; \tag{32}$$

$$\left.\begin{array}{c}\left(\frac{d^2\text{F}}{dy^2}\right)\left(\frac{d^2\text{F}}{dz^2}\right) - \left(\frac{d^2\text{F}}{dy\,dz}\right)^2 > 0; \quad \left(\frac{d^2\text{F}}{dz^2}\right)\left(\frac{d^2\text{F}}{dx^2}\right) - \left(\frac{d^2\text{F}}{dz\,dx}\right)^2 > 0; \\ \left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) - \left(\frac{d^2\text{F}}{dx\,dy}\right)^2 > 0;\end{array}\right\} \tag{33}$$

therefore $\left(\frac{d^2\text{F}}{dx^2}\right)$, $\left(\frac{d^2\text{F}}{dy^2}\right)$, $\left(\frac{d^2\text{F}}{dz^2}\right)$ must be of the same sign; and the singular value of $\text{F}(x, y, z)$ is a maximum or minimum, according as they are negative or positive. There is also another relation between the several partial second derived-functions; $\frac{\text{D}}{dx}\left(\frac{d\text{F}}{dx}\right)$, $\frac{\text{D}}{dy}\left(\frac{d\text{F}}{dy}\right)$, and $\frac{\text{D}}{dz}\left(\frac{d\text{F}}{dz}\right)$ must be of the same sign, and be negative for a maximum value, and positive for a minimum; and

$$\frac{\text{D}}{dx}\left(\frac{d\text{F}}{dx}\right) = \left(\frac{d^2\text{F}}{dx^2}\right) + \left(\frac{d^2\text{F}}{dx\,dy}\right)\frac{dy}{dx} + \left(\frac{d^2\text{F}}{dx\,dz}\right)\frac{dz}{dx}; \tag{34}$$

$$\frac{\text{D}}{dy}\left(\frac{d\text{F}}{dy}\right) = \left(\frac{d^2\text{F}}{dy\,dx}\right)\frac{dx}{dy} + \left(\frac{d^2\text{F}}{dy^2}\right) + \left(\frac{d^2\text{F}}{dy\,dz}\right)\frac{dz}{dy}; \tag{35}$$

$$\frac{\text{D}}{dz}\left(\frac{d\text{F}}{dz}\right) = \left(\frac{d^2\text{F}}{dz\,dx}\right)\frac{dx}{dz} + \left(\frac{d^2\text{F}}{dz\,dy}\right)\frac{dy}{dz} + \left(\frac{d^2\text{F}}{dz^2}\right). \tag{36}$$

Now for convenience of expression, let

$$\left(\frac{d^2\text{F}}{dx^2}\right) = \text{A}, \qquad \left(\frac{d^2\text{F}}{dy^2}\right) = \text{B}, \qquad \left(\frac{d^2\text{F}}{dz^2}\right) = \text{C},$$

$$\left(\frac{d^2\text{F}}{dy\,dz}\right) = \text{E}, \qquad \left(\frac{d^2\text{F}}{dz\,dx}\right) = \text{F}, \qquad \left(\frac{d^2\text{F}}{dx\,dy}\right) = \text{G};$$

and as (34), (35), (36) are to be of the same sign, let us employ a process of reasoning similar to that of Art. 158, and let us assume θ to be the symbol for some quantity which is the same in all; then the following system results:

$$\left.\begin{aligned}(\text{A}-\theta)\,dx+\text{G}\,dy+\text{F}\,dz&=0\\ \text{G}\,dx+(\text{B}-\theta)\,dy+\text{E}\,dz&=0\\ \text{F}\,dx+\text{E}\,dy+(\text{C}-\theta)\,dz&=0\end{aligned}\right\};\qquad(37)$$

whence, by cross-multiplication,

$$(\text{A}-\theta)(\text{B}-\theta)(\text{C}-\theta)-\text{E}^2(\text{A}-\theta)-\text{F}^2(\text{B}-\theta)-\text{G}^2(\text{C}-\theta)$$
$$+2\,\text{EFG}=0;\quad(38)$$

the common Discriminating Cubic, as it is called, and which has three real roots; and, when expanded, becomes

$$\theta^3-(\text{A}+\text{B}+\text{C})\theta^2+(\text{BC}+\text{CA}+\text{AB}-\text{E}^2-\text{F}^2-\text{G}^2)\theta$$
$$-(\text{ABC}+2\text{EFG}-\text{AE}^2-\text{BF}^2-\text{CG}^2)=0.\quad(39)$$

Of this equation the three roots are to be of the same sign, and the result is a maximum or a minimum, according as they are negative or positive; therefore, besides the former conditions (32) and (33), the following expression must be negative for a maximum and positive for a minimum, viz.:

$$\text{ABC}+2\text{EFG}-\text{AE}^2-\text{BF}^2-\text{CG}^2.\qquad(40)$$

Hence, that a function of three variables may have a maximum or a minimum value, the critical values must satisfy $3+3+1$ ($=7$) conditions, viz. three of equations (32), three of equations (33), and one of equation (40).

164.] Lastly, let us consider the general case; and let

$$\text{F}(x_1, x_2, \dots x_n)\qquad(41)$$

be a function of n independent variables, of which the maxima and minima are to be determined; and, for convenience of notation, let

$$\left(\frac{d^2\text{F}}{dx_1^2}\right)=(1,1),\quad\left(\frac{d^2\text{F}}{dx_1\,dx_2}\right)=(1,2),\ \dots\ \left(\frac{d^2\text{F}}{dx_1\,dx_n}\right)=(1,n),$$
$$\left(\frac{d^2\text{F}}{dx_2\,dx_1}\right)=(2,1),\quad\left(\frac{d^2\text{F}}{dx_2^2}\right)=(2,2),\ \dots\ \left(\frac{d^2\text{F}}{dx_2\,dx_n}\right)=(2,n),$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\left(\frac{d^2\text{F}}{dx_n\,dx_1}\right)=(n,1),\quad\left(\frac{d^2\text{F}}{dx_n\,dx_2}\right)=(n,2),\ \dots\ \left(\frac{d^2\text{F}}{dx_n^2}\right)=(n,n);$$

of which it is to be observed, that

$$(1,2)=(2,1),\ \dots\ (n,1)=(1,n),\ \dots.$$

Now the singular value of (41) must arise from the combination of n similar partial singular values due to the separate variation of each of the n variables; and therefore we must have

$$\left(\frac{d\text{F}}{dx_1}\right) = 0, \quad \left(\frac{d\text{F}}{dx_2}\right) = 0, \; \ldots\ldots \; \left(\frac{d\text{F}}{dx_n}\right) = 0; \tag{42}$$

also $\frac{\text{D}}{dx_1}\left(\frac{d\text{F}}{dx_1}\right)$, $\frac{\text{D}}{dx_2}\left(\frac{d\text{F}}{dx_2}\right)$, …… $\frac{\text{D}}{dx_n}\left(\frac{d\text{F}}{dx_n}\right)$ are all to be of the same sign; negative, that is, for a maximum, and positive for a minimum. If θ is the quantity to which each may be equated, then we shall have the following equations,

$$\left.\begin{array}{l}\{(1,1)-\theta\}\,dx_1 + (1,2)\,dx_2 + \ldots\ldots + (1,n)\,dx_n = 0 \\ (2,1)\,dx_1 + \{(2,2)-\theta\}\,dx_2 + \ldots\ldots + (2,n)\,dx_n = 0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ (n,1)\,dx_1 + (n,2)\,dx_2 + \ldots\ldots + \{(n,n)-\theta\}\,dx_n = 0 \end{array}\right\}; \tag{43}$$

whence, by the elimination of the n quantities, $dx_1, dx_2, \ldots dx_n$, there will result an equation in θ of n dimensions, all the roots of which are to be of the same sign; and according as they are positive or negative, will the corresponding value of the function be a minimum or maximum.

The number of conditions which are required to be fulfilled may thus be found: As the total maximum or minimum arises from the combination of the several partial singular values, all the conditions which they involve must separately be satisfied. Hence it is easy to see, that when the equation in θ has been formed, there will be involved, and to be satisfied, in the coefficients of

θ^{n-1}, n conditions,

θ^{n-2}, $\frac{n(n-1)}{1.2}$ - - - -

θ^{n-3}, $\frac{n(n-1)(n-2)}{1.2.3}$ - - - -

. . . .

θ, . . . n - - - -

in the constant term, . . 1 condition;

$$\begin{aligned}\text{the sum of which } &= n + \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)}{1.2.3} + \ldots + n + 1, \\ &= (1+1)^n - 1, \\ &= 2^n - 1.\end{aligned}$$

165.] One remarkable example, wherein the minimum value of a function of many variables is to be discovered, deserves insertion. The problem occurs in the combination of observations, all of which are subject to, and are supposed to be affected with, accidental errors; and the object is to determine the most probable conclusion from the series of given results which are affected with these errors. The process, of which the following is an outline, is generally called the Method of Least Squares.

Suppose that there are n unknown quantities $x_1, x_2, \dots x_n$; and let $u_1, u_2, \dots u_m$ be m other quantities connected with them by m given equations, so that each of the latter is a given function of some or all of the former. Suppose also that the values of $u_1, u_2, \dots u_m$ are capable of being observed; from these observations the values of $x_1, x_2, \dots x_n$ are to be deduced.

Let the observed values of $u_1, u_2, \dots u_m$ be $o_1, o_2, \dots o_m$; the observations then give the m equations,

$$u_1 - o_1 = 0, \quad u_2 - o_2 = 0, \quad \dots \quad u_m - o_m = 0, \tag{44}$$

for the determination of the n unknown quantities.

If m is less than n, these equations are insufficient. If $m = n$ they are generally sufficient, and the solution of the problem is determinate and unique. But if, as is usually the case in practice, m is greater than n, the equations are more than sufficient. Still, if the observations were absolutely accurate, the equations would not be inconsistent, and every sufficient combination of them would give the same values for the unknown quantities. As however the observations are actually liable to error, the equations (44) will in general be inconsistent, and no one set of values of $x_1, x_2, \dots x_n$ can satisfy them all at once. The question is, What set of values are we to adopt?

At the outset it may be observed, that the simplest way of expressing that all the equations (44) subsist at once, would be by the single equation,

$$(u_1 - o_1)^2 + (u_2 - o_2)^2 + \dots + (u_m - o_m)^2 = 0. \tag{45}$$

In the actual case it is impossible to satisfy this equation; but the idea obviously suggests itself of satisfying it as nearly as possible, by choosing the unknown quantities so as to make the expression in the left-hand member of the equation as small as possible.

The question whether this plan really gives the most probable values of the unknown quantities belongs to the Theory of Probabilities, and it would be out of place to discuss it here. The same may be said of the modifications to be introduced when the observations are not all equally liable to error; and of the method of estimating the precision of the results. It may be observed however, that an obvious way of giving greater influence to the better observations is, to multiply each term in the left-hand member of (45) by a positive number representing the goodness or, as it is called, the weight of the corresponding observation; and this is in fact the method indicated by theory; so that the function of which the minimum is to be determined is

$$g_1(u_1-o_1)^2+g_2(u_2-o_2)^2+\ldots+g_m(u_m-o_m)^2; \qquad (46)$$

where $g_1, g_2, \ldots g_m$ are the weights of the several observations; and are proportional respectively to the number of times an observation, of arbitrary fixed liability to error, is to be repeated, in order that the arithmetical mean of its results may be entitled to the same degree of confidence as the single result of the observation in question. The estimation of these weights is the business of the observer; and for our purpose they are to be considered as given constants.

Now if we can find the values of $x_1, x_2, \ldots x_n$ which make the expression (46) a minimum, we may substitute them in the functions $u_1, u_2, \ldots u_m$, and the results may be called the calculated values of these functions; and the differences between these calculated values and the observed values $o_1, o_2, \ldots o_m$ may be called the apparent errors of the observations: they would be the true errors if the calculated values $u_1, u_2, \ldots u_m$ were absolutely correct. Putting $\text{E}_1, \text{E}_2, \ldots \text{E}_m$ for these apparent errors, we have $u_1-o_1=\text{E}_1, \ldots u_m-o_m=\text{E}_m$, and the expression (46) becomes

$$g_1\text{E}_1^2+g_2\text{E}_2^2+\ldots+g_m\text{E}_m^2;$$

and this is to be a minimum. In the case in which the weights of the observations are equal, representing their common value by unity, we have simply $\text{E}_1^2+\text{E}_2^2+\ldots+\text{E}_m^2$, which is to be a minimum; and thus in this case the method consists in determining the unknown quantities so that the sum of the squares of the apparent errors of the observations may be a minimum.

Let us symbolize (46) by Ω; then as $x_1, x_2, \ldots x_n$ are independent variables, and as Ω is to be a minimum,

$$\left(\frac{d\Omega}{dx_1}\right)=0, \quad \left(\frac{d\Omega}{dx_2}\right)=0, \;\ldots\ldots\; \left(\frac{d\Omega}{dx_n}\right)=0;$$

and from these n equations the values of the n unknown quantities $x_1, x_2, \dots x_n$ are to be found. The algebraical solution of these equations is in general impracticable, unless the functions $u_1, u_2, \dots u_m$ are all linear; but as the problems which occur in practice may be reduced to this form, the difficulty does not actually arise. This simplification is effected as follows:

A set of approximate values of the unknown quantities is first obtained in any way that is practicable, or is previously known. Let these be called $\xi_1, \xi_2, \dots \xi_n$; and let their unknown errors be $d\xi_1, d\xi_2, \dots d\xi_n$, so that the true values are $\xi_1+d\xi_1$, $\xi_2+d\xi_2$, $\dots \xi_n+d\xi_n$; and let $d\xi_1, d\xi_2, \dots d\xi_n$ be treated as small quantities, of which the squares, higher powers, and products are to be neglected. Suppose the equation expressing u_1 in terms of $x_1, x_2, \dots x_n$ to be

$$u_1 = \phi(x_1, x_2, \dots x_n);$$

then we have

$$u_1 = \phi(\xi_1 + d\xi_1, \xi_2 + d\xi_2, \dots \xi_n + d\xi_n);$$

and expanding this by (56) of Art. 142, and omitting all terms involving higher powers than the first of the errors, we have

$$u_1 = \phi(\xi_1, \xi_2, \dots \xi_n) + \left(\frac{d\phi}{d\xi_1}\right) d\xi_1 + \left(\frac{d\phi}{d\xi_2}\right) d\xi_2 + \dots + \left(\frac{d\phi}{d\xi_n}\right) d\xi_n;$$

in this equation $\phi(\xi_1, \xi_2, \dots \xi_n)$, $\left(\frac{d\phi}{d\xi_1}\right), \dots \left(\frac{d\phi}{d\xi_n}\right)$ are all given quantities, so that u_1 is reduced to a linear function of the new unknown quantities $d\xi_1, d\xi_2, \dots d\xi_n$. A similar reduction may be effected with $u_2, u_3, \dots u_m$; so that finally the equations

$$u_1 - o_1 = 0, \quad u_2 - o_2 = 0, \quad \dots \quad u_m - o_m = 0,$$

which are given by the observations, are all reduced to the linear form.

If then, as heretofore, we substitute $x_1, x_2, \dots x_n$ for $d\xi_1, d\xi_2, \dots d\xi_n$, and $a, a_1, a_2, \dots k_n$ for the constants, we may assume

$$\left.\begin{aligned} u_1 &= a + a_1 x_1 + a_2 x_2 + \dots + a_n x_n \\ u_2 &= b + b_1 x_1 + b_2 x_2 + \dots + b_n x_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_m &= k + k_1 x_1 + k_2 x_2 + \dots + k_n x_n \end{aligned}\right\}; \qquad (47)$$

and the expression

$$\left.\begin{aligned} g_1(u_1-o_1)^2 + g_2(u_2-o_2)^2 + \dots + g_m(u_m-o_m)^2, \\ \text{or} \qquad g_1 \text{E}_1^2 + \dots + g_m \text{E}_m^2, \end{aligned}\right\} \qquad (48)$$

is to be a minimum.

Here however we may remark that if each of the equations $u_1 - o_1 = 0, \ldots u_m - o_m = 0$, is multiplied by the square root of the weight of the observation by which it was obtained, and if $u_1, o_1, \ldots$ are then written instead of $\{g_1\}^{\frac{1}{2}} u_1$, $\{g_1\}^{\frac{1}{2}} o_1, \ldots$ and $a, a_1, \ldots$ instead of $\{g_1\}^{\frac{1}{2}} a$, $\{g_1\}^{\frac{1}{2}} a_1, \ldots$ the problem will be reduced to the form in which $g_1 = g_2 = \ldots = g_m = 1$, so that this is the only case which need be considered.

Let μ^2 be the sum of the squares of the errors thus modified, which is to be a minimum; then we have

$$\mu^2 = \mathrm{E}_1^2 + \mathrm{E}_2^2 + \ldots + \mathrm{E}_m^2;$$

$$\therefore \quad \mu\, d\mu = 0 = \mathrm{E}_1\, d\mathrm{E}_1 + \mathrm{E}_2\, d\mathrm{E}_2 + \ldots + \mathrm{E}_m\, d\mathrm{E}_m;$$

and replacing $d\mathrm{E}_1, d\mathrm{E}_2, \ldots d\mathrm{E}_m$ by their values from (47) and (48), and arranging the terms as coefficients of $dx_1, dx_2, \ldots dx_n$, we have

$$\begin{aligned} 0 = {} & (a_1 \mathrm{E}_1 + b_1 \mathrm{E}_2 + \ldots + k_1 \mathrm{E}_m)\, dx_1 \\ & + (a_2 \mathrm{E}_1 + b_2 \mathrm{E}_2 + \ldots + k_2 \mathrm{E}_m)\, dx_2 \\ & + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & + (a_n \mathrm{E}_1 + b_n \mathrm{E}_2 + \ldots + k_n \mathrm{E}_m)\, dx_n; \end{aligned}$$

and as $x_1, x_2, \ldots x_n$ are assumed to be independent of each other, the coefficients of each of the differentials in the right-hand member of the equation is equal to zero; so that replacing $\mathrm{E}_1, \mathrm{E}_2, \ldots \mathrm{E}_m$ by their values given in (47) and (48) we have

$$\begin{aligned} & (a_1^2 + b_1^2 + \ldots + k_1^2)\, x_1 + (a_1 a_2 + b_1 b_2 + \ldots + k_1 k_2)\, x_2 + \ldots \\ & \qquad \ldots + (a_1 a_n + b_1 b_n + \ldots + k_1 k_n)\, x_n \\ & \qquad + a_1 (a - o_1) + b_1 (b - o_2) + \ldots + k_1 (k - o_m) = 0, \\ & (a_2 a_1 + b_2 b_1 + \ldots + k_2 k_1)\, x_1 + (a_2^2 + b_2^2 + \ldots + k_2^2)\, x_2 + \ldots \\ & \qquad \ldots + (a_2 a_n + b_2 b_n + \ldots + k_2 k_n)\, x_n \\ & \qquad + a_2 (a - o_1) + b_2 (b - o_2) + \ldots + k_2 (k - o_m) = 0, \\ & . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & (a_n a_1 + b_n b_1 + \ldots + k_n k_1)\, x_1 + (a_n a_2 + b_n b_2 + \ldots + k_n k_2)\, x_2 + \ldots \\ & \qquad \ldots + (a_n^2 + b_n^2 + \ldots + k_n^2)\, x_n \\ & \qquad + a_n (a - o_1) + b_n (b - o_2) + \ldots + k_n (k - o_m) = 0; \end{aligned}$$

whereby we have n linear equations containing n unknown quantities; these may therefore be determined; and the values of them thus found will, according to the method of Least Squares, be affected with the least possible risk of errors. Now without going farther into the subject, and without introducing the convenient symbols which Gauss, to whom we are in great measure indebted for the method, has introduced, I may remark that the practical rules for forming the final simultaneous equations, as appears from the preceding system of equations, are the following. Multiply each equation by the coefficient of x_1 in itself, then the sum of all thus multiplied is the first final equation. Again, multiply each by the coefficient of x_2, and the sum of all thus multiplied is the second final equation: and so on for all the equations. It is evident that we thus obtain n final equations from which the n unknown quantities may be determined. Two examples are subjoined for the purpose of illustrating the process.

166.] Examples of the method of least squares.

Ex. 1. Let there be four linear equations involving three unknown quantities, and of the following form:

$$\begin{aligned} u_1 &= x-y+2z, \\ u_2 &= 3x+2y-5z \\ u_3 &= 4x+y+4z, \\ u_4 &= -x+3y+3z; \end{aligned}$$

and let us suppose that observations are made, and that by them u_1, u_2, u_3, u_4 are found severally to be 3, 5, 21, 14; then the errors of the several equations will be u_1-3, u_2-5, u_3-21, u_4-14: and if μ^2 is the sum of their squares we have

$$\mu^2 = (u_1-3)^2+(u_2-5)^2+(u_3-21)^2+(u_4-14)^2;$$

and as μ^2 is to be a minimum,

$$\mu\,\mathrm{D}\mu = 0 = (u_1-3)\,du_1+(u_2-5)\,du_2+(u_3-21)\,du_3+(u_4-14)\,du_4;$$

in which substituting from the preceding equations, and equating to zero the coefficients of dx, dy, and dz, we have

$$\left.\begin{aligned} u_1+3u_2+4u_3-\ u_4-\ 88 &= 0 \\ -u_1+2u_2+\ u_3+3u_4-\ 70 &= 0 \\ 2u_1-5u_2+4u_3+3u_4-107 &= 0 \end{aligned}\right\};$$

and substituting in terms of x, y, z, we have

$$\left.\begin{aligned} 27x+\ 6y\qquad &= \ 88 \\ 6x+15y+z &= \ 70 \\ y+54z &= 107 \end{aligned}\right\};$$

whence $x = 2.470$, $y = 3.551$, $z = 1.916$; and these are the most probable values of the variables.

Ex. 2. Let there be only one unknown quantity x, for which the values $a_1, a_2, \dots a_m$ have been found by m observations of equal weight. We have then to determine x so that

$$(x-a_1)^2+(x-a_2)^2+\dots+(x-a_m)^2 = u^2$$

may be a minimum. Differentiating this expression and equating to zero the differential, because u is to be a minimum, we have

$$mx-a_1-a_2-\dots-a_m = 0;$$

$$\therefore \quad x = \frac{a_1+a_2+\dots+a_m}{m}.$$

Thus, assuming the validity of the method, the best value for x is the arithmetic mean of the values given by the observations. It was in fact by assuming this result as an axiom, that Gauss first demonstrated the validity of the method. But this is not self-evident, and is not properly axiomatic; and in fact no demonstration has yet been given perfectly free from difficulty, so far as I know.

If the weights of the observations, instead of being equal, were $g_1, g_2, \dots g_m$, the function of which the minimum value is to be determined would be

$$g_1(x-a_1)^2+g_2(x-a_2)^2+\dots+g_m(x-a_m)^2;$$

and the resulting value of x is

$$\frac{g_1a_1+g_2a_2+\dots+g_m a_m}{g_1+g_2+\dots+g_m}.$$

SECTION 5.—*On maxima and minima of functions of variables when all are not independent.*

167.] A problem, which frequently occurs, is the determination of maxima and minima of functions of many variables, when certain relations between the variables are given, so that all those involved in the original functional equation are not independent. Thus suppose that we have to determine the singular values of

$$u = \text{F}(x, y, z, \dots), \tag{49}$$

which is a function of n variables; and suppose besides that m equations connecting these variables are given, viz.

$$\left.\begin{array}{r}\mathrm{F}_1(x, y, z, \ldots\ldots) = 0,\\ \mathrm{F}_2(x, y, z, \ldots\ldots) = 0,\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ \mathrm{F}_m(x, y, z, \ldots\ldots) = 0.\end{array}\right\} \qquad (50)$$

In order to apply the method which has been explained in the preceding Section, it would be necessary to eliminate m variables between $m+1$ equations, by which means u would become a function of $n-m$ variables, all of which would be independent of each other; and then forming the partial derived-functions $\left(\frac{d\mathrm{F}}{dx}\right)$, $\left(\frac{d\mathrm{F}}{dy}\right)$, $\left(\frac{d\mathrm{F}}{dz}\right)$, ... the number of which is $n-m$, and equating each to 0, there would be $n-m$ equations, from which we could, theoretically at least, determine the $n-m$ variables. This method however, though theoretically possible, is frequently attended with great difficulty on account of elimination; and, if the original expressions are symmetrical, the symmetry is destroyed by it; in which case it is better to proceed as follows:

It is plain from what has been said, that as there are $n-m$ variables entirely independent in their variations, we have $n-m$ conditions to make; which will be equivalent to equating to 0 the $n-m$ partial derived-functions with respect to these variables of $\mathrm{F}(x, y, z, \ldots\ldots)$. Differentiating the functions in order, and remembering that $\mathrm{D}u = 0$, because u is to be a maximum or a minimum, we have

$$\left.\begin{array}{r}\mathrm{D}u = 0 = \left(\frac{d\mathrm{F}}{dx}\right)dx + \left(\frac{d\mathrm{F}}{dy}\right)dy + \left(\frac{d\mathrm{F}}{dz}\right)dz + \ldots\ldots\\ 0 = \left(\frac{d\mathrm{F}_1}{dx}\right)dx + \left(\frac{d\mathrm{F}_1}{dy}\right)dy + \left(\frac{d\mathrm{F}_1}{dz}\right)dz + \ldots\ldots\\ 0 = \left(\frac{d\mathrm{F}_2}{dx}\right)dx + \left(\frac{d\mathrm{F}_2}{dy}\right)dy + \left(\frac{d\mathrm{F}_2}{dz}\right)dz + \ldots\ldots\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ 0 = \left(\frac{d\mathrm{F}_m}{dx}\right)dx + \left(\frac{d\mathrm{F}_m}{dy}\right)dy + \left(\frac{d\mathrm{F}_m}{dz}\right)dz + \ldots\ldots\end{array}\right\}; \qquad (51)$$

the meaning of which is, that $x, y, z, \ldots\ldots$ do not vary independently of each other, but consistently with the conditions involved in the last m equations. Hence to eliminate $dx, dy, dz, \ldots\ldots$ multiply these last equations severally by indeterminate quantities, $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_m$, and add to the first; and collecting the coefficients of $dx, dy, dz, \ldots\ldots$ we have

$$\begin{aligned}
&\left\{\left(\frac{d\text{F}}{dx}\right)+\lambda_1\left(\frac{d\text{F}_1}{dx}\right)+\lambda_2\left(\frac{d\text{F}_2}{dx}\right)+\ldots\ldots+\lambda_m\left(\frac{d\text{F}_m}{dx}\right)\right\}dx\\
+&\left\{\left(\frac{d\text{F}}{dy}\right)+\lambda_1\left(\frac{d\text{F}_1}{dy}\right)+\lambda_2\left(\frac{d\text{F}_2}{dy}\right)+\ldots\ldots+\lambda_m\left(\frac{d\text{F}_m}{dy}\right)\right\}dy\\
+&\left\{\left(\frac{d\text{F}}{dz}\right)+\lambda_1\left(\frac{d\text{F}_1}{dz}\right)+\lambda_2\left(\frac{d\text{F}_2}{dz}\right)+\ldots\ldots+\lambda_m\left(\frac{d\text{F}_m}{dz}\right)\right\}dz\\
+&\left\{\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;\right\}\ldots\\
&+\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;=0.
\end{aligned}$$

Which equation is subject to n conditions, viz. $n-m$, on account of $n-m$ independent variables being involved, and m on account of m indeterminate multipliers $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_m$ having been introduced. Let these conditions be, that the coefficient of each differential is equal to 0; therefore

$$\left.\begin{aligned}
&\left(\frac{d\text{F}}{dx}\right)+\lambda_1\left(\frac{d\text{F}_1}{dx}\right)+\lambda_2\left(\frac{d\text{F}_2}{dx}\right)+\ldots\ldots+\lambda_m\left(\frac{d\text{F}_m}{dx}\right)=0\\
&\left(\frac{d\text{F}}{dy}\right)+\lambda_1\left(\frac{d\text{F}_1}{dy}\right)+\lambda_2\left(\frac{d\text{F}_2}{dy}\right)+\ldots\ldots+\lambda_m\left(\frac{d\text{F}_m}{dy}\right)=0\\
&.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.
\end{aligned}\right\};\qquad(52)$$

between which equations $\lambda_1, \lambda_2, \ldots \lambda_m$ are to be eliminated, and $x, y, z, \ldots$ determined; and these will be the values corresponding to a maximum or a minimum value of $\text{F}(x, y, z, \ldots)$.

The sign of the second differential coefficient will determine whether the particular value is a maximum or a minimum: but in most cases where this method is applicable, the form of the function at once decides whether it admits of a maximum or of a minimum.

168.] In the case in which only one equation is given connecting the n variables involved in the given function, whose maximum or minimum value is to be determined, the above results assume a particular form, by means of which the process is much simplified.

$$\text{Let}\qquad u = \text{F}(x, y, z, \ldots) \qquad (53)$$

be the function of which the maximum or minimum value is to be determined; and suppose the variables to be subject to the relation expressed by the equation

$$f(x, y, z, \ldots) = c; \qquad (54)$$

then differentiating (53) and (54), and putting $\text{D}u = 0$, because u is a maximum or a minimum, we have

$$\text{D}\text{F} = 0 = \left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy + \left(\frac{d\text{F}}{dz}\right) dz + \ldots\ldots \tag{55}$$

$$\text{D}f = 0 = \left(\frac{df}{dx}\right) dx + \left(\frac{df}{dy}\right) dy + \left(\frac{df}{dz}\right) dz + \ldots\ldots \tag{56}$$

whence, multiplying (55) by an indeterminate multiplier λ, and subtracting (56) from it, we have

$$\left\{\lambda\left(\frac{d\text{F}}{dx}\right) - \left(\frac{df}{dx}\right)\right\} dx + \left\{\lambda\left(\frac{d\text{F}}{dy}\right) - \left(\frac{df}{dy}\right)\right\} dy + \ldots = 0; \tag{57}$$

and by virtue of the argument of the last Article, equating to zero the several coefficients of $dx, dy, dz, \ldots$ we have

$$\frac{1}{\lambda} = \frac{\left(\frac{d\text{F}}{dx}\right)}{\left(\frac{df}{dx}\right)} = \frac{\left(\frac{d\text{F}}{dy}\right)}{\left(\frac{df}{dy}\right)} = \frac{\left(\frac{d\text{F}}{dz}\right)}{\left(\frac{df}{dz}\right)} = \ldots\ldots \tag{58}$$

that is, the ratio of the coefficients of the same differentials in (55) and (56) is constant.

The algebraical criterion for discriminating between a maximum and a minimum is too complicated to be of any service even in this particular case.

169.] Examples illustrative of the preceding processes.

Ex. 1. To determine the minimum value of $x^2+y^2+z^2$, having given $p = ax+by+cz$, where p, a, b, c are constants.

Let $\quad u^2 = x^2+y^2+z^2, \qquad p = ax+by+cz.$

$$\therefore \quad u\,\text{D}u = x\,dx + y\,dy + z\,dz = 0,$$

$$0 = a\,dx + b\,dy + c\,dz;$$

therefore by reason of the last Article,

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{x^2+y^2+z^2}{ax+by+cz} = \frac{u^2}{p}$$

$$= \frac{\{x^2+y^2+z^2\}^{\frac{1}{2}}}{\{a^2+b^2+c^2\}^{\frac{1}{2}}} = \frac{u}{\{a^2+b^2+c^2\}^{\frac{1}{2}}}, \text{ by Preliminary Theorem I;}$$

$$\therefore \quad u = \frac{p}{\{a^2+b^2+c^2\}^{\frac{1}{2}}};$$

$$x = \frac{ap}{a^2+b^2+c^2}, \quad y = \frac{bp}{a^2+b^2+c^2}, \quad z = \frac{cp}{a^2+b^2+c^2}.$$

The above is the solution of the problem, To find the shortest distance from the origin to a given plane.

Ex. 2. To determine the plane triangle of given perimeter, and of maximum area.

Let the area of the triangle $= u$, and the sides be x, y, z, and the given perimeter $= 2s$.

$$\therefore \quad 2s = x+y+z, \qquad \text{and} \qquad u^2 = s(s-x)(s-y)(s-z);$$

therefore differentiating the former equation, and taking the logarithmic differential of the latter, we have

$$0 = dx+dy+dz,$$

$$-\frac{2\,du}{u} = \frac{dx}{s-x} + \frac{dy}{s-y} + \frac{dz}{s-z} = 0;$$

$$\therefore \quad s-x = s-y = s-z;$$

$$\therefore \quad x = y = z = \frac{2s}{3}; \qquad \text{and the area of the triangle} = \frac{s^2}{3^{\frac{3}{2}}};$$

that is, the triangle is equilateral.

In precisely the same manner it may be shewn that of all figures of a given number of sides and of given perimeter, the equilateral and equiangular one is the greatest.

Hence also it follows that, of all plane figures of given perimeter, the circle is that whose area is the greatest.

For let $2s$ be the given perimeter, and n be the required number of sides;

$$\therefore \quad \text{each side} = \frac{2s}{n};$$

$$\text{and} \quad \therefore \quad \text{the area} = \frac{s^2}{n}\cot\frac{\pi}{n},$$

$$= \frac{s^2}{\pi}\left(\frac{\pi}{n}\cot\frac{\pi}{n}\right).$$

But the last factor continually increases as $\frac{\pi}{n}$ decreases, and attains a maximum when $n = \infty$; in which case the polygon becomes a circle, which is therefore the greatest of all plane figures under a given perimeter.

Ex. 3. To divide a number a into three parts x, y, z, so that $x^m y^n z^p$ may be a maximum.

$$u = x^m y^n z^p, \qquad \therefore \quad \log u = m\log x + n\log y + p\log z;$$

$$a = x+y+z;$$

$$\therefore \quad \frac{\mathrm{D}u}{u} = 0 = \frac{m\,dx}{x} + \frac{n\,dy}{y} + \frac{p\,dz}{z},$$

$$0 = dx + dy + dz;$$

$$\therefore \quad \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a};$$

$$\therefore \quad x = \frac{am}{m+n+p}, \quad y = \frac{an}{m+n+p}, \quad z = \frac{ap}{m+n+p};$$

$$u = m^m n^n p^p \left(\frac{a}{m+n+p}\right)^{m+n+p}.$$

Ex. 4. To determine the greatest quadrilateral figure which can be contained by four given straight lines a, b, c, e.

Fig. 22. Let $\mathrm{AB} = a$, $\mathrm{BC} = b$, $\mathrm{CD} = c$, $\mathrm{DA} = e$; the angle $\mathrm{ABC} = \theta$, $\mathrm{ADC} = \phi$; $u =$ the area.

$$\therefore \quad u = \frac{1}{2}\{ab \sin\theta + ce \sin\phi\};$$

and $$(\mathrm{AC})^2 = a^2 + b^2 - 2ab\cos\theta = c^2 + e^2 - 2ce\cos\phi;$$

$$\therefore \quad \mathrm{D}u = 0 = \frac{1}{2}\{ab\cos\theta\, d\theta + ce\cos\phi\, d\phi\},$$

$$0 = 2ab\sin\theta\, d\theta - 2ce\sin\phi\, d\phi;$$

$$\therefore \quad \frac{\cos\theta}{\sin\theta} = -\frac{\cos\phi}{\sin\phi};$$

$$\therefore \quad \tan\theta = -\tan\phi; \qquad \therefore \quad \theta = 180^\circ - \phi;$$

that is, $\theta + \phi = 180^\circ$, and the opposite angles of the quadrilateral figure are together equal to two right angles, and the quadrilateral is such as can be inscribed in a circle.

Ex. 5. To determine the maximum and minimum values of the central radii vectores of an ellipse, and their relation to a pair of conjugate axes.

Let the ellipse be referred to a pair of conjugate diameters, whose lengths are $2a_1$ and $2b_1$, as the coordinate axes; and let ω be the angle between the axes, and r be the length of any central radius vector; then the equation to the ellipse is

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1.$$

Also $$r^2 = x^2 + 2xy\cos\omega + y^2.$$

$$\therefore \quad \frac{x}{a_1^2}\,dx + \frac{y}{b_1^2}\,dy = 0,$$

and $$(x + y\cos\omega)\,dx + (y + x\cos\omega)\,dy = r\,dr = 0;$$

therefore employing an indeterminate multiplier, we have

$$\left\{\frac{x}{a_1^2}+\lambda(x+y\cos\omega)\right\}dx+\left\{\frac{y}{b_1^2}+\lambda(y+x\cos\omega)\right\}dy=0;$$

and equating to 0 the coefficients of dx and dy,

$$\frac{x}{a_1^2}+\lambda(x+y\cos\omega)=0, \qquad \frac{y}{b_1^2}+\lambda(y+x\cos\omega)=0;$$

whence, multiplying the former by x and the latter by y, and adding,

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}+\lambda(x^2+2xy\cos\omega+y^2)=0;$$

$$\therefore \quad 1+\lambda r^2=0, \qquad \therefore \quad \lambda=-\frac{1}{r^2};$$

and substituting,

$$\left(\frac{1}{a_1^2}-\frac{1}{r^2}\right)x-\frac{\cos\omega}{r^2}y=0, \qquad -\frac{\cos\omega}{r^2}x+\left(\frac{1}{b_1^2}-\frac{1}{r^2}\right)y=0;$$

whence, by cross multiplication,

$$\left(\frac{1}{a_1^2}-\frac{1}{r^2}\right)\left(\frac{1}{b_1^2}-\frac{1}{r^2}\right)-\frac{1}{r^4}(\cos\omega)^2=0;$$

$$\therefore \quad r^4-(a_1^2+b_1^2)r^2+a_1^2b_1^2(\sin\omega)^2=0.$$

Let a and b be the greatest and least values of the radii vectores, so that a^2 and b^2 will be the roots of the last equation; then by the theory of equations,

$$a^2+b^2=a_1^2+b_1^2, \qquad a^2b^2=a_1^2b_1^2(\sin\omega)^2;$$

by means of which the values of a and b may be easily determined.

By a similar process may we determine the analogous relations of the principal axes of an ellipsoid to any system of conjugate axes.

Ex. 6. To inscribe in a sphere the greatest parallelepipedon.

Let $x^2+y^2+z^2=a^2$ be the equation to the sphere; and let u be the content of the parallelepipedon.

$$\therefore \quad u=8xyz;$$

and taking the logarithmic differential,

$$\frac{\mathrm{D}u}{u}=0=\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z};$$

also $\qquad 0=x\,dx+y\,dy+z\,dz;$

$$\therefore \quad x^2 = y^2 = z^2 = \frac{a^2}{3}; \qquad \therefore \quad x = y = z = \frac{a}{\sqrt{3}};$$

and the volume of the parallelepipedon $= \dfrac{8a^3}{3^{\frac{3}{2}}}$.

Ex. 7. To find a point within a triangle, such that the sum of the lines drawn from it to the angular points may be a minimum.

Fig. 20. Let P be the required point; and let $\text{AP} = x$, $\text{BP} = y$, $\text{CP} = z$, $\text{BC} = a$, $\text{CA} = b$, $\text{AB} = c$, $\text{BPC} = \theta$, $\text{CPA} = \phi$, $\text{APB} = \psi$; $u =$ sum of the required lines; wherefore we have

$$u = x + y + z; \tag{59}$$

$$\left.\begin{aligned} a^2 &= y^2 - 2yz\cos\theta + z^2 \\ b^2 &= z^2 - 2zx\cos\phi + x^2 \\ c^2 &= x^2 - 2xy\cos\psi + y^2 \end{aligned}\right\}; \tag{60}$$

$$\theta + \phi + \psi = 2\pi. \tag{61}$$

Whence, by differentiation,

$$\text{D}u = 0 = dx + dy + dz; \tag{62}$$

$$\left.\begin{aligned} 0 &= (y - z\cos\theta)\,dy + (z - y\cos\theta)\,dz + yz\sin\theta\,d\theta \\ 0 &= (z - x\cos\phi)\,dz + (x - z\cos\phi)\,dx + zx\sin\phi\,d\phi \\ 0 &= (x - y\cos\psi)\,dx + (y - x\cos\psi)\,dy + xy\sin\psi\,d\psi \end{aligned}\right\}; \tag{63}$$

$$0 = d\theta + d\phi + d\psi. \tag{64}$$

Now observing that we have six variables involved, which are connected by four equations (60) and (61), two are independent; multiplying therefore the equations (63) severally by x, y, z, and adding, we have

$$\begin{aligned} \{x(y+z) - yz(\cos\phi + \cos\psi)\}dx + \{y(z+x) - zx(\cos\psi + \cos\theta)\}dy \\ + \{z(x+y) - xy(\cos\theta + \cos\phi)\}\,dz \\ + xyz\,\{\sin\theta\,d\theta + \sin\phi\,d\phi + \sin\psi\,d\psi\} = 0. \end{aligned} \tag{65}$$

Whence multiplying (62) and (64) by two indeterminate multipliers, and adding them to (65), and equating to zero the coefficients of the variables, we have

$$\sin\theta = \sin\phi = \sin\psi; \qquad \therefore \quad \theta = \phi = \psi = 120^\circ.$$

Let therefore three segments of circles be described on the sides of the triangle, each containing an angle of 120°; these will meet in a point, which point will be that required.

CHAPTER VIII.

APPLICATION OF THE PRECEDING PRINCIPLES TO THE THEORY OF ALGEBRAICAL EXPRESSIONS.

170.] In the present Chapter I shall take the following to be the type of an algebraical expression of the nth degree, and for convenience of reference shall symbolize it by $f(x)$,

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \dots \pm p_{n-1} x \mp p_n = 0 = f(x), \quad (1)$$

the upper or lower sign being taken according as n is odd or even; n being integral and positive; and no fractional or negative powers of x being involved in the expression; $p_1, p_2, \dots p_n$ being constant coefficients, real or imaginary, and the coefficient of the highest power of x being unity.

A *root* of such an expression is a value which, when substituted for the unknown quantity x, makes the whole to vanish; thus, if a is a root of $f(x)$,

$$f(a) = 0.$$

An imaginary or impossible expression is of the form

$$a + b\sqrt{-1}; \quad (2)$$

which, as was explained in Art. 60, always admits of being put into the form,

$$r(\cos\theta + \sqrt{-1}\sin\theta),$$

of which r is called the Modulus; and

$$r^2 = a^2 + b^2 = (a + \sqrt{-1}\,b)(a - \sqrt{-1}\,b), \quad (3)$$

the latter pair of imaginary factors being called conjugate to each other.

Now of such general algebraical expressions as (1), almost all the properties which are proved in the usual text-books on the Theory of Equations arise from considering them in their *resolved* or *analytical* form; that is, as made up of factors of the form, $x-a$, $x-b$, ... $x-a+\beta\sqrt{-1}$, $x-a-\beta\sqrt{-1}$ ... ; such are the relations which exist between the roots and the coefficients, between two equations the roots of one of which are symmetrical functions of those of the other, &c. But a general treatment of the subject requires the expressions to be considered in their *synthetical* or unresolved state; and we propose in the following Articles to exhibit such properties of them in their compounded state, as fall within the grasp of the preceding principles.

In the first place I must remark on the *continuity* of such expressions; and I shall hereby be led to shew that every equation, be its dimensions odd or even, has a root. In other words, I shall prove that a value, real or imaginary, exists, which when substituted for x in $f(x)$ makes $f(x) = 0$.

If it is thought that an unfair assumption is made in the following Articles in the extension to imaginary quantities of processes of differentiation which have been hitherto applied to possible quantities, let it be borne in mind that the principles of differentiation as unfolded in the first and second Chapters require only the law of continuity to be satisfied in the functions to which they are applied, and that imaginary quantities satisfy that law as well as possible ones. And I would also observe that logarithmic and circular functions are related to each other by means of the exponential equivalents, and that these involve symbols of impossibility; and as we have introduced and operated on each of these in the previous Chapters, our processes have been applied to impossible as well as to possible quantities.

171.] Positive real values may be assigned to x in (1) so great that the corresponding value of x^n shall be greater than the sum of all the following terms.

Let us take the case which is most unfavourable, that namely in which all the terms are positive, and in which consequently their sum is greater than it would be if some of the terms were negative. Let p_k be the greatest of all the positive coefficients $-p_1, p_2, \ldots \mp p_n$; then

$$-p_1x^{n-1}+p_2x^{n-2}-\ldots\mp p_n \text{ is less than } p_k\{x^{n-1}+x^{n-2}+\ldots+x+1\},$$

$$\text{- - - - - - - - - - - is less than } p_k\frac{x^n-1}{x-1},$$

$$\text{- - - - - - - - - - - is less than } p_k\frac{x^n}{x-1},$$

and therefore is less than x^n, if $p_k = x-1$; that is, if $x = p_k+1$, x^n is greater than the sum of all the subsequent terms; and *a fortiori*, if x is greater than p_k+1, x^n is so much greater than the sum of all the terms after x^n: and thus by increasing x, $f(x)$ becomes greater and greater, and in the superior limit when $x=\infty$, $f(x)$ also becomes infinite. Thus in

$$x^3+6x^2+7x+4 = 0,$$

if we substitute 8 for x, x^3 is greater than the sum of all the other terms. Hence it appears that for values of x greater than a certain assignable quantity, as x increases, $f(x)$ increases; and ultimately becomes infinite, when $x=\infty$.

If $x=0$, then $f(x) = \mp p_n$; and if n is odd, $f(x) = -\infty$, when $x = -\infty$; hence if an equation is of uneven dimensions, $f(x)$ changes sign from $-\infty$ to $+\infty$ as x increases from $-\infty$ to $+\infty$; and as such change of sign of $f(x)$ can take place only when $f(x) = 0$, it follows that an equation of odd dimensions has at least one real root: and that root will be positive or negative according as the constant term of the equation is negative or positive.

Hence also it follows that an equation of even dimensions has two real roots if the last term is negative, of which one is positive and the other is negative; because $f(x)$ will change its sign from $+$ to $-$, and again from $-$ to $+$, as x increases from $-\infty$ through 0 to $+\infty$.

As x continuously varies, so $f(x)$ continuously varies; for let h be an increment of x, and let us suppose that $f'(x)$ does not vanish; then by (21), Art. 116,

$$f(x+h)-f(x) = hf'(x+\theta h);$$

the right-hand member of which is infinitesimal, if h is infinitesimal; therefore $f(x+h)-f(x)$ is infinitesimal; that is, $f(x)$ varies continuously as its subject-variable continuously varies.

If $f(x)$ is of even dimensions, $f(x)$ has at least one minimum value; because $f'(x)$ will be of odd dimensions, and will change sign from $-$ to $+$, as x increases from $-\infty$ to $+\infty$; and corresponding to the value of x at which the change of sign takes place will $f(x)$ attain a minimum value. Thus much as to the values of $f(x)$ when x is replaced by a real quantity.

172.] Having thus shewn that $f(x)$ continuously increases up to $+\infty$, as x increases upwards from a certain real value, I will now shew that if for x we substitute in $f(x)$ the most general form of an imaginary expression, viz. $y+z\sqrt{-1}$, $f(y+z\sqrt{-1})$ does not admit of a maximum or a minimum, but continuously either increases or decreases.

Let us replace x in $f(x)$ by $y+z\sqrt{-1}$; then expanding by Taylor's Series, we have

$$f(y+z\sqrt{-1}) = f(y)+\sqrt{-1}\,f'(y)\frac{z}{1} - f''(y)\frac{z^2}{1.2}$$
$$-\sqrt{-1}\,f'''(y)\frac{z^3}{1.2.3} + f^{\text{iv}}(y)\frac{z^4}{1.2.3.4} + \dots\dots; \quad (4)$$

of which right-hand member the possible and impossible parts may be separated, and the whole expression may be equated to $P+Q\sqrt{-1}$; if

$$P = f(y) - f''(y)\frac{z^2}{1.2} + f^{\text{iv}}(y)\frac{z^4}{1.2.3.4}\dots\dots, \quad (5)$$

$$Q = f'(y)\,z - f'''(y)\frac{z^3}{1.2.3} + f^{\text{v}}(y)\frac{z^5}{1.2.3.4.5}\dots\dots; \quad (6)$$

and where P and Q are functions of two *independent* variables y and z. Now differentiating, we have

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dz}\right), \qquad \left(\frac{dP}{dz}\right) = -\left(\frac{dQ}{dy}\right);$$

$$\left(\frac{d^2P}{dy^2}\right) = \left(\frac{d^2Q}{dy\,dz}\right) = -\left(\frac{d^2P}{dz^2}\right), \qquad \left(\frac{d^2Q}{dy^2}\right) = -\left(\frac{d^2P}{dy\,dz}\right) = -\left(\frac{d^2Q}{dz^2}\right);$$

hence, $\left(\frac{d^2P}{dy^2}\right)$ and $\left(\frac{d^2P}{dz^2}\right)$ being necessarily of different signs, the conditions of Art. 158 and 159 cannot be satisfied, and P does not admit of a maximum or minimum value; similarly also it is plain that Q does not admit of a maximum or minimum; hence $f(y+z\sqrt{-1})$ is not a function admitting of a maximum or minimum; and therefore goes on continuously increasing or decreasing, as the variables continuously vary.

173.] Hence we can shew that if $f(x)$ is a function of x of the form (1), one or more values of x of the form $y+z\sqrt{-1}$ exist, which when substituted for x render $f(x)=0$; in other words, we can hence prove that every equation has a root.

Let $y+z\sqrt{-1}$ be replaced by $r(\cos\theta+\sqrt{-1}\sin\theta)$, and $f(y+z\sqrt{-1})$ by $R(\cos T+\sqrt{-1}\sin T)$, r and R being the moduli respectively of $y+z\sqrt{-1}$ and of $f(y+z\sqrt{-1})$, and θ and T being real circular arcs; then, if it can be shewn that

If some value of θ is such as to render $\mathrm{R} = 0$, y and z are such as to make $f(y+z\sqrt{-1}) = 0$, and it is proved that the equation $f(x) = 0$ has a root.

$$\text{Since} \quad f(y+z\sqrt{-1}) = \mathrm{R}(\cos \mathrm{T} + \sqrt{-1}\sin \mathrm{T}), \tag{7}$$

$$\therefore \quad f(y-z\sqrt{-1}) = \mathrm{R}(\cos \mathrm{T} - \sqrt{-1}\sin \mathrm{T}); \tag{8}$$

$$\text{whence} \quad \mathrm{R}^2 = f(y+z\sqrt{-1}) \times f(y-z\sqrt{-1}). \tag{9}$$

Now to take the most general case, let us suppose that all the derived-functions of $f(x)$ up to the mth exclusively vanish when $x = y+z\sqrt{-1}$; viz. that

$$f'(y+z\sqrt{-1}) = 0,\ f''(y+z\sqrt{-1}) = 0, \ldots f^{m-1}(y+z\sqrt{-1}) = 0;$$

whence also,

$$f'(y-z\sqrt{-1}) = 0,\ f''(y-z\sqrt{-1}) = 0, \ldots f^{m-1}(y-z\sqrt{-1}) = 0.$$

Then, by means of Leibnitz's Theorem, see Art. 55, taking the mth differential of R^2 in (9), and neglecting terms which vanish, we have

$$d^m.\mathrm{R}^2 = f^m(y+z\sqrt{-1})\,(dy+dz\sqrt{-1})^m f(y-z\sqrt{-1}) \\ + f^m(y-z\sqrt{-1})\,(dy-dz\sqrt{-1})^m f(y+z\sqrt{-1}).$$

Let R_m and ρ be the moduli respectively of $f^m(y+z\sqrt{-1})$ and $dy+\sqrt{-1}\,dz$; and let T_m and τ be the corresponding arcs, so that

$$f^m(y+z\sqrt{-1}) = \mathrm{R}_m(\cos \mathrm{T}_m + \sqrt{-1}\sin \mathrm{T}_m), \tag{10}$$

$$f^m(y-z\sqrt{-1}) = \mathrm{R}_m(\cos \mathrm{T}_m - \sqrt{-1}\sin \mathrm{T}_m); \tag{11}$$

$$dy \pm dz\sqrt{-1} = \rho(\cos\tau \pm \sqrt{-1}\sin\tau); \tag{12}$$

$$\therefore \quad (dy \pm dz\sqrt{-1})^m = \rho^m(\cos m\tau \pm \sqrt{-1}\sin m\tau); \tag{13}$$

making which substitutions,

$$d^m.\mathrm{R}^2 = 2\,\mathrm{R}\,\mathrm{R}_m\,\rho^m\cos(\mathrm{T}_m - \mathrm{T} + m\tau). \tag{14}$$

Now as $f(y\pm z\sqrt{-1})$ does not admit of a maximum or minimum value, so R^2, being the square of R, and thus necessarily a positive quantity, does not admit of a maximum value; but it must have a minimum value; that derived-function of it therefore, which is the first not to vanish, must be consistent with such a singular value. But on examining (14) τ is arbitrary, and $d^m.\mathrm{R}^2$ will evidently change sign, when τ and therefore when dy and dz receive certain values; as, for instance, change τ to $\tau+\frac{2k+1}{m}\pi$, and the sign of $d^m.\mathrm{R}^2$ will be changed; and such a change is inconsistent with R^2 being a minimum. The apparent difficulty however is obviated if one of the other factors of $d^m.\mathrm{R}^2$ vanishes; but R_m and ρ^m do not vanish, and therefore it follows that the only condition which is consistent with R^2 being a minimum value is, $\mathrm{R}=0$; this therefore is the minimum value; whence, by means of (7),

$$f(y+z\sqrt{-1}) = 0; \tag{15}$$

and therefore there always are some values of y and z such that equation (15) is satisfied; and therefore every equation of the form (1) has a root.

Also from (8) it follows that if $\mathrm{R}=0$, $f(y-z\sqrt{-1})=0$; and therefore if one of a conjugate pair is a root of an equation, the other conjugate is also a root.

If in the general form of the root $z=0$, the root is possible; but if z has a finite value, the root is imaginary.

174.] Taking a to be the *general* symbol for the root of an equation, we may thus prove that the equation is divisible by $x-a$ without a remainder.

Let $f(x)=0$ be the equation; then, since a is a root, $f(a)=0$.

Observing now that Taylor's Series does not fail for a function of x, such as we have assumed $f(x)$ to be, and that the $(n+1)$th derived-function vanishes, because $f(x)$ is algebraical and of n dimensions, by means of (84), Art. 74, we have

$$\begin{aligned} f(x) &= f(a+x-a), \\ &= f(a)+f'(a)\frac{x-a}{1}+f''(a)\frac{(x-a)^2}{1.2}+\dots+f^n(a)\frac{(x-a)^n}{1.2.3\dots n}; \end{aligned} \tag{16}$$

and because $f(a)=0$, the second member is divisible by $(x-a)$.

The converse proposition is evident; viz. that if $x-a$ is a factor of $f(x)$, $f(a)=0$, or a is a root of the equation.

Hence we conclude that if any function of the algebraical form assumed in equation (1), Art. 170, vanishes for a particular value a of the variable, the function has a factor of the form $x-a$; and it is owing to *its* vanishing that the function vanishes.

175.] Hence also it follows, that every equation has as many roots as it has dimensions, and no more.

For dividing $f(x)$ in equation (16) by $(x-a)$, the highest power of x that remains is x^{n-1}, being that which is involved in the last term of it, viz. in $(x-a)^{n-1}$; whence an expression of $n-1$ dimensions results; which again, by virtue of Article 173, has a root, and therefore is again divisible by a factor of the form $x-a$; whereby the expression is depressed to one of $n-2$ dimensions; and if a similar process is repeated $n-1$ times, we shall finally have an expression of one dimension, which will give the last root; and thereby the equation will have been resolved into n factors.

Thus suppose the n roots to be $a_1, a_2, \dots a_n$, then

$$f(x) = (x-a_1)(x-a_2)\dots(x-a_n). \tag{17}$$

Also it is manifest that no other value than one of the n roots can, when substituted for x, make any simple factor, and thereby the whole expression, to vanish; and therefore $f(x)$ has only n roots; some of which however may be equal; and therefore although all the n roots may not be different, yet there can never be fewer than n simple factors.

Again, if the coefficients of the several powers of x in $f(x)$ are real, and $f(x)$ has impossible roots, they must, as observed above, enter in pairs; so that, if a_i and a_j are two imaginary roots which are conjugate to each other,

$$a_i = a+\beta\sqrt{-1} = \rho\{\cos\theta+\sqrt{-1}\sin\theta\},$$
$$a_j = a-\beta\sqrt{-1} = \rho\{\cos\theta-\sqrt{-1}\sin\theta\};$$
$$\therefore\quad (x-a_i)(x-a_j) = (x-a)^2+\beta^2 = x^2-2\rho x\cos\theta+\rho^2,$$

which quadratic expression is essentially positive; and by a similar composition of other conjugate factors we have,

$$f(x)=(x-a_1)(x-a_2)\dots\{(x-a_1)^2+\beta_1{}^2\}\ \{(x-a_2)^2+\beta_2{}^2\}\dots; \tag{18}$$

and therefore $f(x)$ is the product of factors, simple or quadratic.

176.] On the algebraical relation of $f(x)$ to its derived-function.

Let $f(x)$ be a function of the form (1), which has all its coefficients real quantities and all its roots real and unequal; and let the roots of $f(x)$ be $a_1, a_2, \dots a_n$; so that

$$f(x) = x^n-p_1x^{n-1}+p_2x^{n-2}-\dots\pm p_{n-1}x\mp p_n, \tag{19}$$
$$= (x-a_1)(x-a_2)\dots(x-a_n), \tag{20}$$

the upper or lower sign being taken in (19) according as n is odd or even, and the roots being arranged in order of magnitude, viz.

$$a_1 > a_2 > a_3 > \dots > a_n; \tag{21}$$

then, taking the derived-functions of both the members of (19), we have

$$\begin{aligned} f'(x) &= nx^{n-1} - (n-1)p_1 x^{n-2} + \dots \pm p_{n-1}, \\ &= (x-a_2)\ (x-a_3) \dots (x-a_n) + (x-a_1)\ (x-a_3) \dots (x-a_n) \\ &\qquad + (x-a_1)\ (x-a_2) \dots (x-a_n) + \dots \\ &\qquad\qquad \dots + (x-a_1)\ (x-a_2) \dots (x-a_{n-1}). \quad (22) \end{aligned}$$

Let $x = a_1$; then observing that all the parts, except the first, of the second member of (22) disappear, and that by virtue of the arrangement of the roots, as indicated by (21), every factor of the first part is positive, it follows that, if $x = a_1$, $f'(x)$ is positive; similarly, if $x = a_2$, $f'(x)$ is negative; there is therefore some value of x between a_1 and a_2 which makes $f'(x)$ vanish; a root therefore of $f'(x)$ lies between a_1 and a_2. Similarly we have the following results;

if $x = a_1$, $f'(x)$ is positive,
$x = a_2$, $f'(x)$ is negative,
$x = a_3$, $f'(x)$ is positive,
. . . .
$x = a_n$, $f'(x)$ is positive or negative, according as n is odd or even. Hence the roots of $f'(x)$ are real, and lie between the roots of $f(x)$.

Let the roots of $f'(x)$ be $\alpha_1, \alpha_2, \dots \alpha_{n-1}$, arranged in order of descending magnitude, then they stand to the roots of $f(x)$ in the relation indicated in the following table;

$$\begin{array}{ccccccccc} a_1 & & a_2 & & a_3 & \dots & a_{n-1}, & & a_n, \\ & \alpha_1 & & \alpha_2 & & \alpha_3 & \dots & \alpha_{n-1}; & \end{array}$$

whence it appears, that the greatest root of $f(x)$ is greater than the greatest root of $f'(x)$, and the least root of $f(x)$ is less than the least root of $f'(x)$. It is on account of this particular relation of the roots of $f'(x)$ to the roots of $f(x)$, that $f'(x)$ is sometimes called the ***limiting equation*** of $f(x)$.

Hence also it follows, that if all the roots of an equation are real, all the roots of each of its successive derived-functions will be real also.

These results admit of the following geometrical interpretation: Let the curve represented in fig. 23 be that whose equation is $y = f(x)$. As $f(x)$ has n real roots, $f(x)$, and therefore y, $= 0$ at n points corresponding to them; that is, if $\text{OA}_1 = a_1$,

$OA_2 = a_2$, ... the curve cuts the axis of x at A_1, A_2, ... that is, in n points. As $f(x)$ only $= \pm \infty$, when $x = \pm \infty$, the ordinate is finite for all values of x between a_1 and a_2, a_2 and a_3, ...; and by the last Chapter, as $f'(x) = 0$, and changes sign when $x = \alpha_1, = \alpha_2, = \dots$, $f(x)$ is a maximum or minimum corresponding to these roots of $f'(x)$, and therefore we have maxima or minima ordinates at points intermediate to A_1 and A_2, A_2 and A_3, ... A_{n-1} and A_n; that is, $OB_1 = \alpha_1$, $OB_2 = \alpha_2$, ... Also as n is odd or even, so will the curve towards the left, when $x = -\infty$, be below or above the axis of x.

177.] Hence it appears that, if $f(x)$ has m roots equal to each other, $f'(x)$ has $(m-1)$ roots equal to each of the equal roots of $f(x)$; for if $a_1 = a_2 = \dots = a_m$, then $\alpha_1 = \alpha_2 = \dots = \alpha_{m-1}$, as the α's are intermediate to the a's; which proposition is also thus manifest.

Let m roots of $f(x)$ be equal to one another and to a, and let Q_x symbolize the product of the other $n-m$ roots, then

$$f(x) = (x-a)^m Q_x;$$

$$\therefore \quad f'(x) = (x-a)^{m-1} \left\{ m Q_x + (x-a) \frac{d.Q_x}{dx} \right\};$$

and $f'(x)$ has $m-1$ roots equal to a, that is, to each of the m equal roots of $f(x)$.

Hence if $f(x)$ has equal roots, they may be determined by the method of finding the greatest common measure of $f(x)$ and $f'(x)$, and $f(x)$ may be depressed by as many dimensions as it has equal roots.

The latter proof of this proposition manifestly reaches the case of equal imaginary roots which the former may not resolve.

178.] Such information does Infinitesimal Calculus afford as to the continuity of algebraical expressions, as to the possibility of the resolution of $f(x)$ into simple and quadratic factors corresponding to the roots, and as to the relation between the roots of $f(x)$ and of its derived-function. I proceed now to a theorem of great importance; one indeed which theoretically completes this part of the subject; inasmuch as we are hereby enabled to determine the number of real roots, and that of the imaginary roots of an equation. It was discovered a few years ago by M. Sturm, and the memoir containing it was presented by him to the French Institute, and published in the "Mémoires présentés par divers savants a l'Académie des Sciences," Tom. VI, 1835. It is now generally known by the name of "Sturm's Theorem," and is one of the greatest modern discoveries in Algebraical Analysis. The problem is as follows:

Given an equation $f(x)=0$, which has real coefficients; it is required to determine the number of real roots which it contains, and the limits of them.

First let $f(x)$ be cleared of *equal* factors by means of the last Article, so that no two roots of $f(x)$ are equal to each other; then we have the following theorem:

Let $f(x)$ be a function of x of real coefficients, of which $f'(x)$ is the derived-function; let $f(x)$ be operated upon by $f'(x)$ in the way of finding the greatest common measure, but with the peculiarity of the sign of a remainder always being changed before it is made a divisor; and let this process be continued until it terminates by giving a remainder independent of x, and which does not vanish, because $f(x)$ has no equal factors; let the successive remainders thus modified be symbolized by

$$f_1(x), \quad f_2(x), \quad \dots f_{n-1}(x);$$

so that we have the following system of equations;

$$\left.\begin{aligned} f(x) &= f'(x)\, q_1 - f_1(x), \\ f'(x) &= f_1(x)\, q_2 - f_2(x), \\ f_1(x) &= f_2(x)\, q_3 - f_3(x), \\ &\cdots\cdots\cdots\cdots \\ f_{n-3}(x) &= f_{n-2}(x)\, q_{n-1} - f_{n-1}(x); \end{aligned}\right\} \qquad (23)$$

$f_{n-1}(x)$ being the last factor and consequently independent of x.

Let α and β be two numbers of which α is the less, regard being had to its sign; substitute α for x in the series

$$f(x), \quad f'(x), \quad f_1(x), \quad \dots f_{n-1}(x), \qquad (24)$$

and write down in the same order the signs of the results; and count the number of sequences of two terms having contrary signs in this series of results; and suppose A to be that number. Again, substitute β in the same series of functions, and count as before the number of sequences of two terms with contrary signs, and suppose it to be equal to B,

Then there are A − B roots of $f(x)$ lying between α and β; that is, the number of the excess of variations of signs in successive terms, when α is substituted for x, over that when β is substituted for x, is the number of real roots of $f(x)$ greater than α and less than β.

This is Sturm's Theorem; but before I proceed to the proof of it, I must make two observations on the series of functions in (24). (1) No two consecutive functions can vanish for the same value of x; for if two consecutive terms vanish, when $x = c$, say $f_i(c)$ and $f_{i-1}(c)$, then by reason of (23), $f_{i-2}(c) = f_{i-3}(c) = \ldots = 0$, and ultimately $f'(c)$ and $f(c)$ vanish; which last values can coexist only when $f(x)$ has two or more equal roots, and this circumstance is contrary to our first assumption: and therefore we conclude that no two successive functions of the series (24) can simultaneously vanish. (2) No function of the series can be identically zero; for suppose $f_i(x)$ to be always zero; then from (23)

$$f_{i-2}(x) = q_i f_{i-1}(x),$$

and thus the value of x which makes $f_{i-2}(x)$ to vanish will also make $f_{i-1}(x)$ to vanish; that is, two consecutive functions will simultaneously vanish; and this has just been shewn to be impossible.

Now I will proceed to the proof of the theorem; let us suppose a to be a root of $f(x)$; and in the series (24) for x let $a-h$ be substituted; then by equation (16), Art. 114, the series of functions becomes

$$\begin{aligned}
&f(a) - f'(a)\frac{h}{1} + f''(a+\theta h)\frac{h^2}{1.2},\\
&f'(a) - f''(a+\theta h)\,h,\\
&f_1(a) - f_1'(a+\theta h)\,h,\\
&f_2(a) - f_2'(a+\theta h)\,h,\\
&\ldots\ldots\ldots\ldots;
\end{aligned}$$

in each of which we may suppose h to be such an infinitesimal, that the terms involving it must be neglected when added to a finite quantity; then, since a is a root of $f(x)$, $f(a) = 0$, and therefore the signs of the series of terms are the same as those of

$$-f'(a), \quad f'(a), \quad f_1(a), \ldots\ldots. \tag{25}$$

Again, let $a+h$ be substituted for x in the series (24); then the signs of the series of results are the same as those of

$$f'(a), \quad f'(a), \quad f_1(a), \quad f_2(a), \ldots\ldots; \tag{26}$$

whereas then of (25) the first two terms are affected with opposite signs, and the first two terms of (26) have the same signs; therefore by making x increase from a quantity a little below a real root to a quantity a little above it, a variation of sign in the series of functions (24) is lost; that is, what was a sequence of opposite signs has become a sequence of the same signs.

Also a similar loss of variation of signs takes place whenever x passes through a root; and therefore, if we make x to grow by infinitesimal increments from a to β, every time that its value becomes that of a real root of the equation the series of signs of $f(x)$, $f'(x)$, $f_1(x)$, ... loses a variation. And a loss of change of sign occurs *only* under that circumstance. For let us suppose it to occur otherwise; it is evident that it can be only when a sign of one or more of the other functions is changed, and that is, when one or more of them become zero. We have above shewn that two consecutive functions do not simultaneously vanish; let us now suppose that $f_i(x) = 0$, when $x = c$, and that it changes its sign at that critical value; then since from (23)

$$f_{i-1}(c) = q_{i+1} f_i(c) - f_{i+1}(c),$$

$$\therefore \quad f_{i-1}(c) = -f_{i+1}(c); \tag{27}$$

neither of which can vanish; and therefore as x passes through c their signs are the same whether x is infinitesimally less or infinitesimally greater than c. But $f_{i-1}(c)$ and $f_{i+1}(c)$ are, as is clear from (27), of opposite signs; and therefore, whatever are their signs respectively, and whatever change of sign $f_i(x)$ undergoes as x passes through c, the only possible arrangements and changes are (1) from $+ + -$ to $+ - -$, (2) from $+ - -$ to $+ + -$, (3) from $- + +$ to $- - +$, (4) from $- - +$ to $- + +$; and in each of these cases no change of sign is lost: in all there is one and only one sequence of two terms affected with contrary signs. We conclude therefore that a loss of change of sign can take place only in the first two functions of the series, and only when x passes through a root of $f(x)$.

Hence we finally conclude that there are as many real roots between a and β as there are more variations of sign for a than for β.

In calculating the successive values of the series of functions (24) we may observe that any function may be multiplied or divided by a positive but not by a negative quantity; because the sign would thereby be changed, and it is from the signs that the proposition is deduced.

179.] Ex. 1. $x^3-x^2-4x+3 = 0 = f(x)$.

$$\therefore \quad 3x^2-2x-4 = f'(x);$$

$$\begin{array}{rl}
3x^2-2x-4) & 3x^3-3x^2-12x+9 \ (x \\
 & \underline{3x^2-2x^2-\ 4x\qquad} \\
 & \qquad -\ x^2-\ 8x+9; \\
\text{or} & \qquad -3x^2-24x+27 \ (-1 \\
 & \qquad \underline{-3x^2+\ 2x+\ 4} \\
 & \qquad\qquad -26x+23;
\end{array}$$

$$\therefore \quad f_1(x) = 26x-23.$$

$$\begin{array}{rl}
26x-23) & 78x^2-52x-104 \ (3x \\
 & \underline{78x^2-69x} \\
 & \qquad 17x-104 \\
 & \\
 & \qquad 442x-2704 \ (17 \\
 & \qquad \underline{442x-\ 391} \\
 & \qquad\qquad -2313;
\end{array}$$

$$\begin{aligned}
\therefore \quad f(x) &= x^3-x^2-4x+3, \\
f'(x) &= 3x^2-2x-4, \\
f_1(x) &= 26x-23, \\
f_2(x) &= 2313.
\end{aligned}$$

The signs of which series of functions, corresponding to the following values, are

when	$x = -2$,	$-, +, -, +$,
	$x = -1$,	$+, +, -, +$,
	$x = 0$,	$+, -, -, +$,
	$x = 1$,	$-, -, +, +$,
	$x = 2$,	$-, +, +, +$,
	$x = 3$,	$+, +, +, +$.

Hence, as a change of sign is lost in passing from $x = -2$ to $x = -1$, a root lies between -2 and -1; and as changes are again lost in passing from $x = 0$ to $x = 1$, and from $x = 2$ to $x = 3$, two other roots lie severally between 0 and 1, and between 2 and 3. Thus the equation has three real roots, the positions of which have been determined.

180.] Hence follows a method of determining the whole number of real and imaginary roots of an expression of the form $f(x) = 0$.

Whatever is the number and value of the real roots, they must be between $-\infty$ and $+\infty$; let us therefore form as above the series of functions

$$f(x),\quad f'(x),\quad f_1(x),\quad f_2(x),\ \dots f_{n-1}(x),$$

the number of which is generally $n+1$, $f(x)$ being of n dimensions; and let $+\infty$ or a very large quantity be substituted for x, so that the signs of the functions are the same as the signs of the first terms; let m be the number of variations of signs of consecutive terms of this series.

And now let $-\infty$ be substituted for x, then the signs of the functions of even dimensions will be the same as before, but the signs of those of odd dimensions will be the contrary.

There will therefore in the latter substitution be as many variations of succeeding terms as there were permanences in the former, that is, there will be $n-m$ variations.

And as all the real roots are comprised within these limits, their number, by Sturm's Theorem, is $n-m-m$ or $n-2m$, and therefore the number of imaginary roots is $2m$.

There exist therefore as many pairs of imaginary roots as there are variations in the signs of the first terms of the functions $f(x)$, $f'(x)$, $f_1(x)$, $\dots f_{n-1}(x)$.

Ex. 1.
$$\begin{aligned} f(x) &= x^3-3px+2q, \\ f'(x) &= 3x^2-3p, \\ f_1(x) &= px-q, \\ f_2(x) &= p^3-q^2. \end{aligned}$$

The series of signs of the first terms is the same as that of

$$1,\quad 1,\quad p,\quad p^3-q^2.$$

If therefore p is negative, there is one variation, and therefore only one real root; and if p is positive, there is one real root when p^3 is less than q^2, and when p^3 is greater than q^2 all the roots are real.

Before I pass on to another subject, I would observe, as it was originally remarked by Sturm himself, that the preceding process would be equally applicable if $f'(x)$ were replaced by any other function of x of $n-1$ dimensions, of which no two or more roots are equal, and of which no root is equal to a root of $f(x)$. But as such a function could only generally be found, if we knew the roots of $f(x)$, and as these are not known, so the only available function with which we may first operate on $f(x)$ in the way of finding the greatest common measure is $f'(x)$, because its relation to $f(x)$ fulfills all the conditions which the process requires. Hence it is not quite left to our choice to take for its first divisor any function of $n-1$ dimensions that we please.

The functions in (24) are called the Sturmian functions. The general forms of them were given without proof by Mr. Sylvester in the Philosophical Magazine, Dec. 1839: and were subsequently demonstrated by Sturm himself in Liouville, Tome VII, p. 356. Other inquiries connected with them will be found in Liouville, Tome XII, p. 54, by M. Borchardt; Tome XIII, p. 269, by Mr. A. Cayley; in a paper by Mr. A. Cayley lately (Feb. 17, 1857) read to the Royal Society. The insertion of the results is evidently beyond the scope of the present work, and the student desirous of further information must have recourse to the original memoirs.

181.] Fourier's Theorem.

The following process was arranged by Fourier to separate the real and impossible roots of an equation; but as it only indicates a number which the sought number of real roots does not exceed, the discovery of M. Sturm renders it almost useless; however, as it advantageously exhibits the relations between the successive derived-functions in an algebraical point of view, it is right to insert it in a treatise on Infinitesimal Calculus.

Let $f(x)$ be a function of x of the form (1), Art. 170, and let it be cleared of equal factors; and let us suppose all its coefficients to be real; let the several derived-functions of it be formed, whereby we have a series,

$$f(x), \quad f'(x), \quad f''(x), \ldots f^{n}(x); \tag{28}$$

and let a and β be two numbers of which a is the less; then there cannot be more real roots of $f(x)$ between a and β than the excess of the number of alterations of sign in the above series of functions, when a is substituted for x, over the number resulting from the substitution of β.

For suppose a to be root of the equation $f(x) = 0$; then, since $f(a) = 0$, by the substitution of $a-h$ for x in the above functions, the series (28) becomes, when h is infinitesimal,

$$-f'(a)\,h,\quad f'(a),\ \dots f^n(a);$$

and when $a+h$ is substituted for x,

$$f'(a)\,h,\quad f'(a),\ \dots f^n(a);$$

and thus, supposing none of the derived-functions to vanish, the passage of the substituted quantity, from a value a little below a real root to one a little above it, causes a variation of sign to be exchanged for a permanence; and *pari ratione* as such a variation will be lost whenever the substituted quantity passes through a root, it follows that, as many real roots as there are lying between a and β, so many losses of variations of signs at least will there be in the series of functions, when we pass gradually from a to β.

At least, I say; for the series of signs may also be affected by the vanishing of any of the subsequently derived-functions; for suppose b to be such as, when substituted for x, to cause $f^r(x)$ to vanish, and h to be an infinitesimal; then, if $b-h$ is substituted for x, we have for $f^{r-1}(x)$, $f^r(x)$, $f^{r+1}(x)$,

$$f^{r-1}(b),\quad -f^{r+1}(b)\,h,\quad f^{r+1}(b); \tag{29}$$

and when $b+h$ is substituted for x,

$$f^{r-1}(b),\quad f^{r+1}(b)\,h,\quad f^{r+1}(b). \tag{30}$$

If $f^{r-1}(b)$ and $f^{r+1}(b)$, viz. the first and last terms of (29) and (30), are of contrary signs, then we shall have a variation and a continuance both in (29) and (30), so that no change will be lost. But if $f^{r-1}(b)$, $f^{r+1}(b)$ are of the same sign, then in (29) we shall have two variations, and in (30) two continuations, so that *two* changes of sign will be lost.

Similarly it may be shewn, that if many successive derived-functions vanish for a particular value of x, an even number of variations of sign may disappear.

There may therefore be losses of variation of sign in the series of functions given in (28) at other values of x than roots of $f(x)$, but variations must nevertheless be exchanged for continuations at the roots; therefore the Theorem gives only a number which is *not less* than the number of real roots. The advantage of Sturm's Theorem is, that it gives the exact number of real and of imaginary roots.

182.] The rule commonly known by the name of Des Cartes' Rule of Signs is a particular case of Fourier's Theorem; viz. in the general equation $f(x) = 0$, the number of positive roots cannot exceed the number of variations of signs of the successive terms, and the negative roots the number of continuations of signs.

Firstly, let $x = 0$, in the series of functions (28); then the signs of $f(x), f'(x), \dots f^n(x)$ are the same as those of the several and successive terms of $f(x)$ taken from right to left; and when $x = \infty$, the signs of the functions are all positive. Hence there can be no more real positive roots than there are changes of sign in the successive terms of $f(x)$.

Secondly, let $x = -\infty$, then the series of functions form only variations of sign, of which there are of course n, and therefore the number exceeds the number of variations, when $x = 0$, by the number of permanences in the terms of the equation. Hence the number of negative roots cannot exceed the number of continuations of signs.

183.] Taylor's Series also furnishes a method, which was invented by Newton, for finding a number greater than the greatest root of an equation.

Let $f(x) = 0$ be an equation of n dimensions of the form (1), Art. 170, and for x let us substitute $y+h$; so that

$$f(y+h) = f(h) + f'(h)\frac{y}{1} + f''(h)\frac{y^2}{1.2} + \dots + f^n(h)\frac{y^n}{1.2.3\dots n}.$$

Suppose such a value to be given to h as to make $f(h)$, $f'(h), \dots f^n(h)$ all positive, then by Fourier's Theorem no root of the equation can lie between h and $+\infty$; therefore h is greater than the greatest positive root.

184.] Taylor's Series is also useful for approximating to a root of an equation.

Suppose two values a and β, of which a is the less, and the difference between which is small, to have been found, which, when substituted for x, give results with different signs; then a root of the equation lies between them. To determine it, let us suppose the root to be $a+h$; then $f(a+h)=0$; but by Art. 116, equation (21),

$$f(a+h) = f(a) + f'(a+\theta h)\,h,$$

$$\therefore \quad h = -\frac{f(a)}{f'(a+\theta h)};$$

and neglecting θh when added to a, since h is small, we have

$$h = -\frac{f(a)}{f'(a)}; \tag{31}$$

in which case however $f'(a)$ ought to be large in comparison of $f(a)$, otherwise the result is inconsistent with our supposition of h being small.

And here I must conclude this inquiry. I have investigated the theory of algebraical expressions so far as the principles of Infinitesimal Calculus cast light on them; and I have incidentally given one or two theorems which enable us to determine approximate values of the roots. A systematic inquiry into the discovery of roots of equations would take me through the theory of quadratic, cubic, and biquadratic equations in their most general form; and would carry me on to the great discovery of Abel, that the roots of an equation of a degree higher than the fourth cannot generally be expressed in an algebraical form; and to make the subject complete I should also have to discuss the theory of simultaneous equations, and of elimination; and I should in the course of such discussion enter on the subject of determinants. All this is manifestly beyond the scope of the present work. To the student however who desires information on the profound investigations of the ablest mathematicians in these subjects, I would recommend the study of (1) Cours d'Algèbre Supérieure, par J. A. Serret, 2$^{\text{me}}$ edit., Paris, 1854; (2) some memoirs in the works of Abel, edited by Holmboe, Christiania, 1839; (3) the memoirs of Evariste Galois in Liouville's Journal. He will also in M. Serret's treatise find references to the different sources whence he may derive information on these and kindred subjects.

DIFFERENTIAL CALCULUS.

PART II.

GEOMETRICAL APPLICATIONS.

CHAPTER IX.

ON GEOMETRY.

SECTION 1.—*On the adjustment of the Principles of Geometry and Infinitesimal Calculus.*

185.] IT will by this time have become tolerably plain to the attentive reader, that the characteristic property of Number, which is the foundation of Infinitesimal Calculus, is that of continuous and infinitesimal growth; and that Differentiation is the mathematical expression of the Law of Continuity. Now our object in the following pages is to apply the propositions which have been proved above to questions of pure geometry; and therefore it is necessary so to modify or enlarge the principles of that science, as to adjust them to those of Infinitesimal Calculus.

As it is not however our intention to write a treatise on the principles or difficulties of Elementary Geometry, we shall rather enuntiate axioms and definitions, and state results, than prove propositions, leaving the last to be effected by our applications; neither shall we discuss the methods by which we have arrived at them, in the belief that a rational understanding of the first Chapter of the present Treatise is sufficient to explain them. In accordance with illustrations therein given, we have introduced the ideas of *motion* and of *limits;* motion perhaps as having to do with the generation of geometrical quantities, but chiefly as involving the property of infinitesimal divisibility, which is necessary to a due conception of the latter property of *limits*; motion however, as we have introduced it, does not encroach on the subject of mechanics, wherein we treat of motion as the effect of certain causes, and discuss its circumstances, as, for example, the particular law of force which produces it, the velocity with which the moving material changes position, which necessarily involves time, and so on: but in what follows we consider motion as a simple act, a primary conception as a quality of matter; and if it tends, as it does, to give clearness to our first geometrical conceptions, it is nothing but a servile adherence to an inferior, though customary method, which would hinder us from introducing it.

It is conceived that all geometrical quantity, whether linear, superficial, or spatial, is from its very nature capable of increase or decrease to an infinite extent. A line may be very long, nay of an infinite length, or very short; space may be very small, such as, so to speak, it would require a microscope of almost infinite power to render visible, or it may be very large. Whenever such quantities vary, they vary in accordance with the law of continuity; they cannot pass from one magnitude to another without passing through all intermediate magnitudes; they grow larger and larger, or less and less. This capability of increase or decrease is involved in our idea of geometrical quantity; it is necessary to its completion; and if it is omitted, our notions fall short of the properties of the subject-matter of the science.

There are however limits within which this variation is included; the superior limit of geometrical magnitude of the concrete kind called *space* is infinite space: so of superficial and linear magnitudes, the superior limits are respectively infinite superficies, and a line of infinite length.

The inferior limit of all these is the same, the geometrical zero, a *point*.

186.] Of the definitions of geometrical quantities founded on such notions, the following are useful for our present object.

I. A *point* is the inferior limit of geometrical space.

II. A *sphere* is the locus of a point of space, which is always at the same distance from a given point.

III. A *plane* is the surface of a sphere, the radius of which is infinitely great.

IV. A *circle* is the locus of a point, which is always at the same distance from a given point, all the points being in one plane.

V. A *straight line* is the arc of a circle, the radius of which is infinitely great.

VI. A *triangle* is a plane figure contained by three straight lines meeting one another, two and two.

VII. And if the triangle is isosceles, the sides of that triangle, having a finite base and the vertex at an infinite distance, are *parallel straight lines.*

As this is not intended to be an accurate treatise on the principles of geometry, many words are used which have not been defined, as line, locus, &c.; these however are to be taken in their ordinary significations: and it is to be observed, with respect to these definitions and conceptions, that the surfaces, lines, &c. they refer to, are only approximations to the accurate ones. But they are such approximations as may differ from the real ones by quantities as small as we please; and as these small quantities may be infinitesimals, such that it would require an infinity of them to make a finite quantity, and as we do not take an infinite number of them, these differences must, in conformity with what has been said in the first Chapter, be neglected, and our definitions are rigorously exact.

Having defined a plane, as we have done, to be the limiting spherical surface when the radius becomes infinitely great, it follows that the extreme positive side of the plane, when continued, runs into the extreme negative side; that is, having traced the plane as far as we can on the positive side, we meet it again on the negative; and although the surface appears to be discontinuous, it is not in reality so: the positive side being continued into the negative, and the apparent discontinuity arising from the defect in our power of apprehending and symbolizing such quantities. Thus then, if we have any continuous curve traced on the plane, and the curve runs off to the extreme positive side of the plane, we ought not to consider it to stop or to have points of discontinuity, but we must consider the branches of it to be continued, and must look for them on the negative side of the plane. We may borrow from the figure of the earth, and our mode of determining position on its surface, an illustration of what is here intended. We measure, say from the meridian of Greenwich, degrees along the equator to 180° east longitude; and then, instead of proceeding further on and measuring in the same direction, we measure backwards, and reckon degrees of longitude west: and what would be 181° east longitude becomes 179° west. If then east corresponds to the positive direction, west does to the negative.

It is worth remarking how exactly our notion of a plane coincides with the definition which I have given. We speak of the surface of water as a plane; whereas it is a portion of the surface of a sphere, whose radius is very large compared with the area we take, say, 4000 miles compared with a few inches.

So again as to our conception of a straight line. A straight line being a particular case of a circle, is a continuous line; it does not terminate at positive infinity or at negative infinity; but the two branches of the line are connected with one another, running, if we may so speak, round the circle of which the radius is infinity, and joining together. If then we take any given point on the circle as the origin, the distance to the opposite extremity of the diameter of the circle is positive infinity, and we do not measure or follow the line further in this direction, but considering the line to be continued beyond that point, we meet it on the opposite side, and measure it backwards. There is no point of discontinuity in the line: the line proceeds in the same direction; it has been positive infinity; the pole or extremity of the diameter of the circle has been passed, and then the line becomes negative infinity. The illustration above given from the figure of the earth aptly illustrates our meaning in this case. Considering any meridian to be the very large circle, and taking any place on it to be the origin, the "antipodes" to it becomes either positive infinity or negative infinity, according as we measure in the positive or negative direction; the sign of the quantity changes immediately after the pole has been passed, and what was positive infinity becomes negative infinity. Therefore in this point of view infinity is not a quantity incapable of increase, for the line may be continued round and round the meridianal circle as often as we please; there is no limit to the quantity: the limit is to our powers of symbolizing such quantities.

It is worth observing too, that the definition given of parallel straight lines enables us to avoid the difficulty connected with our first introduction to the theory and properties of such lines. If we shew, as Playfair has done, that the exterior angles of a triangle are together equal to four right angles, it follows that the interior angles are equal to two right angles: but if the base of a triangle remains finite, and the vertex is removed further and further, the vertical angle becomes less and less, and diminishes without limit; in which case the sum of the base angles is equal to two right angles, and the sides become parallel straight lines; whence the properties of such lines, which are enuntiated in the xxixth proposition of the first book of Euclid, immediately follow.

A good illustration of this theory occurs in the Phenomena of Parallax. If the angles subtended at the centre of the earth by the sun and any fixed star, whose parallax has not been discovered, are observed when the earth is in perihelion and at aphelion, it is found that, notwithstanding the extreme delicacy of our instruments, the sum of these two angles is exactly equal to two right angles. Taking then the two positions of the earth to be the extremities of the base of a triangle, and the line passing through the sun's centre and terminated by them to be the base, and the fixed star to be the vertex, it appears that, although the base of the triangle is 190,000,000 of miles, the angle subtended by it at the vertex is too small to be measured, and the two lines drawn to the star from the earth, at the two positions of it, are to all appearance parallel straight lines.

187.] In corroboration also of what has here been stated, the following are a few out of a great many striking instances:

In differentiating tan θ, we have

$$\frac{d.\tan\theta}{d\theta} = (\sec\theta)^2,$$

which is necessarily a positive quantity; and therefore by Theorem I, Art. 110, θ and $\tan\theta$ are always increasing and decreasing simultaneously, and therefore as θ increases $\tan\theta$ increases. Now as θ approaches to 90°, $\tan\theta$ becomes $+\infty$; and immediately after θ has passed 90°, $\tan\theta$ becomes $-\infty$, indicating that negative infinity is positive infinity increased; that is, as θ has increased and passed through 90°, $\tan\theta$ has increased from $+\infty$ to $-\infty$. And so again, as θ increases from

90° to 180°, $\tan\theta$ is continually increasing from $-\infty$ to 0, and passes through 0, and increases to $+\infty$, which is the value of $\tan\theta$ when $\theta = 270°$; and so on, as θ increases, $\tan\theta$ is continually increasing, travelling, if we may so say, round the circle of which the straight line along which $\tan\theta$ lies is conceived to be the limit, when the radius of the circle is infinitely great. It is impossible not to remark how exactly this illustration agrees with what has been said in Chapter I on the order of infinitesimals. For corresponding to every 180° through which θ turns, $\tan\theta$ passes from 0 to ∞, and on through ∞ to 0 again; that is, the path through which $\tan\theta$ has travelled is infinite, although θ has passed over only a finite angle; and therefore, when θ has revolved through 360° and 540° and 720°, and so on, $\tan\theta$ has travelled over a length of line equal to twice, three times, four times, &c., the infinite length corresponding to a revolution of θ through 180°; and thus we have infinities bearing a finite ratio to each other. Conceive moreover θ to have revolved an infinite number of times through 180°, then the distance over which $\tan\theta$ will have travelled will be an infinity of infinities, that is, will be $(\text{infinity})^2$; and thus we obtain different orders of infinity.

Again, suppose the following problem to be given: To find the maximum and minimum values of y, when

$$y = \frac{(x+2)^2}{(x-3)^3};$$

$$\therefore \quad \frac{dy}{dx} = -\frac{(x+2)(x+12)}{(x-3)^4};$$

$y = \infty$, when $x = 3$; but as $\frac{dy}{dx}$ does not change its sign, this value of y is neither a maximum nor a minimum. How then is the result to be interpreted? As follows: Since $\frac{dy}{dx}$ is negative, y decreases as x increases, and when x is a little less than 3, $y = -\infty$: but when x is a little greater than 3, $y = +\infty$; therefore, as x has passed through 3, the value of y has changed from $-\infty$ to $+\infty$, but y has decreased during this progressive increase of x, therefore $+\infty$ is $-\infty$ decreased; therefore y has not reached a minimum or a maximum value when $x = 3$, because it has not become $-\infty$, and then returned, but it has gone on decreasing. And if we draw a graphical representation of the curve corresponding to the equation, such as that in fig. 24, the phenomena explain themselves. The curve on the negative side of the axis of y is of the form CB, where OB $= 2$; and if OA $= 3$, the curve is continually approaching the line drawn through A parallel to the axis of y, and when x is nearly 3, y is $-\infty$: but when x is greater than 3, y is $+\infty$; that is, the curve has crossed the asymptote at the pole of the circle of infinite radius opposite to A, and has returned in the direction EF, the branch in the direction of E being a continuation of that in the direction of D. Similarly the branch in the direction F would, if produced, unite itself to that in the direction C, having crossed the axis of x at the pole opposite to O.

In corroboration of this theory, it will appear that whenever a curve is of the form fig. 24, if the criteria, which will be discussed in the next Chapter, are applied at points such as those where the branch E meets the branch D, and crosses the asymptote, we have the characteristics of a point of inflexion; and if the curve be such as in fig. 25, we have the characteristics of a point of *embrassement*; and whenever such as is represented in fig. 26, the conditions of a maximum ordinate.

And so again whenever a branch of a curve continues to infinity, it always returns in some way or another; and in whatever manner a rectilineal asymptote is drawn, no branch of the curve ever goes off asymptotic to it without returning in one of the ways indicated in the figures 24, 25, 26; and it seems impossible to account for such phenomena except on the theory explained above, viz. that the plane and the straight line are respectively the superior limits of the sphere and the circle, when the radii become infinitely large*.

SECTION 2.—*On the extension of symbols of direction.*

188.] In algebraical geometry, and therefore in the applications of the Differential Calculus to the theory of plane curves, we meet with symbols of two distinct characters; symbols of quantity, such as $a, b, c, \ldots\ldots x, y, z, \theta, \phi, \psi, \ldots\ldots$ when symbolical respectively of lines and angles: and symbols of *direction*, $+$, $-$, $+\surd(-)$, $-\surd(-)$, &c. Our object is so to enlarge our power of interpreting symbols of this second kind as to comprehend those which are usually called Impossible or Imaginary, of which however we shall discuss only two, viz. $+\surd -$, $-\surd -$, or as they may be written, in accordance with the index law, $+(-)^{\frac{1}{2}}$, $-(-)^{\frac{1}{2}}$[*].

---

* For a further elucidation of many points in elementary geometry more or less connected with the present subject, I would refer the reader to a small Treatise on the Difficulties of Elementary Geometry, by F. W. Newman, M.A., formerly Fellow of Balliol College, Oxford; Longman and Co., London, 1841.

As to symbols of quantity, it is to be observed that, when we symbolize a line by a, we do not mean that a is the absolute length of the line; for all lengths can only be relative, and there must be some modulus or standard to compare them with: but we intend a line which is in length a times some arbitrary, though for the time fixed, standard unit. So a line symbolized by b is a line b times in length some unit. Thus then a, b are numerical quantities; not concrete magnitudes, but abstract quantuplicities, the subject-matter of arithmetical algebra, and therefore subject to its laws; they do not designate the absolute lengths of lines, but the number of times a certain concrete unit is to be taken. So again if an area is symbolized by ab, a and b are abstract numbers, which must be multiplied together by the laws of arithmetical algebra, and their product is the number of times the superficial unit is to be taken. Let it therefore be carefully borne in mind that this is the meaning of the several symbols of quantity, whether constant or variable, which we shall use in the following Chapters. Suppose that we have a line symbolized by a, and that we fix upon a certain point as the origin from which lines are to be measured, any line drawn from it, equal in length to a times the linear unit, will fulfil the requirements of the single symbol a. But inasmuch as an indefinite number of equal lines may be drawn from any one point, thus far we have no means of determining which of all such lines is intended; hence arises the necessity of some other symbols to indicate direction, or, as they are called, *symbols of direction* or *affection*. One or two of the most simple cases of these we proceed to explain, feeling assured that the principle of explanation is so entirely in harmony with the usual meaning of + and − that it ought not to be omitted in an elementary treatise; and also chiefly because it enables us to shew that an algebraical curve, though apparently discontinuous and confined within certain fixed limits, is not in reality so, but extends to infinity in all directions. Other parts of the theory, some of which are as yet not sufficiently established, we omit, as unsuited to our present object.

---

[*] For a fuller explanation of the principles of explaining these and such like symbols, we would refer to Études Philosophiques sur la Science du Calcul, par M. F. Vallès, 8vo. Paris, 1841; and for the general theory of the meaning of $(+)^{\frac{p}{q}}$ to Dr. Peacock's Algebra, 8vo. Cambridge, vol. i. 1842, vol. ii. 1845; to Mr. Warren's Treatise on the Square Root of Negative Quantities, 8vo. Cambridge, 1828; and to many papers in the Cambridge Mathematical Journal, of which for the most part Mr. Gregory was the author.

189.] Suppose o, fig. 27, to be the point from which lines are to be measured, and OA $= a$ times the linear unit to be drawn from o towards the right hand. Now since, as we said above, any line drawn from o, a times the linear unit in length, will be symbolized by a, it is necessary to fix on some *originating* direction; suppose this to be OA, and any line measured from o towards A to be affected with the symbol of direction +; if then a line, after it has undergone any operation or a series of operations, comes into the position OA, it is still to be symbolized by + : and, if the line is a, by $+a$. Such an operation we might conceive to be a reciprocating one, the line at one time being in the position OA, and at another in the position O′A′, having moved sideways, and assumed all intermediate positions. Or we may conceive that the line OA, see fig. 28, has revolved round the point o, and, having turned in the plane of the paper through 360°, has again come into its original position, and so on continually; and it is manifest that as often as it has revolved through any multiple of 360°, it has assumed its original position OA, and is therefore to be symbolized by $+a$. So also there are many conceivable ways in which the line may have moved, and that periodically, and at the end of a complete period be in the position OA. But have we any other customary mode of indicating direction, to serve as a guide which of these conceivable operations to take? We have. Whenever a line equal in length to a is measured from o towards the left, we symbolize it by $-a$; if therefore either (−) were a symbol for the operation of one oscillation having been performed on the line, i. e. the line having passed into the position O′A′, see fig. 27: or (−) symbolized the line OA, fig. 28, having been turned through 180°, either would account for the negative sign of affection, and $(-)$ would be the symbol of the operation; but under the first hypothesis, the line at one stage of the process will be half on the positive side of the origin and half on the negative; if therefore the operation is continuous, which it is, in passing from $+$ to $-$, there should be some symbol to indicate that particular stage; it does not however appear that we have any symbol of the kind; and such a motion, and a line in such a state, we do not use nor contemplate in our ordinary geometrical conceptions. Let us therefore consider whether we have not symbols to indicate a line in any intermediate position between OA and OA$_1$, conceiving the line to pass from the one position to the other by means of a revolution through 180°.

As we said before, whenever the line is measured from O in the direction OA, it is to be affected with a $+$ sign. Taking therefore O as the origin of line, and OA as the *direction line* from which symbols and operations of affection are to be originated, whenever a line, as, for example, OA, has turned an integral number of times through 360°, it is to be affected with the sign with which it started. If therefore it was affected with the $+$ sign at first, indicating that it started from OA, and if $+$ is the symbol of turning through 360°, after one revolution the symbol of affection is $+$ on the back of $+$, that is, according to the index law, $+^2$; similarly after two revolutions, $+^3$; and after $(n-1)$ revolutions, $+^n$. Supposing therefore that the line which is of the length a, when along the originating direction OA, is unaffected with any sign: $+a$ means that the line has turned through 360°, and has come again into the position whence it started; and so $+^n a$ means that a line of length a has revolved n times from the direction of origination, and is in the position OA; whence it appears, in accordance with the arithmetical meaning and law of $+$, that $+$ is, for symbolical purposes of direction, equivalent to $+^n$, n being a whole number.

In conformity then with the algebraical law of indices $+^{\frac{1}{2}}$ is the symbol of that operation, which, being performed twice successively, brings the symbol into the value $+$; that is, if $+$ signifies turning the line through 360°, $(+)^{\frac{1}{2}}$ indicates turning it through 180°; but $-$ symbolizes this operation,

$$\therefore \quad +^{\frac{1}{2}} = -, \text{ and } (-)^2 = +;$$

or the operation symbolized by $(-)$ performed twice successively is equivalent to the operation signified by $+$, and means turning a line through 360°. Similarly again $(+)^{\frac{2n+1}{2}}$ is equivalent to $-$; for it is equivalent to $+^{n+\frac{1}{2}} = +^{n}(+)^{\frac{1}{2}} = +^{n}-$; and this coincides with the usual ambiguity in the sign of $+^{\frac{1}{2}}$, for it may as far as the form $+^{\frac{1}{2}}$ teaches, be either $+$ or $-$. If therefore the $+$, whose root is to be extracted, is raised to an even power, its root is to be affected with a positive sign; but if the $+$ is $+^{2n+1}$, then the square root is $+^{n}+^{\frac{1}{2}}$, which is equivalent to $-$, and the root must be affected with the negative sign. Hence also it is plain that $\sqrt{(-a)} \times \sqrt{(-a)}$, which equals $\sqrt{a^2}$, can only be $-a$; because the $+$, with which a^2 is affected under the radical, is of only the first power.

Therefore we have shewn that in symbolical geometry

$$\left.\begin{array}{ll} \text{1st,} & +^{n} \quad = +, \\ \text{2d,} & +^{\frac{2n+1}{2}} = -. \end{array}\right\}$$

190.] So again $+^{\frac{1}{4}}$ symbolizes that operation which, being performed twice successively, is equivalent to $+^{\frac{1}{2}}$, that is, to $(-)$; and, being performed four times successively, is equivalent to $+$;

$$\therefore \quad +^{\frac{1}{4}} = (-)^{\frac{1}{2}};$$

and therefore, as $-$ indicates that a line is to be turned through 180°, so $(-)^{\frac{1}{2}}$ means that a line is to be turned through 90°. Whenever then a line is affected with $(-)^{\frac{1}{2}}$, which is equivalent to $+^{\frac{1}{4}}$, as its symbol of direction, that line is to be drawn at right angles to the original direction of origination, viz. in the direction OA_2, see fig. 28; and whenever the symbol of direction is $+^{\frac{3}{4}}$, which $= +^{\frac{1}{2}}+^{\frac{1}{4}} = -(-)^{\frac{1}{2}}$, the line which is affected with it is to be drawn in the direction OA_3. Similarly $+^{\frac{4n+1}{4}}$ indicates a line drawn in the direction OA_2, and $+^{\frac{4n+3}{4}}$ a line drawn in the direction OA_3. So also $+^{\frac{1}{n}}$ means that the line with which it is affected is to be drawn at an angle of $\dfrac{360°}{n}$ to the originating direction OA.

191.] We have inserted the above method of explaining symbols of direction, which are usually termed Impossible and

Imaginary and passed over in silence, because it is clearer to the perception than another method which has received copious elucidation from Dr. Peacock and Mr. Warren: that viz. in which $\cos\theta + \sqrt{-1}\sin\theta$ is considered as the symbol, and whereby, when it is affixed to a line, say ρ, the direction is indicated in which the line is to be drawn; thus

$$\rho(\cos\theta + \sqrt{-1}\sin\theta) \tag{1}$$

represents a line of length ρ drawn at an angle θ to the originating direction. The two methods coincide at those points which will be most useful in the sequel; thus let $\theta = 0$, then the line represented by (1) is ρ, and coincident with the zero operation, that is, with the line of origination; let $\theta = 90^\circ$, the line becomes $\rho\sqrt{-1}$, and is at right angles to the originating direction; let $\theta = 180^\circ$, and the line is $-\rho$, that is, is the originating line produced backwards; let $\theta = 270^\circ$, and (1) becomes $-\sqrt{-1}\,\rho$, and is in a direction at right angles to and below the originating direction; and if $\theta = 360^\circ$, the line becomes $+\rho$, and lies in its original direction.

In accordance then with the interpretation of $\sqrt{-1}$ which such a symbol as (1) thus used involves, it will be observed that (1) correctly represents two sides of a *rectangle*; that is, fig. 29, if OP $= \rho$ and POM $= \theta$, OM $= \rho\cos\theta$ and PM $= \rho\sin\theta$, and as PM is affected with $\sqrt{-1}$, it is to be measured in a direction PM, which is perpendicular to OM; which lines therefore cannot be added, or subtracted, as they are not in the same line, but we may by an extension of interpretation suppose (1) to represent the diagonal OP of the parallelogram, of which OM and MP are the two containing sides.

It is also to be observed, that

$$\cos\theta + \sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}}, \tag{2}$$

and that therefore $e^{\theta\sqrt{-1}}$ may be used as a symbol of direction; wherein θ expresses the angle of inclination to the originating line of the line which the symbol affects.

192.] To apply these principles to the delineation of plane curves from their equations, suppose $y = f(x)$ to be the equation to the curve; since x and y have already preoccupied the two directions at right angles to each other in the plane of the paper, which is, and conveniently so, called the *plane of reference*, we must seek for some other course by which a line, which has been measured in the positive direction, may be made to turn through 180° into the negative. Such we shall have if it is made to revolve in a plane to which the other axis is perpendicular; as, for instance, let x revolve in a plane at right angles to the axis of y, then, whenever x is affected with $\pm(-)^{\frac{1}{2}}$, it is to be measured in a plane passing through the axis of y, and perpendicular to the axis of x. Similarly if y is affected with $\pm(-)^{\frac{1}{2}}$, it is to be drawn in the plane passing through the axis of x, and perpendicular to the axis of y. Thus it appears that an equation between x and y may represent not only a curve in the plane of the paper, but also curves in the planes at right angles to it, passing through the axes of x and y.

Let us consider the following examples:

The equation to the ellipse, referred to its centre as origin, and principal axes as coordinate axes, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

whence we have $$y=\pm\frac{b}{a}\{a^2-x^2\}^{\frac{1}{2}}, \tag{3}$$

and $$x=\pm\frac{a}{b}\{b^2-y^2\}^{\frac{1}{2}}; \tag{4}$$

and therefore neither y nor x is affected with $\pm\sqrt{-}$, as long as x is less than $\pm a$, and y is less than $\pm b$.

But let x be greater than $\pm a$, then we may write (3) in the form

$$y=\pm\sqrt{-}\,\frac{b}{a}\{x^2-a^2\}^{\frac{1}{2}}; \tag{5}$$

which equation, short of the symbol $\sqrt{-}$, represents an hyperbola whose transverse axis is $2a$, and conjugate axis $2b$, and whose asymptotes are as drawn in fig. 30; but which hyperbola, when the $\sqrt{-1}$ is introduced, is in the plane containing the line A′OA, and perpendicular to the plane of the paper, and is delineated by the dotted line; also as the equations to the asymptotes are

$$y=\pm\sqrt{-1}\,\frac{b}{a}\,x, \tag{6}$$

they lie in the same plane as the curves, and are represented by the lines OL and OL′.

Similarly, when y is greater than $\pm b$, we have

$$x = \pm\sqrt{-1}\,\frac{a}{b}\,(y^2-b^2)^{\frac{1}{2}}, \tag{7}$$

which represents an hyperbola in the plane passing through the line BOB′ and perpendicular to the plane of the paper, and which is delineated by the dotted lines of the figure, viz. SBS′ and TB′T′; and the equations to its asymptotes are

$$x = \pm\sqrt{-}\,\frac{a}{b}\,y. \tag{8}$$

Thus the general equation to the ellipse not only represents the ellipse in the plane of the paper, but also two hyperbolæ in planes containing the coordinate axes and perpendicular to the plane of reference.

Similarly the equation

$$x^2+y^2 = a^2,$$

in addition to the circle in the plane of xy, expresses also two rectangular hyperbolæ in planes perpendicular to it, and containing the axes of x and y.

Again, consider the equation to the parabola, viz.

$$y^2 = 4mx;$$

$$\therefore\ y = \pm 2(mx)^{\frac{1}{2}};$$

and therefore the curve is in the plane of the paper for all positive values of x; but let x be negative, and we have

$$y = \pm 2\sqrt{-}\,(mx)^{\frac{1}{2}};$$

which expresses another equal parabola, but turned in the opposite direction and in a plane perpendicular to that of the paper, as is indicated by the dotted curve of fig. 31.

Many other examples of the same kind will occur in the sequel. The explanation of such impossible symbols is necessary to a due adjustment of geometrical interpretation to the law of continuity; for no *algebraical* formula can, so far as is known, give points of discontinuity, neither therefore ought the geometrical representative to exhibit such; but it does so, unless we interpret those quantities which are affected with $\pm\sqrt{-1}$.

The preceding Section is but a mere sketch of a method of extensive application, and of only one part of it, viz. of that which relates to $\pm\sqrt{-}$; in solving cubic equations, and in tracing the curves which they represent, we shall meet with such symbols as $(+)^{\frac{1}{3}}$, $(-)^{\frac{1}{3}}$, &c., which indicate that branches of the curve exist in planes inclined at 120°, &c. to the plane of the paper; but the full development would occupy more space than can be given to it in an elementary treatise.

Section 3.—*On the generation of some plane curves of higher orders, and on their equations.*

193.] The reader is supposed to be familiar with the principal properties of the straight line, the circle, and the three conic sections as exhibited in their algebraical equations; yet, as more copious illustration will be required in the succeeding Chapters than they afford, it is necessary to insert an account of the modes of description, and the equations of some curves of a higher order; most of which possess historical interest from the labour bestowed on them by ancient mathematicians.

In the first place let it be observed, that the equation to a curve may sometimes be expressed by means of a subsidiary angle: the elimination of which from the given equations will produce the better-known equation to the curve.

Thus the equation to the ellipse may be put in the forms,

$$\left.\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}\right\}; \qquad \therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1. \tag{9}$$

The hyperbola may be expressed by

$$\left.\begin{aligned} x &= a\sec\theta \\ y &= b\tan\theta \end{aligned}\right\}; \qquad \therefore \quad \frac{x^2}{a^2}-\frac{y^2}{b^2} = 1. \tag{10}$$

Other examples of the same kind will be found in the sequel.

194.] The Cissoid of Diocles. Fig. 34.

Definition.—If at equal distances from O and A, the two extremities of a diameter of a circle, two ordinates MQ and NS are drawn, and if OS is drawn cutting MQ in P, the locus of the point P is the Cissoid of Diocles.

Let OC = CB = CA = a = the radius of the circle; OM = x, MP = y.

Then, by the geometry, OM : MP :: ON : NS;

but $\quad$ ON = OA − AN = OA − OM = $2a-x$,

$\quad$ NS = MQ = $\{\text{OM} \times \text{MA}\}^{\frac{1}{2}} = (2ax-x^2)^{\frac{1}{2}}$;

substituting which values in the above proportion, we have

$$\frac{y}{x} = \frac{\{2ax-x^2\}^{\frac{1}{2}}}{2a-x}; \tag{11}$$

$$y^2 = \frac{x^3}{2a-x}, \qquad \therefore \quad y = \pm \frac{x^{\frac{3}{2}}}{\{2a-x\}^{\frac{1}{2}}}. \tag{12}$$

The equation represents the curve described in fig. 34: the dotted part being that out of the plane of the paper, and when y is affected with $\pm\sqrt{-}$; and as the equation to the fundamental curve, viz. the circle, also expresses a rectangular hyperbola out of the plane of reference, that part arises from the hyperbola having been operated upon in a manner analogous to the circle in the above generation of the curve.

The equation will be subsequently completely analysed; but certain salient points of it are at once evident from the geometrical description. Thus the curve lies equally above and below the axis of x; it passes through O and B, and has for an asymptote the line drawn through A and perpendicular to OCA; it lies out of the plane of the paper to the left of O and to the right of A; in the fig. OE $= a$, OD $= 3a$.

If (12) is expressed in terms of polar coordinates, the equation is

$$r = 2a \sin\theta \tan\theta. \tag{13}$$

195.] The Witch of Agnesi. Fig. 35.

DEF.—In the ordinate MQ of a circle a point P is taken, so that MP : MQ :: OA : OM; the locus of the point P is the Witch of Agnesi.

Let OC $=$ CA $= a$, OM $= x$, MP $= y$.

Then, by the definition, MP : MQ :: OA : OM; but

$$\text{MQ} = \{2ax-x^2\}^{\frac{1}{2}}; \qquad \therefore \quad y : \{2ax-x^2\}^{\frac{1}{2}} :: 2a : x; \tag{14}$$

$$\therefore \quad y^2 = 4a^2\frac{2a-x}{x}, \qquad y = \pm 2a\left\{\frac{2a-x}{x}\right\}^{\frac{1}{2}}. \tag{15}$$

The dotted parts of the curve are out of the plane of reference, and arise from an analogous operation being performed on that rectangular hyperbola, out of the plane of the paper, which the equation to the fundamental curve also represents.

Although we are obliged to reserve the complete discussion of equation (15) until the next Chapter, yet it appears that the curve cuts the axis of x at A, and that the axis of y is an asymptote; and that the ordinate is affected with $\pm\sqrt{-}$ whenever x is negative, and whenever x is greater than $2a$; in the fig. $OB = OB' = 2a$.

196.] The Conchoid of Nicomedes. Fig. 36.

DEF.—A point A and a straight line EOE′ being given, from A a straight line AQP is drawn cutting OE′ in Q, and P is such that QP is always equal to a given straight line OB; the locus of the point P, in the different positions of AQP, is the Conchoid of Nicomedes.

From A draw AO at right angles to EOE′, and let $OA = a$; let the straight line $QP = OB = b$; $OM = x$, $MP = y$.

Then, by the geometry, $\frac{AO}{OQ} = \frac{MP}{MQ} = \frac{AO+MP}{OQ+MQ} = \frac{a+y}{x}$;

$$\therefore \quad OQ = \frac{ax}{a+y}.$$

Also $$PQ^2 = MP^2 + MQ^2,$$

$$b^2 = y^2 + \left(x - \frac{ax}{a+y}\right)^2;$$

$$\therefore \quad x^2y^2 = (b^2-y^2)(a+y)^2, \tag{16}$$

$$x = \pm\frac{a+y}{y}(b^2-y^2)^{\frac{1}{2}}. \tag{17}$$

From which equation it appears that $x = \infty$, when $y = 0$, and therefore that EOE′ is an asymptote.

The line EOE′ is called the *rule* of the conchoid, and PQ or OB the *modulus*. If the line b is measured from Q towards A instead of along AQ produced, then another curve is generated which is called the Inferior Conchoid, and is represented in the figure.

(1) If b is less than a, the upper and lower conchoids, as shewn in the figure, are somewhat similar in form.

(2) If $b = a$, the lower conchoid passes through A, and is somewhat like the lower conchoid drawn in the figure, but without the loop.

(3) If b is greater than a, the lower conchoid has an oval or loop, a double point of which is at A, and is that drawn in the figure.

The equation may be easily expressed in polar coordinates.

Let A be the pole, and $PAO = \theta$, $AP = r$;

$$\therefore \quad r = a\sec\theta \pm b, \tag{18}$$

the upper and lower signs referring respectively to the upper and lower conchoids.

197.] The Lemniscata of James Bernoulli. Fig. 37.

DEF.—If from the centre of an equilateral hyperbola perpendiculars are drawn to the tangents, the locus of the points of intersection is the Lemniscata.

Let x and y be the current coordinates of the lines PQ and OP; and let x', y' be the coordinates to Q, the point on the hyperbola at which the tangent is drawn; the equations to the hyperbola and the tangent are respectively

$$x'^2 - y'^2 = a^2, \qquad (19)$$

$$xx' - yy' = a^2; \qquad (20)$$

whence the equation to OP is $y = -\frac{y'}{x'}x$;

$$\therefore \quad \frac{x}{x'} = -\frac{y}{y'}; \qquad (21)$$

and multiplying each term of (20) by one or other of these equalities, we have

$$x^2 + y^2 = \frac{a^2 x}{x'} = -\frac{a^2 y}{y'};$$

$$\therefore \quad x' = \frac{a^2 x}{x^2 + y^2}, \qquad y' = -\frac{a^2 y}{x^2 + y^2};$$

and therefore, by means of (19),

$$(x^2 + y^2)^2 = a^2(x^2 - y^2). \qquad (22)$$

The curve, as is manifest from the generation of it, consists of two ovals, meeting in a double point at O; the tangents to which are coincident with the asymptotes of the hyperbola, and form angles of 45° on each side of OA.

The polar equation is

$$r^2 = a^2 \cos 2\theta. \qquad (23)$$

198.] The Logarithmic Curve. Fig. 32.

No better definition of the curve can be given than that expressed by its equation,

$$y = a^x; \qquad (24)$$

which means that the abscissa is the logarithm of the ordinate to the base a.

Hence, when $x = 0$, $y = a^0 = 1$; when $x = 1$, $y = a$; when $x = \infty$, $y = \infty$; when $x = -\infty$, $y = 0$.

Therefore OA $= 1$; and as the ordinate recedes further from Oy, it increases and ultimately becomes infinite; and as x decreases, that is, increases negatively, y decreases, and the axis of x is an asymptote to the curve.

199.] The Catenary. Fig. 33.

The catenary is the curve in which a perfectly flexible and uniform, though heavy, string hangs, when suspended in vacuo from two points.

Let OM $= x$, MP $= y$, OC $= c$; its equation is

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}; \tag{25}$$

where e is the Napierian logarithmic base; but as a knowledge of mechanics is requisite for a determination of the equation, the discussion of it will be found in Chapter V of the third volume of our work. It is manifest however that when $x = 0$, $y = c$; and as the equation is not altered when $-x$ is written for $+x$, that the curve is symmetrical with respect to the axis of y.

200.] The Tractory, or Equitangential Curve. Fig. 38.

DEF.—If AP is a curve, such that PT, the length of the tangent intercepted between the point of contact and the axis of x, is always equal to OA; then the locus of P is the equitangential curve.

Let OM $= x$, MP $= y$, OA $=$ PT $= a$; then the definition of the curve above given leads, as will be seen in the next Chapter, to an equation of the form,

$$\frac{dy}{dx} = -\frac{y}{\{a^2-y^2\}^{\frac{1}{2}}}; \tag{26}$$

and the equation to the curve is that of which (26) is the derived-function, and is therefore, as will be seen in Vol. II, Chap. VI,

$$x = a \log\left\{\frac{a+(a^2-y^2)^{\frac{1}{2}}}{y}\right\} - \{a^2-y^2\}^{\frac{1}{2}}. \tag{27}$$

This curve is sometimes considered as generated by attaching one end of a string of constant length a to a weight at A, and by moving the other end along Ox; the weight is supposed to trace out the curve: hence arises the name *Tractory* or *Tractrix*. But the mode of generation is incorrect, unless we also consider the friction produced by traction to be infinitely great, so that the weight's momentum, which is caused by its motion, may be instantly destroyed.

201.] The Cycloid. Figs. 39 and 40.

DEF.—A cycloid is the curve traced out by a point in the circumference of a circle, as the circle rolls along a fixed straight line.

(α) Let the given straight line, fig. 39, be taken as the axis of x, and the radius of the rolling circle be a, and the origin be at the point O, where the generating point P is in contact with the fixed line; and let RPQ be a position of the generating circle, such that OQ is equal to the arc PQ. Let E be the point in the line OAE, at which the generating point is again in contact with it, so that OAE = the circumference of the circle $= 2\pi a$. Bisect OE in A, and at A draw the ordinate AB = 2CP; then by the geometry of the figure, B is the highest point.

Let $\text{OM} = x$, $\text{MP} = y$, $\text{CP} = \text{CQ} = a$, $\text{PCQ} = \theta$.

Then, since OQ = the arc PQ, $\text{OQ} = a\theta$;

$$\begin{aligned} \therefore\quad x &= \text{OM}, & y &= \text{MP}, \\ &= \text{OQ} - \text{MQ}, & &= \text{CQ} - \text{CN}, \\ &= a\theta - a\sin\theta; & &= a - a\cos\theta; \end{aligned}$$

$$\therefore\quad \left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a \operatorname{versin}\theta \end{aligned}\right\}; \tag{28}$$

which two equations are those to the cycloid, when the starting-point is the origin; and if θ is eliminated, we have

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - \{2ay - y^2\}^{\frac{1}{2}}. \tag{29}$$

Since $\sin\theta$ and $\operatorname{versin}\theta$ have the same values whenever θ is increased by 2π, or by 4π, ... it appears from (28) that the values of y recur whenever x is increased by $2\pi a$, or by $4\pi a$ ... ; hence there is a series of curves similar and equal to OBE placed along the straight line OAE, parts of which at O and E are drawn in the figure; this is evident from the mode of generation of the curve.

The line OAE is called *the base*, and AB *the axis*, and B *the highest point* of the cycloid.

(β) It is also frequently convenient to refer the cycloid to the highest point as origin, and to *its axis* as the axis of x, in which case its equation may be found as follows.

Fig. 40. Let RPT′ be the circle in its generating position, P being the generating point, the arc PR being equal to the line

BR; from P let MP be drawn at right angles to OA, and let OQA be a semicircle described on the axis OA.

Let OM $= x$, MP $= y$, OC $=$ CQ $=$ CA $= a$, QCO $= \theta$.

Then since AB $=$ the semi-circumference $=$ RPT$'$, of which the parts RB and arc RP are equal, therefore AR $=$ the arc PT$'$ $=$ the arc OQ, on account of the similar positions and equality of the two semicircles; whence

$$\begin{aligned} y &= \text{MP}, & x &= \text{OM}, \\ &= \text{MN}+\text{NP}, & &= \text{OC}-\text{CM}, \\ &= \text{AR}+\text{MQ}, & &= a - a\cos\theta, \\ &= \text{arc OQ} + \text{CQ} \sin \text{QCM}, & &= a \operatorname{versin}\theta; \\ &= a\theta + a\sin\theta; \end{aligned}$$

whence we have,

$$\left.\begin{aligned} y &= a(\theta + \sin\theta) \\ x &= a(1-\cos\theta) \end{aligned}\right\}; \tag{30}$$

which two equations, taken simultaneously, are those to the cycloid; and by the elimination of θ we have

$$y = a \operatorname{versin}^{-1}\frac{x}{a} + \{2ax - x^2\}^{\frac{1}{2}}. \tag{31}$$

202.] The Companion to the Cycloid. Fig. 41.

It appears from the first of equations (30) of the last Article, that the ordinate to the cycloid is equal to the sum of the ordinate of the circle, viz. MQ of fig. 40, and of a part produced, viz. PQ, which is equal to the intercepted arc OQ; but if the ordinate to a circle is produced until the whole is equal to the intercepted arc of the circle, the locus of the extremity is called the Companion to the Cycloid.

Let OM $= x$, MP $= y$, OC $=$ CA $= a$, QCO $= \theta$; then, since MP is equal to the arc OQ,

$$\left.\begin{aligned} y &= a\theta \\ x &= a(1-\cos\theta) \end{aligned}\right\}; \tag{32}$$

which two equations are those to the companion to the cycloid; and, if θ is eliminated, we have

$$y = a \operatorname{versin}^{-1}\frac{x}{a}. \tag{33}$$

203.] Epitrochoidal and Hypotrochoidal Curves. Figs. 42, 46.

DEF.—An epitrochoid is the curve generated by a point within or without the circumference of a circle, which rolls on, that is, outside, another circle of given radius.

If the generating circle rolls inside the given circle, the generated curve is called the Hypotrochoid.

And if the generating point is on the circumference of the rolling circle, it is called an Epicycloid or Hypocycloid, according as it rolls without or within the fixed circle.

We shall consider the epitrochoid to be the normal case, and deduce the equations to the other curves from its equations by changing the signs and values of the constants.

Let O, the centre of the fixed circle, be the origin, and Q be the centre of the generating circle, and P the generating point; and suppose B, in the line QBP, to have been originally in contact with the fixed circle at A, and let OA be the axis of x; see fig. 42.

Let $\text{OM} = x$, $\text{MP} = y$, $\text{QOA} = \theta$, $\text{OR} = \text{OA} = a$, $\text{QR} = \text{QB} = b$, $\text{QP} = mb$. Then, since one circle rolls on the other, the arc $\text{AR} =$ the arc BR;

$$\therefore \quad \text{RQB} = \frac{a}{b}\theta, \qquad \text{QPL} = 180^\circ - \frac{a+b}{b}\theta;$$

$$\left.\begin{aligned}\therefore \quad x &= \text{OM}, \\ &= \text{ON} + \text{LP}, \\ &= (a+b)\cos\theta - mb\cos\left(\frac{a+b}{b}\right)\theta; & (34) \\ y &= \text{MP}, \\ &= \text{NQ} - \text{QL}, \\ &= (a+b)\sin\theta - mb\sin\left(\frac{a+b}{b}\right)\theta; & (35)\end{aligned}\right\}$$

and (34) and (35), taken simultaneously, are the equations to the epitrochoid.

If the generating circle rolls inside instead of outside the fixed circle the sign of b must be changed, and the curve is an hypotrochoid; the equations to which are, fig. 43,

$$\left.\begin{aligned}x &= (a-b)\cos\theta + mb\cos\frac{a-b}{b}\theta \\ y &= (a-b)\sin\theta - mb\sin\frac{a-b}{b}\theta\end{aligned}\right\}. \qquad (36)$$

204.] And if $m = 1$, the generating point is on the circumference of the rolling circle, and the curves become respectively the epicycloid and the hypocycloid; and the equations are

$$\left.\begin{aligned} x &= (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta \\ y &= (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta \end{aligned}\right\}; \qquad (37)$$

and
$$\left.\begin{aligned} x &= (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\right)\theta \\ y &= (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\right)\theta \end{aligned}\right\}; \qquad (38)$$

the curves expressed by which are those dotted respectively in figs. 42 and 43.

When a and b are commensurable numbers, the branches of the curve re-enter after a certain number of revolutions of the generating circle: in which cases the subsidiary angle θ may be eliminated, and the equation expressed in an algebraical form; but when a and b are incommensurable, the branches never re-enter, and the equation can only be expressed in a transcendental form equivalent to the above equations.

Some varieties of the above curves, in which the equations assume particular forms, are subjoined.

205.] Suppose that the generating circle of the epicycloid is equal to the fixed circle, then $a = b$; and equations (37) become

$$x = 2a\cos\theta - a\cos 2\theta, \qquad y = 2a\sin\theta - a\sin 2\theta;$$

whence, squaring and adding,

$$x^2+y^2 = 5a^2-4a^2\cos\theta,$$
$$x^2+y^2-a^2 = 4a^2(1-\cos\theta).$$

Again,
$$x = 2a\cos\theta - a\{2(\cos\theta)^2-1\};$$
$$\therefore\quad x-a = 2a\cos\theta(1-\cos\theta),$$
$$y = 2a\sin\theta(1-\cos\theta);$$
$$\therefore\quad (x-a)^2+y^2 = 4a^2(1-\cos\theta)^2;$$
$$\therefore\quad (x^2+y^2-a^2)^2 = 4a^2\{(x-a)^2+y^2\}; \qquad (39)$$

which is the equation to the curve expressed in rectangular coordinates; O, the centre of the fixed circle, being the origin. Fig. 44.

Let us change the origin to A, and transform the equation to polar coordinates, by putting $x = a + r\cos\phi$, $y = r\sin\phi$; whence

$$r = 2a(1-\cos\phi). \qquad (40)$$

The curve is called the Cardioid, from its heart-like shape.

206.] In the equations to the hypocycloid, let $b = \frac{a}{4}$; in which case equations (38) become

$$x = \frac{a}{4}\{3\cos\theta + \cos 3\theta\} = a(\cos\theta)^3,$$

$$y = \frac{a}{4}\{3\sin\theta - \sin 3\theta\} = a(\sin\theta)^3;$$

$$\therefore \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}; \qquad (41)$$

see fig. 45.

Again, in the equations to the hypotrochoid, let $b = \frac{a}{2}$; in which case equations (36) become

$$\left.\begin{aligned} x &= \frac{a}{2}(1+m)\cos\theta \\ y &= \frac{a}{2}(1-m)\sin\theta \end{aligned}\right\}; \qquad (42)$$

$$\frac{x^2}{\frac{a^2}{4}(1+m)^2} + \frac{y^2}{\frac{a^2}{4}(1-m)^2} = 1; \qquad (43)$$

which equation represents an ellipse, the axes of which are

$$a(1+m) \text{ and } a(1-m);$$

see fig. 46.

Again, in equations (38) of the hypocycloid, let $b = \frac{a}{2}$; whence we have

$$\left.\begin{aligned} x &= a\cos\theta \\ y &= 0 \end{aligned}\right\}; \qquad (44)$$

which equations express a straight line on the axis of x, of length $2a$, which is coincident with the diameter of the circle.

SECTION 4.—*On certain general properties of curves as expressed by algebraical equations of the* n*th degree.*

207.] In the following Chapter I shall enter on a formal inquiry into the properties of plane curves referred to rectangular coordinates, as exhibited by means of differentiation; but preliminary to that investigation it is necessary to explain certain general properties of a particular class of such curves, those, viz., which are expressed by algebraical equations; because these will receive further illustration hereafter; and it cannot be that such illustration will be adequately comprehended, unless the finite and more elementary properties which are illustrated are first known. The matter of the present section is collected from various sources; chiefly from the works of Cramer and Plücker, the Journals of Liouville and Crelle, the Nouvelles Annales de Mathématiques of Terquem and Gerono, and the Cambridge and Dublin Mathematical Journal. This last Journal contains many papers on the subject by Professor Boole, Mr. Cayley, and the Rev. George Salmon. I must however add that the following is only a sketch of the most elementary parts of a subject which contains matter of the deepest reflection; and which is daily receiving large additions from the most eminent mathematicians of the present age. But it will be sufficient for the object of this work.

An algebraical equation of the nth degree in terms of two variables x and y contains a constant term, and all possible combinations of the various powers of x and y, so that the sum of the indices in no term exceeds n: thus it is of the form,

$$\begin{aligned} &a_0 \\ &+ a_1 x + b_1 y \\ &+ a_2 x^2 + b_2 xy + c_2 y^2 \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ a_n x^n + b_n x^{n-1} y + c_n x^{n-2} y^2 + \ldots + j_n x y^{n-1} + k_n y^n = 0. \quad (45) \end{aligned}$$

This is the form of the equation which will generally hereafter be expressed as

$$\text{F}(x, y) = 0. \quad (46)$$

As the equation stands at present, the number of its terms is

$$1 + 2 + 3 + \ldots + (n+1);$$

the sum of which series is $\frac{(n+1)(n+2)}{2}$; but as every term has a coefficient the whole series may be divided by any one, and thus the number of *arbitrary* constants will be diminished by one; and therefore in the equation of the nth degree in terms of two variables the number of coefficients which admit of arbitrary determination is

$$\frac{(n+1)(n+2)}{2} - 1 = \frac{n(n+3)}{2}. \quad (47)$$

It is often convenient to express (45) in another form: (45) consists of a series of homogeneous expressions which are, beginning from the constant term, of the orders 0, 1, 2, ... n

respectively: let $u_0, u_1, u_2, \dots u_n$ represent the homogeneous expressions which are severally of the dimensions $0, 1, 2, \dots n$; then (45) becomes

$$u_0 + u_1 + u_2 + \dots + u_n = \text{F}(x, y) = 0, \tag{48}$$

the number of determinable constants in which is $\frac{n(n+3)}{2}$.

If in (48) $u_0 = 0$, that is, if there is no constant term, then the whole vanishes when $x = y = 0$; and thus the origin is on the curve. It is also to be observed that the degree of a curve cannot be altered by the substitution of any linear functions of new variables for the original variables; and as change of origin and change of coordinate axes require only such linear substitutions, it is evident that the degree of a curve is not affected by such substitutions.

208.] Hereafter also it will be found convenient to express (45) in another form. As it stands it is neither symmetrical nor homogeneous: the absence of these properties may appear at first sight to be of small importance; but as many conclusions, both numerical and geometrical, will be drawn from the *form* of expressions, it is desirable that they should be symmetrical at first, because symmetrical expressions, when operated on in differentiation, give rise to new symmetrical expressions; whereas unsymmetrical expressions do not. And if besides it is possible to make them homogeneous the results will be much simplified by means of Euler's Theorems of such functions given in Art. 82. Both these advantages may be obtained by the following method due to, I believe, M. Otto Hesse, Professor at Kœnigsberg. Let us assume $\frac{x}{z}$ and $\frac{y}{z}$, instead of x and y, to be the coordinates of a point in the plane; and thus in the equation to the plane curve let us replace x and y by $\frac{x}{z}$ and $\frac{y}{z}$; as we ordinarily express a point whose coordinates are x and y by (x, y), so I shall now express it by (x, y, z); after this substitution (45) becomes

$$\begin{aligned} & a_0 z^n \\ & + (a_1 x + b_1 y) z^{n-1} \\ & + (a_2 x^2 + b_2 xy + c_2 y^2) z^{n-2} \\ & + \dots\dots\dots \\ & + a_n x^n + b_n x^{n-1} y + \dots + j_n x y^{n-1} + k_n y^n = 0; \end{aligned}$$

and (48) becomes

$$u_n + u_{n-1}z + u_{n-2}z^2 + \ldots + u_1 z^{n-1} + u_0 z^n = \text{F}(x, y, z) = 0, \quad (49)$$

which are homogeneous expressions of the nth degree: and to which therefore the simplifications contained in Euler's Theorem, Art. 82, are applicable. Similarly a third variable will be introduced in other cases; thus if a and b are the coordinates of a point, I shall now take $\frac{a}{c}$ and $\frac{b}{c}$ to be the coordinates, and shall express the point by the notation (a, b, c); and analogously with other variables. We can pass from expressions involving three variables to the equivalent ones which involve only two, by replacing the third variable by unity. One or two examples are subjoined in which equations are given in terms of three variables.

Thus it is evident that the general form of the equation of a straight line is

$$\text{A}x + \text{B}y + \text{C}z = 0. \quad (50)$$

The general form of the equation of a conic is

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 + 2\text{E}yz + 2\text{G}zx + 2\text{H}xy = 0. \quad (51)$$

The equation of a straight line passing through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$x(y_1 z_2 - z_1 y_2) + y(z_1 x_2 - x_1 z_2) + z(x_1 y_2 - y_1 x_2) = 0; \quad (52)$$

the coefficients in which are the several determinants of the coordinates of the two given points taken two and two together.

The advantages arising from the symmetry of algebraical expressions in terms of three variables will be shewn hereafter. I must however here observe that as (49) is homogeneous and of n dimensions,

$$\begin{aligned} x\left(\frac{d\text{F}}{dx}\right) + y\left(\frac{d\text{F}}{dy}\right) + z\left(\frac{d\text{F}}{dz}\right) &= n\,\text{F}(x, y, z) \\ &= 0; \end{aligned}$$

and therefore if $\left(\frac{d\text{F}}{dx}\right) = 0$ and $\left(\frac{d\text{F}}{dy}\right) = 0$, then also $\left(\frac{d\text{F}}{dz}\right) = 0$. Thus in (51)

$$\begin{aligned} \left(\frac{d\text{F}}{dx}\right) &= 2(\text{A}x + \text{H}y + \text{G}z), \\ \left(\frac{d\text{F}}{dy}\right) &= 2(\text{H}x + \text{B}y + \text{E}z), \\ \left(\frac{d\text{F}}{dz}\right) &= 2(\text{G}x + \text{E}y + \text{C}z); \end{aligned}$$

and therefore if of these three equations the first two vanish, the third also vanishes; and thus if $z=1$, and $\mathrm{A}x+\mathrm{H}y+\mathrm{G}=0=\mathrm{H}x+\mathrm{B}y+\mathrm{E}$, then also $\mathrm{G}x+\mathrm{E}y+\mathrm{C}=0$.

209.] Another property of homogeneous functions must also be proved, as we shall find it to be important in our subsequent investigations as to the degree of certain curves. I will here however take the problem in its most general form.

Let $\mathrm{F}_1=0$, $\mathrm{F}_2=0$, ... $\mathrm{F}_n=0$, be a system of n homogeneous equations involving n variables x_1, x_2, ... x_n, in the most general form and with literal coefficients; and let us suppose these equations to be respectively of the degrees m_1, m_2, ... m_n; then we have n equations involving $n-1$ different variables; for by reason of the homogeneity of each, the highest power of x_n, whatever that is, may be divided through, and the number of variables will thereby become $n-1$; these equations therefore are not independent; a relation must exist amongst them; and this relation must be capable of expression in terms of the coefficients of the equations. Of this relation I propose to determine a property as to the power in which the coefficients of the several equations will be found. The relation is technically called the Resultant; and if it contains no extraneous factors, the complete resultant; and if the extraneous factors are omitted, the reduced resultant*. Let $m_1 m_2 m_3 \ldots m_n=\mathrm{M}$. Now as the function $\mathrm{F}_1=0$ is homogeneous and of m_1 dimensions, it is capable of resolution into m_1 linear factors, each of which is of the form $a_1 x_1+a_2 x_2+\ldots+a_n x_n$; and similarly is each of the other given expressions capable of resolution into m_2, m_3, ... m_n factors respectively of a similar form. As these equations are simultaneously true, to arrive at the most general result we must combine each of the component factors of $\mathrm{F}_1=0$ with each and every one of $\mathrm{F}_2=0$, $\mathrm{F}_3=0$, ... $\mathrm{F}_n=0$; let us take one of each set; then eliminating the unknown quantities, which are $n-1$ in number, between these n equations, we obtain a condition of the nth degree into which each coefficient enters in the first degree; and this process may be performed in M different ways, and the product of these M different quantities is the complete resultant, and is evidently homogeneous in terms of the original coefficients. The coefficients of $\mathrm{F}_1=0$, enter into each term in the degree $\frac{\text{M}}{m_1}$, those of $\text{F}_2 = 0$ in the degree $\frac{\text{M}}{m_2}$, and so with the others: to prove this, let us fix our thoughts by assuming $m_1 = 3$; so that

---

* See Crelle, Band XXXIV, p. 30.

$$\text{F}_1 = (a_1x_1 + a_2x_2 + \ldots + a_nx_n)\,(b_1x_1 + \ldots + b_nx_n)\,(c_1x_1 + \ldots + c_nx_n)$$
$$= \text{A}x_1^3 + \text{B}x_1^2x_2 + \ldots$$

In the product of the M different results of elimination a_1, b_1, c_1, say, will enter symmetrically; and therefore each one will enter in the power $\frac{\text{M}}{3}$ only, and therefore in the product, A which is equal to $a_1b_1c_1$, will enter in the power $\frac{\text{M}}{3}$; and the same is true of all the other coefficients, and of the coefficients of the other equations; so that the degree of the complete resultant is

$$\text{M}\left(\frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_n}\right).$$

Thus much at present as to curves expressed by equations in terms of three variables, and homogeneous. We shall return to the subject hereafter. Now I shall proceed with the investigations of other properties of curves as expressed by equations in terms of two variables.

210.] The curve which is expressed by an equation of the nth degree is said to be of the nth degree or of the nth order, the two words *order* and *degree* being used indifferently. Suppose now that we have two curves, one of the nth and the other of the mth degree; these will generally intersect in mn points; for let us take the equation to the curve as given in (45); then if y is eliminated from it and from a similar equation of the mth degree, the result will be an expression in terms of x of the mnth degree; the roots of which will be the abscissæ to the mn points in which the two curves intersect; the points in which they intersect, I say; because if all the roots are real, the curves will intersect in mn points in the plane of reference; and if the roots are imaginary, the points of intersection will be imaginary so far as they are exhibited in the plane of reference; but the equation of the mnth degree will just as truly have mn roots. Thus two curves of the second degree may intersect in four points; a curve of the second degree may intersect a curve of the third degree in six points.

As the equation of a straight line is of the first degree, so does a straight line meet a curve of the nth degree in n points which may be either real or imaginary. Thus a straight line may cut a curve of the second degree in two points, a curve of the third degree in three points, and so on: and even if the straight line is at an infinite distance, yet it intersects a curve of the nth degree just as truly in n points.

The following theorem due to Newton and called now by his name, is such a simple application of this property, that I must insert it.

Let us take the general equation (45), or its abridged form (48); let us replace x and y by lr and mr; where r is the radius vector of a point on the curve from the origin, and l and m are projective coefficients: then we have an equation of the nth degree in terms of r, the roots of which are the n distances, from the origin, of the n points where the curve is met by the straight line $mx - ly = 0$. Through the origin let two transversals be drawn, of which let the equations be $m_1 x - l_1 y = 0$ and $m_2 x - l_2 y = 0$; and let them meet the curve in the points $\text{P}_1, \text{P}_2, \dots \text{P}_n, \text{Q}_1, \text{Q}_2, \dots \text{Q}_n$; then by the theory of equations

$$\text{OP}_1.\text{OP}_2 \dots \text{OP}_n = \pm \frac{a_0}{l_1{}^n a_n + l_1{}^{n-1} m_1 b_n + \dots + k_n m_1{}^n},$$

$$\text{OQ}_1.\text{OQ}_2 \dots \text{OQ}_n = \pm \frac{a_0}{l_2{}^n a_n + l_2{}^{n-1} m_1 b_n + \dots + k_n m_2{}^n};$$

and therefore by division

$$\frac{\text{OP}_1.\text{OP}_2 \dots \text{OP}_n}{\text{OQ}_1.\text{OQ}_2 \dots \text{OQ}_n} = \frac{l_2{}^n a_n + l_2{}^{n-1} m_2 b_n + \dots + k_n m_2{}^n}{l_1{}^n a_n + l_1{}^{n-1} m_1 b_n + \dots + k_n m_1{}^n};$$

and as both the numerator and the denominator of this ratio are unaltered by a change of origin, provided that the directions of the lines OP and OQ are the same, so we infer that if from a point in the plane of a curve two transversals are drawn in given directions, the ratio of the product of the segments of the one taken from the given point to the product of the segments of the other is the same whatever is the place of the point.

We have also another important theorem first given by Cotes in his Harmonia Mensurarum. Let each radius vector OP drawn from O cut the curve in the points $\text{P}_1, \text{P}_2, \dots \text{P}_n$; and let P be taken such that

$$\frac{n}{\text{OP}} = \frac{1}{\text{OP}_1} + \frac{1}{\text{OP}_2} + \dots + \frac{1}{\text{OP}_n};$$

then the locus of P will be a straight line. For substituting lr and mr for x and y in (45), and dividing through by r^n we have,

$$\frac{a_0}{r^n}+\frac{a_1 l+b_1 m}{r^{n-1}}+\ldots=0;$$

$$\therefore\quad \frac{1}{\text{OP}_1}+\frac{1}{\text{OP}_2}+\ldots+\frac{1}{\text{OP}_n}=-\frac{a_1 l+b_1 m}{a_0};$$

and therefore by the given condition, if $\text{OP}=r$, and r refers to the new locus,

$$\frac{n}{r}=-\frac{a_1 l+b_1 m}{a_0};$$

$$\therefore\quad a_1 x+b_1 y+n a_0=0,$$

which is the equation to a straight line.

211.] As a curve of the nth degree has $\frac{n(n+3)}{2}$ determinable constants in its equation, so may these be determined if a sufficient number of independent conditions is given. Hereafter we shall meet with many different ways in which they may be determined. It is evident however that they may generally be found, if the coordinates of $\frac{n(n+3)}{2}$ points through which the curve is to pass are given, because in this case we may replace successively x and y in equation (45) by the given coordinates, and we shall have n linear equations, by means of which the arbitrary constants may be determined. This is clearly a definite and generally an unique problem: and thus only one curve of the nth degree can be drawn passing through the assigned points. Hence a curve of the second degree may pass through five points; a curve of the third degree through nine points; a curve of the fourth degree through fourteen points; and so on.

If however the points have certain relative positions, it may be that a *proper* curve of the nth degree cannot be drawn through them. That is, although we have shewn that the coefficients of the algebraical equation of the nth degree may be found in terms of the coordinates of the given points, yet the resulting equation may be susceptible of resolution into factors of lower degrees; so that the result may be the combination of two or more curves of lower degrees, the sum of which is equal to n; and thus not be the equation of a proper curve of the nth degree. Thus if $n = 2$, the number of points which is sufficient for the determination of the constants is 5; but if three of these are in a straight line, no conic can be drawn which shall pass through them, because a conic cannot cut a straight line in more points than two; and thus the only equation of the second degree which can satisfy such a system of points is that composed of two straight lines passing respectively through the three and the other two points.

Generally of the number of points which are sufficient for the determination of a proper curve of the nth degree not more than nm can be on a curve of the mth degree; because two curves of the nth and mth degrees respectively cannot intersect in more than nm points. And generally too the number of points which may be on a curve of the mth degree, so that a proper curve of the nth degree may pass through them, is less than mn. For let us, for instance, suppose the curve of the nth degree to be made up of two curves of the mth and $(n-m)$th degrees respectively. Under any circumstances the possibility of such a resolution requires one condition amongst the constants, so that the number of coefficients yet remaining to be determined is $\frac{n(n+3)}{2} - 1$. For the determination of the curve of the $(n-m)$th degree, $\frac{(n-m)(n-m+3)}{3}$ coefficients are required; so that only $nm - \frac{(m-1)(m-2)}{2}$ remain, and these may manifestly be on the curve of the mth degree.

Besides this case, wherein the resulting equation admits of resolution into other equations of lower degrees, it may be that the assigned number of points will not yield a definite result. For if the values of a certain number of unknown quantities are determined by means of an equal number of linear equations of the form (45), each will be expressed by a fraction; and the constants of the equation may be such as to make both the numerator and the denominator of any one or more to vanish, in which case the quantities are indeterminate; and thus the curve is not determined; and the number of curves passing through the given points may be infinite.

Thus, suppose the number of points through which a curve is to pass to be less by one than that required for the complete determination of the curve; that is, suppose a curve of the nth degree to pass through $\frac{n(n+3)}{2}-1$ different points; let $u=0$ and $v=0$, be two equations of the nth degree of the form (45), which represent two curves passing through these $\frac{n(n+3)}{2}-1$ points; then, if k is an undetermined constant, $u+kv=0$ is the equation to a curve of the nth degree passing through all the points of intersection of $u=0$ and of $v=0$, and therefore clearly passing through the $\frac{n(n+3)}{2}-1$ given points; and as k is undetermined, the number of curves of the nth degree passing through these is also indeterminate. Now let us suppose the curve $u+kv=0$ to pass through another point, so that it becomes completely determined; then if u' and v' are the values of u and v, when x and y are replaced by the coordinates of this last point, we have $u'+kv'=0$, whence we know k, and the curve is completely determined. There is however one case in which k will not have the required determinate value; and that is when the last point, through which the curve is to pass, and by means of which k is determined, is the point of intersection of $u=0$ and $v=0$, because in this case $u'=0$ and $v'=0$, and k takes an indeterminate form. This is also manifest from the equation $u+kv=0$; for this curve manifestly passes through *all* the points of intersection of $u=0$ and $v=0$; that is, through n^2 points, and yet is not determinate, because its equation contains an undetermined constant k. Hence we conclude that all curves of the nth degree, which pass through $\frac{n(n+3)}{2}-1$ given points, also pass through as many others as n^2 is in excess of this number; that is, pass through $\frac{(n-1)(n-2)}{2}$ other fixed points. And of this equation we have the following particular cases. All curves of the third degree which pass through the same eight points also pass through a common ninth point. And thus if eight points are given, through which a curve of the third degree is to pass, the curve is not determined if the ninth point is the other of the intersection of the two curves $u=0$, $v=0$, of the third degree. All curves of the fourth degree which pass through the same thirteen given points also pass through three other given points. Hence also it follows that $\frac{n(n+3)}{2}$ given points are not always sufficient to determine a curve of the nth degree, and that one other additional point at least may be required for the purpose.

It is to be observed that these results are true, whether the equations of the nth degree represent proper curves of the nth degree, or systems of factors of lower degrees. Thus if the constants of an equation of the second degree are to be determined by making the curve pass through five given points, of which four are in one straight line, the resulting equation will be composed of two simple factors of the first degree, but all the arbitrary constants in it cannot be determined, because the fifth point is not sufficient to determine the position of the second straight line.

212.] Thus although two curves of the nth degree intersect in n^2 points, yet any n^2 points taken arbitrarily may not be the points of intersection of two curves of the nth degree: but $\frac{n(n+3)}{2}-1$ of them being given, the remainder, viz. $\frac{(n-1)(n-2)}{2}$ will be fixed. Similarly two curves of the mth and nth degrees respectively intersect in mn points; but mn points taken arbitrarily on the curve of the mth degree will not be the points of intersection of it with the curve of the nth degree.

Generally, however, every curve of the nth degree which passes through $nm-\frac{(m-1)(m-2)}{2}$ points on a curve of the mth degree, will also pass through $\frac{(m-1)(m-2)}{2}$ other and fixed points on that curve; because this number of points is, see Art. 211, less by one than the number which is required for the absolute determination of the curve; and therefore these points are the points of intersection of all curves of the nth degree which satisfy the required conditions. An extension of several of the preceding properties of algebraic curves will be found in a memoir by Mr. A. Cayley, Fellow of Trinity College, Cambridge, in Vol. III of the Cambridge Mathematical Journal, p. 211.

CHAPTER X.

ON PROPERTIES OF PLANE CURVES, AS DEFINED BY EQUATIONS REFERRED TO RECTANGULAR COORDINATES.

SECTION 1.—*Single tangents and normals, and their properties.*

213.] In the following investigations it will be convenient to use sometimes the explicit, and at other times the implicit form of the equation to a curve in terms of two variables x and y. The general forms I shall take to be

$$y = f(x); \tag{1}$$

$$u = \mathrm{F}(x,y) = c; \tag{2}$$

and their differentials and derived functions will be expressed by the symbols which have been used in the former part of the volume; sometimes also the equations will be given in terms of three variables, as in Art. 208.

And as we are about to exhibit certain properties of geometrical space by means of Differential Calculus, I must say a few words on the mode by which a branch of the science of number is applied to the proof of geometrical truths. Differential Calculus does not of itself yield geometrical results: its results are numerical; but if a geometrical truth is so imagined as to be capable of expression in an analogous algebraical form; as, for instance, if we so imagine the fourth proposition of the second book of Euclid, as to express it in the analogous algebraical form $(a+b)^2 = a^2+2ab+b^2$; or if a geometrical definition is so framed that we can express the numerical analogue of it with algebraical symbols; then we have certain algebraical formulæ, which are numerical truths, from which certain other numerical truths, contained explicitly or implicitly in them, may be deductively inferred. For this purpose of deductive inference, we are about to employ the Differential Calculus. It is evident therefore that the first step in the inquiry is, so to imagine the geometrical truths, or so to frame the geometrical definitions, that they may be translated into corresponding algebraical expressions; this has been done to a certain extent in Section 3 of the preceding chapter. Thus they take an algebraical form, and become the subject matter of Infinitesimal Calculus: and we are hereby enabled to deduce from them many properties, which are again to be translated into geometrical propositions. In accordance with this mode of inquiry many of the following Articles will begin with a geometrical definition.

It is true also that we may sometimes begin with an algebraical equation; in that case however the equation is only the symbolical expression of a geometrical proposition, which has been, or is capable of being, geometrically imagined. As, for instance, we may investigate certain geometrical properties of the curve whose equation is (31), Art. 201; but that equation is only the symbolical expression of the geometrical definition of the cycloid.

214.] We propose to find in the first place the general equation of a tangent to a given curve at a given point, which we define as follows:

The tangent to a curve at a given point is that straight line which passes through the point, and another point on the curve which is infinitesimally near to the former point.

Let ξ and η be the current coordinates of the tangent line; and let (x, y) be the point of contact on the given curve; and let us suppose the straight line at first to pass through it, and through the point $(x+\Delta x, y+\Delta y)$, which is at a finite distance from it. Then we have the three equations

$$\left.\begin{aligned} \text{A}\xi+\text{B}\eta+\text{C} &= 0 \\ \text{A}x+\text{B}y+\text{C} &= 0 \\ \text{A}.\Delta x+\text{B}.\Delta y &= 0 \end{aligned}\right\}; \tag{3}$$

whence $$(\xi-x)\,\Delta y-(\eta-y)\,\Delta x = 0. \tag{4}$$

Suppose the two points to approach infinitesimally near to each other, in which case Δy and Δx become respectively dy and dx, and the line whose equation is (4) becomes a tangent; and we have
$$(\xi-x)\,dy-(\eta-y)\,dx = 0,$$

and $$\eta-y = \frac{dy}{dx}(\xi-x); \tag{5}$$

or, as it may be written,

$$\frac{\eta-y}{dy} = \frac{\xi-x}{dx}. \tag{6}$$

If therefore the equation to a curve is

$$y = f(x), \qquad \text{so that} \quad \frac{dy}{dx} = f'(x),$$

then the equation (5) to the tangent is

$$\eta - y = f'(x)\,(\xi - x). \tag{7}$$

Thus, for instance, if the equation to the parabola is

$$y = 2m^{\frac{1}{2}}x^{\frac{1}{2}}, \qquad \text{then} \quad \frac{dy}{dx} = \frac{m^{\frac{1}{2}}}{x^{\frac{1}{2}}};$$

and the equation to the tangent is

$$\eta - y = \left(\frac{m}{x}\right)^{\frac{1}{2}}(\xi - x). \tag{8}$$

215.] If the equation to the curve is given in the implicit form,

$$u = \mathrm{F}(x, y) = c; \tag{9}$$

then

$$\mathrm{D}u = \left(\frac{d\mathrm{F}}{dx}\right)dx + \left(\frac{d\mathrm{F}}{dy}\right)dy = 0; \tag{10}$$

and if this substitution is made in (5), the equation to the tangent becomes,

$$(\xi - x)\left(\frac{d\mathrm{F}}{dx}\right) + (\eta - y)\left(\frac{d\mathrm{F}}{dy}\right) = 0. \tag{11}$$

If the equation to the curve is a homogeneous function of n dimensions; then by the property of such functions, proved in Art. 82, equation (112), we have

$$x\left(\frac{d\mathrm{F}}{dx}\right) + y\left(\frac{d\mathrm{F}}{dy}\right) = n\mathrm{F}; \tag{12}$$

and the equation (11) to the tangent becomes,

$$\xi\left(\frac{d\mathrm{F}}{dx}\right) + \eta\left(\frac{d\mathrm{F}}{dy}\right) = n\mathrm{F}. \tag{13}$$

Generally, either (11) or (13) is the most convenient form for the equation of the tangent.

Thus if the equation to the ellipse is given in the form

$$\mathrm{F}(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\left(\frac{d\mathrm{F}}{dx}\right) = \frac{2x}{a^2}, \qquad \left(\frac{d\mathrm{F}}{dy}\right) = \frac{2y}{b^2};$$

and as the equation is homogeneous and of two dimensions, $n = 2$; and (13) becomes, after division by 2,

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} = 1.$$

216.] Again, let the equation to the curve be given in terms of three variables, as (49), Art. 208, and be homogeneous, and of the form $\mathrm{F}(x,y,z) = 0$; let the coordinates to any point on a line be ξ, η, ζ, and let this line pass through two points at an infinitesimal distance apart, viz. (x, y, z), $(x+dx, y+dy, z+dz)$; then we shall have the following system of equations;

$$\left.\begin{aligned} \mathrm{A}\xi+\mathrm{B}\eta+\mathrm{C}\zeta &= 0 \\ \mathrm{A}x+\mathrm{B}y+\mathrm{C}z &= 0 \\ \mathrm{A}.dx+\mathrm{B}.dy+\mathrm{C}.dz &= 0 \end{aligned}\right\}; \tag{14}$$

whence we have

$$\xi(y\,dz-z\,dy)+\eta(z\,dx-x\,dz)+\zeta(x\,dy-y\,dx) = 0. \tag{15}$$

But by reason of the equation to the curve, and by reason of the homogeneity of that equation, we have

$$\left.\begin{aligned} dx\left(\frac{d\mathrm{F}}{dx}\right)+dy\left(\frac{d\mathrm{F}}{dy}\right)+dz\left(\frac{d\mathrm{F}}{dz}\right) &= 0 \\ x\left(\frac{d\mathrm{F}}{dx}\right)+\ y\left(\frac{d\mathrm{F}}{dy}\right)+\ z\left(\frac{d\mathrm{F}}{dz}\right) &= 0 \end{aligned}\right\}; \tag{16}$$

whence
$$\frac{\left(\frac{d\mathrm{F}}{dx}\right)}{z\,dy-y\,dz} = \frac{\left(\frac{d\mathrm{F}}{dy}\right)}{x\,dz-z\,dx} = \frac{\left(\frac{d\mathrm{F}}{dz}\right)}{y\,dx-x\,dy}; \tag{17}$$

whence, and from (15), we have

$$\xi\left(\frac{d\mathrm{F}}{dx}\right)+\eta\left(\frac{d\mathrm{F}}{dy}\right)+\zeta\left(\frac{d\mathrm{F}}{dz}\right) = 0; \tag{18}$$

which is the equation to the tangent.

Thus, if the equation to a conic in the homogeneous form in terms of three variables is

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{E}yz+2\mathrm{G}zx+2\mathrm{H}xy = 0,$$

the equation to its tangent at the point (x, y, z) is

$$\xi(\mathrm{A}x+\mathrm{G}z+\mathrm{H}y)+\eta(\mathrm{B}y+\mathrm{E}z+\mathrm{H}x)+\zeta(\mathrm{C}z+\mathrm{G}x+\mathrm{E}y) = 0,$$

and which takes the ordinary form, if $\zeta = z = 1$.

217.] To find the equation to the normal to a plane curve at a given point. The normal is defined as follows:

The normal to a plane curve at a given point is the straight line perpendicular to the tangent, and which passes through the point of contact.

Let ξ, η be the current coordinates to the normal, and let

(x, y) be the point of contact of the tangent; then the equation to a line passing through (x, y) and perpendicular to that whose equation is (5), is

$$\eta - y = -\frac{dx}{dy}(\xi - x); \tag{19}$$

which may also be written in the form

$$(\eta - y)\,dy + (\xi - x)\,dx = 0. \tag{20}$$

And if the equation to the curve is an implicit function, it becomes

$$\frac{\eta - y}{\left(\frac{d\mathrm{F}}{dy}\right)} = \frac{\xi - x}{\left(\frac{d\mathrm{F}}{dx}\right)}. \tag{21}$$

Thus, if the equation to an ellipse is

$$\mathrm{F}(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation to the normal is

$$\frac{b^2}{y}(\eta - y) = \frac{a^2}{x}(\xi - x).$$

Hence also it follows, that the equation to a line passing through the origin, and perpendicular to the tangent, is

$$\frac{\eta}{\left(\frac{d\mathrm{F}}{dy}\right)} = \frac{\xi}{\left(\frac{d\mathrm{F}}{dx}\right)}; \tag{22}$$

by means of which, in combination with equation (11), and that to the curve, the locus of the point of intersection of the tangent with the perpendicular on it from the origin may be determined.

Thus if the equation to the rectangular hyperbola is $xy = k^2$, the equations to the tangent and to the perpendicular on the tangent from the origin are respectively

$$\frac{\xi}{x} + \frac{\eta}{y} = 2, \qquad \text{and} \qquad \frac{\eta}{x} = \frac{\xi}{y};$$

between which and the equation to the curve, if we eliminate x and y, we have

$$\xi^2 + \eta^2 = 2k(\xi\eta)^{\frac{1}{2}}.$$

The equation to the analogous curve with reference to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is $$(\xi^2 + \eta^2)^2 = a^2\xi^2 + b^2\eta^2.$$

218.] Let Δs be the distance between the two points on the curve through which the cutting line of Art. 214 passes, that is, let it be the length of the chord joining them; then

$$\Delta s^2 = \Delta x^2 + \Delta y^2;$$

and let the two points approach infinitesimally near to one another; in which case, according to the notation of Art. 17, $\Delta x, \Delta y, \Delta s$ become respectively dx, dy, ds, and we have

$$ds^2 = dx^2 + dy^2; \tag{23}$$

and ds becomes the distance between these two points, which are infinitesimally near to each other; that is, it becomes an element of the curve, or an infinitesimal arc; or, as we shall call it, a length-element of the curve: it is in fact the small portion of the tangent line which is common to the tangent and the curve, the tangent indeed being the length-element produced. Or, under another mode of considering the curve, that is, of conceiving it to be generated by a point moving according to a given law, ds is the distance between two successive positions of the point; and if these two positions are taken so near to each other, that only an infinitesimal instant of time has elapsed during the passage from one to the other, it is impossible to conceive but that the moving point has passed in a straight line from one to the other; the length of which straight line is ds.

If then we use the character τ to symbolize the angle made by the tangent with the axis of x, that is, the angle PTM in fig. 47, we have, from equation (5),

$$\tan\tau = \frac{dy}{dx}; \tag{24}$$

whence

$$\frac{dy}{\sin\tau} = \frac{dx}{\cos\tau} = \pm\, ds, \tag{25}$$

the last equality following from Preliminary Theorem I; the numerator and denominator of the preceding equalities having been squared and added, and subsequently the square root having been extracted.

Also again, by equation (10),

$$\frac{dy}{\pm\left(\frac{dF}{dx}\right)} = \frac{dx}{\mp\left(\frac{dF}{dy}\right)} = \frac{ds}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}}; \tag{26}$$

therefore, from (25) and (26),

In (25) the + sign alone should be used.

$$\cos\tau = \pm\frac{dx}{ds} = \frac{\pm\left(\frac{d\mathrm{F}}{dy}\right)}{\left\{\left(\frac{d\mathrm{F}}{dx}\right)^2+\left(\frac{d\mathrm{F}}{dy}\right)^2\right\}^{\frac{1}{2}}}; \tag{27}$$

$$\sin\tau = \pm\frac{dy}{ds} = \frac{\mp\left(\frac{d\mathrm{F}}{dx}\right)}{\left\{\left(\frac{d\mathrm{F}}{dx}\right)^2+\left(\frac{d\mathrm{F}}{dy}\right)^2\right\}^{\frac{1}{2}}}. \tag{28}$$

Hence also, if ψ is the angle between the normal and the axis of x, viz. the angle PGM, in fig. 47,

$$\tan\psi = \frac{dx}{dy}; \tag{29}$$

$$\therefore\quad \sin\psi = \pm\frac{dx}{ds} = \frac{\pm\left(\frac{d\mathrm{F}}{dy}\right)}{\left\{\left(\frac{d\mathrm{F}}{dx}\right)^2+\left(\frac{d\mathrm{F}}{dy}\right)^2\right\}^{\frac{1}{2}}}; \tag{30}$$

$$\cos\psi = \pm\frac{dy}{ds} = \frac{\mp\left(\frac{d\mathrm{F}}{dx}\right)}{\left\{\left(\frac{d\mathrm{F}}{dx}\right)^2+\left(\frac{d\mathrm{F}}{dy}\right)^2\right\}^{\frac{1}{2}}}. \tag{31}$$

The preceding equations frequently render it convenient to deduce from the equation to a curve the relations between ds, dx, and dy: of which some examples, giving rise to differential expressions, are subjoined.

Ex. 1. $y^2 = 4mx$; $\quad\therefore\quad 2y\,dy = 4m\,dx$;

$$\therefore\quad \frac{dy}{2m} = \frac{dx}{y} = \frac{ds}{\{4m^2+y^2\}^{\frac{1}{2}}};$$

$$\therefore\quad \frac{dy}{ds} = \frac{2m}{\{4m^2+y^2\}^{\frac{1}{2}}} = \frac{m^{\frac{1}{2}}}{\{x+m\}^{\frac{1}{2}}},$$

$$\frac{dx}{ds} = \frac{y}{\{4m^2+y^2\}^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{\{x+m\}^{\frac{1}{2}}}.$$

Ex. 2. To find the relations between dx, dy and ds, in the equation to the cycloid.

(a) Let the starting point be the origin; therefore, by equation (29), Art. 201,

$$x = a\,\mathrm{versin}^{-1}\frac{y}{a} - \{2ay-y^2\}^{\frac{1}{2}};$$

$$\therefore \quad dx = \frac{y\,dy}{\{2ay-y^2\}^{\frac{1}{2}}};$$

and
$$\frac{dx}{y} = \frac{dy}{\{2ay-y^2\}^{\frac{1}{2}}} = \frac{ds}{\{2ay\}^{\frac{1}{2}}}. \tag{32}$$

(β) Let the highest point be the origin; therefore, by equation (31), Art. 201,

$$y = a \operatorname{versin}^{-1}\frac{x}{a} + \{2ax-x^2\}^{\frac{1}{2}},$$

$$dy = \frac{(2a-x)\,dx}{\{2ax-x^2\}^{\frac{1}{2}}} = \left\{\frac{2a-x}{x}\right\}^{\frac{1}{2}} dx;$$

$$\frac{dy}{\{2a-x\}^{\frac{1}{2}}} = \frac{dx}{x^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}}. \tag{33}$$

Ex. 3. To find the relation between dx, dy and ds, in the equation to the catenary.

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\};$$

$$dy = \frac{1}{2}\left\{e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right\} dx, \tag{34}$$

$$= \frac{\{y^2-c^2\}^{\frac{1}{2}}}{c}\,dx; \tag{35}$$

$$\therefore \quad \frac{dy}{\{y^2-c^2\}^{\frac{1}{2}}} = \frac{dx}{c} = \frac{ds}{y}. \tag{36}$$

219.] Let us now consider the general geometrical results which are contained in the preceding equations to the tangent and the normal.

Let us assume the curve drawn in fig. 47 to be the *typical form* of all curves; of which PT is the tangent line at the point P, PG the normal line; MT the subtangent; MG the subnormal; OY the perpendicular from the origin on the tangent line; OT, OT′ respectively the intercepts of the axes of x and y by the tangent line; and the lines, PT and PG, the parts of the tangent and normal lines intercepted between the point of contact and the axis of x, are called respectively the *tangent* and the *normal*.

Let $\quad$ OM $= x$, $\quad$ MP $= y$, $\quad$ OY $= p$,

OT $= \xi_0$, $\quad$ OT′ $= \eta_0$.

Then, by the equations to the tangent, (5) and (11),

$$\text{if } \eta = 0, \quad \xi_0 = x - y\frac{dx}{dy} = \frac{x\left(\frac{d\mathrm{F}}{dx}\right) + y\left(\frac{d\mathrm{F}}{dy}\right)}{\left(\frac{d\mathrm{F}}{dx}\right)}; \tag{37}$$

$$\xi = 0, \quad \eta_0 = y - x\frac{dy}{dx} = \frac{x\left(\frac{d\mathrm{F}}{dx}\right) + y\left(\frac{d\mathrm{F}}{dy}\right)}{\left(\frac{d\mathrm{F}}{dy}\right)}; \tag{38}$$

the numerators of these last values generally admit of simplification by Euler's Theorem; but of this hereafter.

Without however deducing the values of the other geometrical lines of the figure from the preceding equations, we will express them in the following manner; which is preferable, because it addresses itself more directly to geometrical construction and to the eye.

In fig. 47, $\mathrm{PTM} = \tau$, $\mathrm{PGM} = \psi$;

$$\therefore \quad \tan \mathrm{PTM} = \tan \mathrm{MPG} = \frac{dy}{dx}, \tag{39}$$

$$\tan \mathrm{PGM} = \tan \mathrm{TPM} = \frac{dx}{dy}. \tag{40}$$

$$\left.\begin{aligned} \therefore \quad \text{Subtangent} &= \mathrm{MT} = \mathrm{MP} \tan \mathrm{MPT} = y\frac{dx}{dy}, \\ \text{Subnormal} &= \mathrm{MG} = \mathrm{MP} \tan \mathrm{MPG} = y\frac{dy}{dx}; \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} \text{Tangent} &= \mathrm{PT} = y \sec \tan^{-1}\frac{dx}{dy} = y\left\{1 + \frac{dx^2}{dy^2}\right\}^{\frac{1}{2}} = y\frac{ds}{dy}, \\ \text{Normal} &= \mathrm{PG} = y \sec \tan^{-1}\frac{dy}{dx} = y\left\{1 + \frac{dy^2}{dx^2}\right\}^{\frac{1}{2}} = y\frac{ds}{dx}; \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned} \xi_0 &= \mathrm{OT} = \mathrm{OM} - \mathrm{MT} = x - y\frac{dx}{dy}, \\ \eta_0 &= \mathrm{OT}' = \mathrm{MP} - \mathrm{PR} = y - x\frac{dy}{dx}; \end{aligned}\right\} \tag{43}$$

$$\begin{aligned} p &= \mathrm{OY} = \mathrm{MP} \sin \mathrm{MPT} - \mathrm{OM} \sin \mathrm{PTM}, \\ &= y\frac{dx}{ds} - x\frac{dy}{ds} = \frac{y\,dx - x\,dy}{\{dx^2 + dy^2\}^{\frac{1}{2}}}. \end{aligned} \tag{44}$$

Also by means of equations (27) and (28),

$$p = \pm \frac{x\left(\frac{d\text{F}}{dx}\right) + y\left(\frac{d\text{F}}{dy}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2 + \left(\frac{d\text{F}}{dy}\right)^2\right\}^{\frac{1}{2}}}; \tag{45}$$

and therefore, if $\text{F}(x, y)$ is a homogeneous function of n dimensions,

$$p = \pm \frac{n\text{F}}{\left\{\left(\frac{d\text{F}}{dx}\right)^2 + \left(\frac{d\text{F}}{dy}\right)^2\right\}^{\frac{1}{2}}}.$$

Hereby also we may put in other forms the equation to the tangent given in equation (5),

$$\begin{aligned} \eta\, dx - \xi\, dy &= y\, dx - x\, dy, \\ \eta \frac{dx}{ds} - \xi \frac{dy}{ds} &= y \frac{dx}{ds} - x \frac{dy}{ds}, \\ &= p, \\ &= \pm \frac{x\left(\frac{d\text{F}}{dx}\right) + y\left(\frac{d\text{F}}{dy}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2 + \left(\frac{d\text{F}}{dy}\right)^2\right\}^{\frac{1}{2}}}. \end{aligned} \tag{46}$$

220.] From the equations to the tangent and normal it appears, that whenever x or y is affected with $\pm \sqrt{-}$, such signs will remain in the equations, and therefore η and ξ will be similarly affected; and therefore whenever the curve is out of the plane of reference, the tangent and normal also are.

Also, as $\frac{dy}{dx}$ is the trigonometrical tangent of the angle made with the axis of x by the tangent to the curve, at all points at which $\frac{dy}{dx}$ has a finite value the curve is inclined to the axis of x at a finite angle: and if $\frac{dy}{dx}$ is positive, then x and y are simultaneously increasing or decreasing, see Art. 110, and the curve is such as the logarithmic curve of fig. 32, or the cycloid of fig. 40; and if $\frac{dy}{dx}$ is negative, as x increases y decreases, and *vice versâ*, and the curve is such as the equitangential curve in fig. 38.

If $\frac{dy}{dx} = 0$, the tangent, and therefore the curve at the point of contact, is parallel to the axis of x; and if $\frac{dy}{dx}$ changes its sign at such a point, there is a maximum or minimum ordinate, such as is drawn in figs. 12 and 13; but if $\frac{dy}{dx}$ does not change sign, then the form of the curve will be such as in figs. 14 and 15, according as $\frac{dy}{dx}$ is positive or negative.

If $\frac{dy}{dx} = \infty$, the tangent, and therefore the curve at the point of contact, is perpendicular to the axis of x, and may be such as oy in fig. 39, or as is drawn in one or other of the diagrams of fig. 48; that is, if $\frac{dy}{dx}$, in passing through ∞, changes sign from $+$ to $-$, the point of the curve may be such as is represented in (α): if it changes sign from $-$ to $+$, it may be that represented in (β); but if $\frac{dy}{dx}$ does not change sign and is positive throughout, the curve is that indicated in (γ), and if it is negative throughout, it is that indicated in (δ).

If at any point of a curve $\frac{dy}{dx} = \frac{0}{0}$; that is, if $\left(\frac{d\text{F}}{dx}\right) = 0$ and $\left(\frac{d\text{F}}{dy}\right) = 0$, the direction of the tangent at the point will be indeterminate as far as the form of $\frac{dy}{dx}$ defines it; but it may be evaluated, and the means of doing so will be discussed in Section 4 of the present Chapter.

221.] Illustrative examples on the preceding Articles.

Ex. 1. Properties of the ellipse.

$$\text{F}(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\because \quad \left(\frac{d\text{F}}{dx}\right) = \frac{2x}{a^2}, \qquad \left(\frac{d\text{F}}{dy}\right) = \frac{2y}{b^2};$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}.$$

Hence the equation to the tangent is

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} = 1;$$

the equation to the normal is

$$\frac{a^2}{x}(\xi - x) = \frac{b^2}{y}(\eta - y), \qquad \text{or,} \quad \frac{a^2 \xi}{x} - \frac{b^2 \eta}{y} = a^2 - b^2.$$

The intercept of the axis of x by the tangent $= \xi_0 = \dfrac{a^2}{x}$,

- - - - - - - - - - - - y - - - - - - $= \eta_0 = \dfrac{b^2}{y}$;

The subtangent $= y\dfrac{dx}{dy} = -\dfrac{a^2y^2}{b^2x}$,

The subnormal $= y\dfrac{dy}{dx} = -\dfrac{b^2}{a^2}x$;

The perpendicular from origin on tangent $= \dfrac{a^2b^2}{\{b^4x^2+a^4y^2\}^{\frac{1}{2}}}$.

Ex. 2. Properties of the Cissoid of Diocles.

The equation is $\quad y^2 = \dfrac{x^3}{2a-x}$;

$$\therefore \quad \frac{dy}{dx} = \pm \frac{x^{\frac{1}{2}}(3a-x)}{(2a-x)^{\frac{3}{2}}};$$

therefore at the origin, $\dfrac{dy}{dx} = 0$, and the curve touches the axis of x; when $x = 3a$, $\dfrac{dy}{dx} = 0$, and changes sign, and therefore the ordinates are severally a maximum and a minimum; when $x = 2a$, $\dfrac{dy}{dx} = \infty$, and therefore the curve is perpendicular to the axis of x; when $x = a$, $y = a$, $\dfrac{dy}{dx} = 2$, and therefore the curve cuts its fundamental circle at $\tan^{-1}2$. These several values of the tangent are exhibited in fig. 34.

Ex. 3. Let the curve be the hypocycloid whose equation is, Art. 206,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

The equation to the tangent is

$$\frac{\xi}{x^{\frac{1}{3}}} + \frac{\eta}{y^{\frac{1}{3}}} = a^{\frac{2}{3}};$$

and therefore $\quad \xi_0 = a^{\frac{2}{3}}x^{\frac{1}{3}}, \quad$ and $\quad \eta_0 = a^{\frac{2}{3}}y^{\frac{1}{3}}$;

$$\therefore \quad \eta_0^2 + \xi_0^2 = a^{\frac{4}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = a^2;$$

and therefore the length of the tangent line intercepted between the coordinate axes is constant.

Ex. 4. To find the differential equation to the equitangential curve; see Art. 200.

According to the definition, the line PT of fig. 38 is to be of constant length; let the length be a;

$$\therefore \quad a = y\left\{1+\frac{dx^2}{dy^2}\right\}^{\frac{1}{2}}, \qquad \therefore \quad \frac{dy}{dx} = -\frac{y}{\{a^2-y^2\}^{\frac{1}{2}}};$$

the negative sign being taken because, according to the form drawn in fig. 38, y decreases as x increases.

Ex. 5. Properties of the logarithmic curve.

$$y = a^x;$$

$$\frac{dy}{dx} = a^x \log a = y \log a;$$

$$\therefore \quad \text{the subtangent} = y\frac{dx}{dy} = \frac{1}{\log a} = \text{a constant.}$$

Also when $x = 0$, $y = 1$; in which case $\frac{dy}{dx} = \log a$, which is the tangent of the angle at which the curve is inclined to the axis of x, at the point where it cuts the axis of y.

Ex. 6. Properties of the cycloid.

(a) Firstly, let the starting point be the origin; the equation to the curve is,

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - \{2ay-y^2\}^{\frac{1}{2}}.$$

$$\therefore \quad \frac{dx}{dy} = \frac{y}{\{2ay-y^2\}^{\frac{1}{2}}},$$

$$\frac{dy}{dx} = \frac{\{2ay-y^2\}^{\frac{1}{2}}}{y} = \frac{2a-y}{\{2ay-y^2\}^{\frac{1}{2}}}.$$

Therefore in fig. 49,

$$\{2ay-y^2\}^{\frac{1}{2}} = \text{LP}, \quad \text{and} \quad 2a-y = \text{LQ}, \quad \text{and} \quad \frac{dy}{dx} = \tan \text{PTM},$$

$$\therefore \quad \tan \text{PTM} = \frac{\text{QL}}{\text{PL}} = \tan \text{QPL};$$

therefore the tangent at P also passes through the point Q.

Hence also the normal at the point P passes through G, the other extremity of the diameter of the generating circle; as is also manifest from the above value of $\frac{dx}{dy}$.

The length of the normal $= \text{PG} = y\frac{ds}{dx} = \{2ay\}^{\frac{1}{2}}$; see (42);

$$\therefore \quad \text{PG}^2 = \text{QG} \times \text{GL}.$$

(β) Secondly, let the highest point be the origin; see fig. 50; then

$$y = \{2ax - x^2\}^{\frac{1}{2}} + a \text{ versin}^{-1} \frac{x}{a},$$

$$\frac{dy}{dx} = \frac{2a - x}{\{2ax - x^2\}^{\frac{1}{2}}} = \frac{\{2ax - x^2\}^{\frac{1}{2}}}{x};$$

$$\therefore \quad \tan \text{PTM} = \frac{\text{MQ}}{\text{OM}} = \tan \text{QOM};$$

therefore PT is parallel to the chord OQ. Hence also OT′ = PQ = the arc OQ; and hence too the normal PG is parallel to the chord QA.

222.] We must return however to the general equations of the tangent and the normal given in Articles 214–217, for they require closer consideration.

The general equation (11) to the tangent is

$$\xi \left(\frac{dF}{dx}\right) + \eta \left(\frac{dF}{dy}\right) = x \left(\frac{dF}{dx}\right) + y \left(\frac{dF}{dy}\right). \qquad (47)$$

Let us suppose the equation to the curve to be expressed in the form (48) of Art. 207; that is, let

$$F(x, y) = u_0 + u_1 + u_2 + \ldots + u_n = 0, \qquad (48)$$

where $u_0, u_1, \ldots u_n$, are homogeneous functions of $0, 1, 2, \ldots n$ dimensions respectively in terms of x and y; then

$$\left.\begin{aligned} \left(\frac{dF}{dx}\right) &= \left(\frac{du_1}{dx}\right) + \left(\frac{du_2}{dx}\right) + \ldots + \left(\frac{du_n}{dx}\right) \\ \left(\frac{dF}{dy}\right) &= \left(\frac{du_1}{dy}\right) + \left(\frac{du_2}{dy}\right) + \ldots + \left(\frac{du_n}{dy}\right) \end{aligned}\right\}; \qquad (49)$$

so that the right-hand member of (47) becomes

$$x \left(\frac{du_1}{dx}\right) + y \left(\frac{du_1}{dy}\right) + x \left(\frac{du_2}{dx}\right) + y \left(\frac{du_2}{dy}\right) + \ldots + x \left(\frac{du_n}{dx}\right) + y \left(\frac{du_n}{dy}\right); \quad (50)$$

but by Euler's Theorems, Art. 82, this expression is equal to

$$u_1 + 2u_2 + \ldots + nu_n;$$

and therefore, by reason of (48), to

$$-\{u_{n-1} + 2u_{n-2} + \ldots + (n-1)u_1 + nu_0\}; \qquad (51)$$

so that (47) becomes

$$\xi \left(\frac{dF}{dx}\right) + \eta \left(\frac{dF}{dy}\right) + u_{n-1} + 2u_{n-2} + \ldots + (n-1)u_1 + nu_0 = 0; \quad (52)$$

which is the equation to the tangent in the reduced form; and is of $n-1$ dimensions in terms of x and y.

If the equation to the curve is expressed in terms of three variables x, y, z, this result is evident immediately: the equation to the tangent in this case is given by (18), and $\left(\frac{dF}{dx}\right)$, $\left(\frac{dF}{dy}\right)$, $\left(\frac{dF}{dz}\right)$ are plainly of $n-1$ dimensions in terms of x, y and z.

If $z = \zeta = 1$, (18) and (52) are identical; in which case

$$\zeta\left(\frac{dF}{dz}\right) = u_{n-1} + 2u_{n-2} + \ldots + (n-1)u_1 + nu_0.$$

Let us return to (52). If the origin is on the curve, $u_0 = 0$; and if the tangent is in this case drawn at the origin, then omitting the terms which vanish, and replacing $\left(\frac{dF}{dx}\right)$ and $\left(\frac{dF}{dy}\right)$ by their values from (49), we have

$$\xi\left(\frac{du_1}{dx}\right) + \eta\left(\frac{du_1}{dy}\right) = 0; \qquad (53)$$

and if we replace ξ and η by x and y, remembering that x and y are in this case the current coordinates of the tangent, (53) becomes, by reason of Euler's Theorem,

$$u_1 = 0; \qquad (54)$$

so that if $u_1 + u_2 + \ldots + u_n = 0$ is the equation to a curve, $u_1 = 0$ is the equation to the tangent at the origin.

Thus if the equation to a conic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = 0,$$

the equation to the tangent at the origin is

$$Dx + Ey = 0.$$

Since a straight line may cut a curve of the nth degree in n points, it follows that if two of these points become coincident, that is, if a line touches a curve at a given point, it can cut it in only $n-2$ other points. Thus if a straight line touches a conic, it cannot meet the curve again. If a straight line touches a curve of the third order, it must cut it at some other point. Generally, if a line touches a curve of an even order, it will either meet it again in an even number of points or not at all. And if a line touches a curve of an odd order it will meet it again in an odd number of points, and in one point at least.

Since a tangent passes through two points of a curve, it is manifest that if a curve is to touch a given straight line at a given point, this circumstance is generally equivalent to two conditions in the determination of the constants of the curve. See Art. 211. Thus as five conditions will determine a central conic, it is generally impossible to construct one which shall touch the three sides of a triangle at given points. As four conditions determine a parabola, so a parabola is determinate if it touches two given straight lines at given points.

223.] Now (ξ, η) in (52) is any point on that tangent which touches the curve at (x, y); let us suppose (ξ, η) to be a fixed point, and tangents to be drawn from it to the curve; then as (52) is of $n-1$ dimensions in terms of x and y, all the points of contact lie in a curve of the $(n-1)$th degree; and therefore there may be as many points of contact, and consequently as many tangents as there are points of intersection of (52) with the original curve: that is, there may be $n(n-1)$ tangents drawn from a given point to a curve of the nth degree. If the $n(n-1)$ roots of the equation which determines the points of intersection are all real, they give points at which tangents may actually be drawn to the curve from (ξ, η); but the tangents which correspond to imaginary roots cannot be drawn, at least to those branches of the curve which are in the plane of reference. Hence the theorem gives the number which the tangents drawn from a given point and in the plane of reference cannot exceed. Since $n(n-1)$ is an even number, all the roots may be imaginary, entering as pairs of conjugate roots, and therefore it is possible that no tangents can be drawn to a given curve from points within certain portions of space in the plane of reference: for ξ and η enter into the equation (52), and therefore the reality of the roots of the equation which determines the points of contact is dependent on their values. This is geometrically evident: because from all points towards which a curve is concave no tangents can be drawn to that relatively concave part: whereas for all those points towards which a curve is convex two tangents can be drawn to that relatively convex part. And if the point whence the tangents are drawn varies, the two tangents become coincident when it is on the curve. And thus from a point on the curve only $n(n-1)-2$ tangents can be drawn to the curve. This is in accordance with the algebraical theorem, that if a pair of conjugate roots of an equation gradually varies, and becomes real by reason of the vanishing of the imaginary part, they are equal at the values where the transition from their being imaginary to their being real takes place.

Although the tangents corresponding to the imaginary roots may not be capable of geometrical exhibition, yet as they are important in reference to a subject which we shall presently investigate, their algebraical existence must not be forgotten. These theorems are also true when the point from which the tangents are drawn is at an infinite distance; that is, in other words, when the tangents are all parallel to each other.

If $n = 2$, the original curve is a conic; and $n(n-1) = 2$; therefore not more than two tangents can be drawn from the same point to a conic; and the point whence the tangents are drawn may have such a position that two may be drawn, or only one, or none: thus as to an ellipse; from all points outside the curve two real tangents can be drawn: from all points on the curve, only one: and from points within the curve, none.

If $n = 3$, $n(n-1) = 6$; and therefore six is the greatest number of tangents that can be drawn from a given point to a curve of the third degree; and if the point from which the tangents are drawn is on the curve, only four tangents can be drawn. We shall hereafter have various modifications of these propositions.

Plane curves are also arranged according to the number of tangents which can be drawn to them from a given point: and according to this character they are said to be of a certain *class*. Thus if a curve is capable of having m tangents drawn to it from a given point, it is said to be of the mth class. Thus, if a curve is of the nth degree, it is of the $n(n-1)$th class. A curve of the second degree is of the second class; a curve of the third degree is of the sixth class.

It is also manifest by general reasoning that not more than $n(n-1)$ tangents can be drawn to a curve of the nth degree from a point in its plane. A tangent meets a curve in two coincident points; and as a straight line drawn from a given point cannot cut a curve in more than n points, so the number of pairs of points of intersection of the line with the curve may be $\frac{n(n-1)}{2}$; and when two points of intersection become coincident, the cutting line becomes a tangent; and as each pair of points may become coincident in two different ways, so the number of tangents may be $n(n-1.)$

*224.] Here we must investigate another method of finding the equations of the $n(n-1)$ tangents to a plane curve, which will give the equations of all in a synthetic form; and the process too being general will lead us on the way to other more general theorems of algebraic curves. For the sake of symmetry and homogeneity I shall use the equation to the curve in the triliteral form, and shall express it by the equation,

$$\text{F}(x', y', z') = 0. \tag{55}$$

Take any two points A, B in the plane, (ξ, η, ζ) and (x, y, z), and, in the line joining them, consider a point P, (x', y', z'), which divides the distance between them in the ratio $\mu : \nu$; so that AP : BP :: $\nu : \mu$;

$$x' = \frac{\nu x + \mu\xi}{\mu+\nu}, \quad y' = \frac{\nu y + \mu\eta}{\mu+\nu}, \quad z' = \frac{\nu z + \mu\zeta}{\mu+\nu}; \tag{56}$$

let us suppose P, (x', y', z') to be on the curve (55); then substituting (56) in (55), and suppressing the common denominator $(\mu+\nu)^n$, which may be done because (55) is homogeneous, we have

$$\text{F}(\nu x + \mu\xi, \; \nu y + \mu\eta, \; \nu z + \mu\zeta) = 0; \tag{57}$$

which is an equation of the nth degree in terms of the ratio $\mu : \nu$. The roots of this are the ratios in which the line joining (ξ, η, ζ) and (x, y, z) are cut by the curve at each of the n points where it meets the curve; and if these n values are substituted in (56), we shall have the coordinates to the n points of section.

Let (57) be expanded by the theorem given in (56), Art. 142; then, as (57) is homogeneous and of the nth degree, we have

$$\begin{aligned}\nu^n \text{F}(x, y, z) &+ \nu^{n-1}\mu \left\{\xi\left(\frac{d\text{F}}{dx}\right) + \eta\left(\frac{d\text{F}}{dy}\right) + \zeta\left(\frac{d\text{F}}{dz}\right)\right\} \\ &+ \frac{1}{1.2}\nu^{n-2}\mu^2 \left\{\xi^2\left(\frac{d^2\text{F}}{dx^2}\right) + \eta^2\left(\frac{d^2\text{F}}{dy^2}\right) + \zeta^2\left(\frac{d^2\text{F}}{dz^2}\right)\right. \\ &\qquad \left. + 2\eta\zeta\left(\frac{d^2\text{F}}{dy\,dz}\right) + 2\zeta\xi\left(\frac{d^2\text{F}}{dz\,dx}\right) + 2\xi\eta\left(\frac{d^2\text{F}}{dx\,dy}\right)\right\} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots = 0.\end{aligned} \tag{58}$$

Let us take a more convenient notation; and let

$$\left.\begin{aligned} p &= \text{F}(x,y,z), \\ p_1 &= \xi\left(\frac{dp}{dx}\right)+\eta\left(\frac{dp}{dy}\right)+\zeta\left(\frac{dp}{dz}\right), \\ p_2 &= \xi\left(\frac{dp_1}{dx}\right)+\eta\left(\frac{dp_1}{dy}\right)+\zeta\left(\frac{dp_1}{dz}\right), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ p_n &= \xi\left(\frac{dp_{n-1}}{dx}\right)+\eta\left(\frac{dp_{n-1}}{dy}\right)+\zeta\left(\frac{dp_{n-1}}{dz}\right) \end{aligned}\right\}; \qquad (59)$$

so that (58) becomes

$$\nu^n p + \nu^{n-1}\mu p_1 + \frac{1}{1.2}\nu^{n-2}\mu^2 p_2 + \ldots + \nu\mu^{n-1}p_{n-1} + \mu^n p_n = 0; \qquad (60)$$

which is again of the nth degree in terms of the ratio $\mu : \nu$, and of which the roots refer to the n points of intersection of the straight line with the curve.

The extreme point (x, y, z) of this cutting line is not fixed; it may therefore be any point in the plane of the curve: let us assume it to be on the curve: then it is manifest that $p = 0$, and also that one of the roots of (60) is equal to zero, or that the equation is divisible by μ.

If the line meets the curve in two coincident points, let us suppose them to be (x, y, z) the extremity of the cutting line; then two values of μ are equal to zero, and (60) must be divisible by μ^2. In this case $p = 0$, and $p_1 = 0$: and as these are simultaneous, the points to which these equal and vanishing values of μ correspond are the points of intersection of the two curves $p = \text{F}(x,y,z) = 0$, and $p_1 = 0$. But if a line drawn from (ξ, η, ζ) to the curve meets the curve in two coincident points, that line is a tangent: therefore the equation $p_1 = 0$, which is of the first order in terms of ξ, η, ζ, is that of the tangent of the curve; and this expressed at length is

$$\xi\left(\frac{d\text{F}}{dx}\right)+\eta\left(\frac{d\text{F}}{dy}\right)+\zeta\left(\frac{d\text{F}}{dz}\right) = 0; \qquad (61)$$

which is the same equation as that found above in Art. 216, and is of the $(n-1)$th order in terms of x, y, z; so that the points at which tangents drawn from (ξ, η, ζ) touch the original curve lie on a curve of the $(n-1)$th order. And this result admits of extension; for from (59) it appears that $p_2 = 0$, which is an equation of the $(n-1)$th order in terms of x, y, z, is that which passes through the $(n-2)$ points at which tangents drawn from (ξ, η, ζ) touch the curve $p_1 = 0$; similarly, $p_3 = 0$ is the curve of the $(n-3)$rd order which passes through the $(n-3)$ points of contact of lines drawn from (ξ, η, ζ) with the curve $p_2 = 0$; and similarly $p_{n-1} = 0$ is the equation to a straight line in terms of x, y, z which passes through the points of contact of the tangents drawn from (ξ, η, ζ) to the conic $p_{n-2} = 0$. The further consideration of (59) and (60) must be deferred to a subsequent Section.

225.] In connexion with the present subject it remains for us still to investigate more generally the equation of the system or the pencil of tangents which can be drawn from a given point (ξ, η, ζ) to a curve of the nth degree.

Let the equation to the curve be $\mathrm{F}(x, y, z) = 0$; then we have the following equations,

$$\mathrm{F}(x, y, z) = 0; \tag{62}$$

$$\xi \left(\frac{d\mathrm{F}}{dx}\right) + \eta \left(\frac{d\mathrm{F}}{dy}\right) + \zeta \left(\frac{d\mathrm{F}}{dz}\right) = 0, \tag{63}$$

$$x(\eta z' - \zeta y') + y(\zeta x' - \xi z') + z(\xi y' - \eta x') = 0; \tag{64}$$

of which (63) is the equation to the tangent, and (64) is the equation of a straight line passing through (ξ, η, ζ) and another point (x', y', z'), and of which line therefore x', y', z' may be the current coordinates. Now (62) is a homogeneous equation of n dimensions, (63) is also homogeneous and of $n-1$ dimensions in terms of x, y, z; and (64) is of one dimension in terms of the same variables; therefore, if x, y, z are eliminated, the resultant is, see Art. 209, of $n(n-1)$ dimensions in terms of $\eta z' - \zeta y'$, $\zeta x' - \xi z'$, $\xi y' - \eta x'$; of n dimensions in terms of ξ, η, ζ; and apparently of $2n-1$ dimensions in terms of the coefficients of $\mathrm{F} = 0$, because these coefficients enter in the nth degree by reason of (63) and in the $(n-1)$th degree by reason of (62); but the resultant has a linear rational factor with reference to the coefficients of $\mathrm{F} = 0$, because (x', y', z') is on the tangent, and

$$x' \left(\frac{d\mathrm{F}}{dx}\right) + y' \left(\frac{d\mathrm{F}}{dy}\right) + z' \left(\frac{d\mathrm{F}}{dz}\right) = 0; \tag{65}$$

this factor is therefore to be omitted; and the resultant consequently is of the $2(n-1)$th degree with respect to the coefficients of $\mathrm{F} = 0$; and is the equation of the system of tangents which can be drawn to the curve from the point (ξ, η, ζ). As an example of this process let us take the ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0; \tag{66}$$

and the equation to its tangent is

$$\frac{\xi x}{a^2}+\frac{\eta y}{b^2}+\frac{\zeta z}{c^2}=0; \tag{67}$$

and the equation to the line through (ξ, η, ζ) and (x', y', z') is

$$x(\zeta y_1-\eta z_1)+y(\xi z_1-\zeta x_1)+z(\eta x_1-\xi y_1)=0; \tag{68}$$

so that from (67) and (68) we have

$$\frac{x}{x'\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}\right)-\xi\left(\frac{\xi x'}{a^2}+\frac{\eta y'}{b^2}+\frac{\zeta z'}{c^2}\right)}=\ldots=\ldots \tag{69}$$

substituting which in (66) we have

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}\right)\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}\right)-\left(\frac{\xi x'}{a^2}+\frac{\eta y'}{b^2}+\frac{\zeta z'}{c^2}\right)^2=0; \tag{70}$$

which is a quadratic equation in terms of x', y', z', and gives therefore two values for the point to which a line drawn from (ξ, η, ζ) will be a tangent to the ellipse: two tangents therefore can be drawn to an ellipse from a given point.

Further investigations in connexion with this subject will be found in a paper by Joachimsthal in Crelle's Journal, Vol. XXXIII, p. 371, and in a paper by Mr. Cayley in the same Journal, Vol. XXXIV, p. 30.

226.] The equation to the tangent given in (18) or (61), which is equivalent also to equation (52), must still be considered in a more general way, and with reference to the curve which passes through the $n(n-1)$ points of contact: for, as before observed, these equations are of the $(n-1)$th degree in terms of x, y, and z, and of the first degree in reference to ξ, η, ζ. This curve requires a peculiar name on account of many properties which it possesses; and it is called the Polar, or the first polar, with reference to the point (ξ, η, ζ) which is called the Pole. The original curve $\mathrm{F}(x, y, z)=0$ is called the Base-curve; and thus Base-curve, Pole, and Polar are correlative terms. In (59), if (x, y, z) is a point on the curve $p=\mathrm{F}(x, y, z)=0$, p is the base-curve: $p_1=0$ or

$$\xi\left(\frac{d\mathrm{F}}{dx}\right)+\eta\left(\frac{d\mathrm{F}}{dy}\right)+\zeta\left(\frac{d\mathrm{F}}{dz}\right)=0 \tag{71}$$

is the equation of the first polar, with reference to (ξ, η, ζ) as the pole.

Thus if the base-curve is a conic of which the equation is

$$\text{A}x^2+\text{B}y^2+\text{C}z^2+2\text{E}\,yz+2\text{G}\,zx+2\text{H}\,xy = 0,$$

the equation of the first polar is

$$\xi(\text{A}x+\text{H}y+\text{G}z)+\eta(\text{H}x+\text{B}y+\text{E}z)+\zeta(\text{G}x+\text{E}y+\text{C}z) = 0,$$

which is of the first degree in terms of x, y, z, and is therefore the equation to a straight line.

Similarly if the base-curve is of the third degree, the first polar is a conic, and all the six points of contact of tangents to the curve drawn from a given point are on a conic.

Let us consider the modifications which (71) undergoes for particular positions of the pole. Let the origin be the pole; then $\xi = \eta = 0$; and (71) becomes

$$\left(\frac{d\text{F}}{dz}\right) = 0, \tag{72}$$

and this is the polar of the origin. Thus the polar of the origin of the conic whose equation is (51), Art. 208, is

$$\text{G}x+\text{E}y+\text{C}z = 0. \tag{73}$$

Again, let us suppose the pole to be at an infinite distance, so that all the tangents drawn from it to the curve are parallel, and let us suppose $\frac{b}{a}$ to be the tangent of the angle which all these parallel tangents make with the axis of x; then the equation to the polar is

$$a\left(\frac{d\text{F}}{dx}\right)+b\left(\frac{d\text{F}}{dy}\right) = 0, \tag{74}$$

which is of the $(n-1)$th degree; and as the points of contact are the $n(n-1)$ points, common to this and to the original curve, it follows that all the points of contact of parallel tangents lie on a curve of the $(n-1)$th degree.

If all the tangents are parallel to the axis of x, $b = 0$, and therefore $\left(\frac{d\text{F}}{dx}\right) = 0$; similarly if the tangents are parallel to the axis of y, $a = 0$, and therefore $\left(\frac{d\text{F}}{dy}\right) = 0$. Thus $\left(\frac{d\text{F}}{dx}\right) = 0$, and $\left(\frac{d\text{F}}{dy}\right) = 0$ are the equations to the polars of $\text{F}(x, y, z) = 0$, when the pole is at an infinite distance on the axes of x and y respectively. One case of these equations requires notice: if

$F(x, y, z)$ is the equation of a conic, $\left(\frac{dF}{dx}\right)$ and $\left(\frac{dF}{dy}\right)$ are expressions of the first degree, and represent straight lines. Now if $\left(\frac{dF}{dx}\right) = 0$, at the points common to it and the conic, the tangents of the conic are parallel to the axis of x; and as all chords of a conic parallel to these parallel tangents are bisected by the line passing through the points of contact, so $\left(\frac{dF}{dx}\right) = 0$ is the equation of a line bisecting all chords parallel to the axis of x; it is therefore a diameter. Similarly $\left(\frac{dF}{dy}\right) = 0$ is the equation to a diameter bisecting all chords parallel to the axis of y: accordingly the point of intersection of these two lines is the centre of the conic. See Ex. 2, Art. 141.

If $F(x, y, z)$ is of three dimensions, (74) is the equation to a conic; and therefore the six points in a curve of the third degree at which lines parallel to a given straight line touch the curve lie in a conic.

227.] Suppose that the pole is not a fixed point, but moves in a given curve in the plane of the base-curve: then as the position of the pole moves, that also of the polar is changed. Let us call the curve along which the pole moves, the Directrix: then, if the directrix is a continuous curve, the successive corresponding polars will have positions infinitesimally consecutive, and will doubtless in their consecutive intersections generate another curve: curves generated in this way are called Envelopes; but the general theory of them must be reserved to a future part of our treatise. See Chapter XIII. Without entering on the general theory I am here able to consider the most simple case; that, viz. in which the directrix is a straight line.

Let the equation to the directrix be

$$a\xi + b\eta + c\zeta = 0; \tag{75}$$

the equation to the first polar is

$$\xi\left(\frac{dF}{dx}\right) + \eta\left(\frac{dF}{dy}\right) + \zeta\left(\frac{dF}{dz}\right) = 0; \tag{76}$$

whence we have

$$\frac{\xi}{b\left(\frac{dF}{dz}\right) - c\left(\frac{dF}{dy}\right)} = \frac{\eta}{c\left(\frac{dF}{dx}\right) - a\left(\frac{dF}{dz}\right)} = \frac{\zeta}{a\left(\frac{dF}{dy}\right) - b\left(\frac{dF}{dx}\right)}; \tag{77}$$

and as no other relation is given between ξ η and ζ, these quantities are indeterminate; and thus we have

$$b\left(\frac{d\text{F}}{dz}\right)-c\left(\frac{d\text{F}}{dy}\right)=c\left(\frac{d\text{F}}{dx}\right)-a\left(\frac{d\text{F}}{dz}\right)=a\left(\frac{d\text{F}}{dy}\right)-b\left(\frac{d\text{F}}{dx}\right)=0; \quad (78)$$

which are two equations of the $(n-1)$th degree in terms of x, y, z: these are the equations to two curves each of which is of the $(n-1)$th order; and these curves do not involve ξ, η, ζ: they are therefore the same for all positions of the pole in the directrix; and they intersect in $(n-1)^2$ points; and therefore we conclude that all the first polars of a curve of the nth degree pass through $(n-1)^2$ points, if the directrix of the pole is a straight line. A particular case of this is the well-known theorem, that all the polars of a conic corresponding to poles on a given straight line pass through one and the same point.

228.] The first polar is a curve of the $(n-1)$th degree. To it let a process with reference to the same pole be applied similar to that by which the first polar is derived from the base-curve: then another curve will be derived from it, and this may from analogy be called the second polar, being as it is the polar of the aforesaid first polar, and with reference to the same pole. Similarly may other polars, the third, the fourth, &c. be derived. On referring to the series of equations (59) $p_1 = 0$ is the equation of the first polar; and as $p_2 = 0$ bears the relation to it, with reference to (ξ, η, ζ), which $p_1 = 0$ does to the base-curve, $p_2 = 0$ is the equation to the second polar: similarly $p_3 = 0$ is the equation to the third polar; and so on; and as $p = 0$ is an equation of the nth degree in terms of x, y, z, so if curves are formed according to the scheme in (59), $p_{n-1} = 0$ is of the first degree in terms of x, y, z; and is therefore as to these coordinates the equation of a straight line. Similarly $p_{n-2} = 0$ is the equation of a conic; therefore the $(n-1)$th polar is a straight line, and the $(n-2)$th is a conic. These results are true, whatever is the base curve. I may also observe that by reason of the symmetry of the function in (57),

$$\begin{aligned} p_{n-2} = {} & x^2\left(\frac{d^2\text{F}}{d\xi^2}\right)+y^2\left(\frac{d^2\text{F}}{d\eta^2}\right)+z^2\left(\frac{d^2\text{F}}{d\zeta^2}\right) \\ & +2yz\left(\frac{d^2\text{F}}{d\eta\, d\zeta}\right)+2zx\left(\frac{d^2\text{F}}{d\zeta\, d\xi}\right)+2xy\left(\frac{d^2\text{F}}{d\xi\, d\eta}\right) = 0, \quad (79) \end{aligned}$$

$$p_{n-1} = x\left(\frac{d\text{F}}{d\xi}\right)+y\left(\frac{d\text{F}}{d\eta}\right)+z\left(\frac{d\text{F}}{d\zeta}\right) = 0; \quad (80)$$

which are respectively the equations to the polar conic and the polar straight line. It is also to be observed that the successive polars $p_1, p_2, \dots p_{n-1}$, are in terms of ξ, η, ζ severally of the orders $1, 2, \dots (n-1)$ *.

229.] The general equation of the normal to a plane curve given in (21), Art. 217, is

$$(\eta-y)\left(\frac{d\text{F}}{dx}\right)-(\xi-x)\left(\frac{d\text{F}}{dy}\right) = 0, \tag{81}$$

which is evidently of n dimensions in terms of x and y, and does not generally admit of reduction. If (ξ, η) is a point from which normals are drawn to a curve $\text{F}(x, y)$ of the nth degree, (x, y) in (81) is the point at which the normal meets the curve: and as (81) is of n dimensions, it follows that there are as many points of intersection of normals with the curve as there are points of intersection of (81) with the original curve: that is, the number of points is n^2: therefore from a given point (ξ, η) n^2 normals may be drawn to a curve: this is clearly the largest number; and the number which can actually be drawn will be the same as that of the real roots in the equation of n^2 dimensions which arises from the combination of (81) and of $\text{F}(x,y) = 0$. As ξ and η enter into this equation by means of (81), so will the nature of the roots depend on them; and therefore there may be certain districts of the plane of reference, in which if the point (ξ, η) is taken, all the roots may be real: certain lines of demarcation on which if (ξ, η) is, two or an even or any number of the roots may become equal; and certain districts again beyond these lines for which pairs of the roots will be imaginary. Thus to a central conic from all points within certain districts four normals can be drawn; from points on a certain line, which is called the Evolute, two of the four roots which give the four normals will be equal and only three normals can be drawn; and from all points in the district beyond this line only two normals can be drawn. To the parabola not more than three normals can be drawn from any point in its plane, the fourth normal being that to the infinite branch of the curve. Thus if

---

* For other theorems as to successive polars, see a memoir by Steiner in Crelle's Journal, Vol. XLVII, p. 1; and the French translation of it in Liouville, Vol. XVIII.

the equation of the parabola is $y^2 = 4ax$, the equation of its normal is

$$\eta - y = -\frac{y}{2a}(\xi - x);$$

and if we eliminate x from these two equations we have

$$y^3 - 4a(\xi - 2a)y - 8a^2\eta = 0; \qquad (82)$$

which is a cubic; so that if the roots of this are all real, three normals may be drawn to a parabola; if two of the roots are equal, only two normals; and if two roots are imaginary, only one normal can be drawn: and as one root of a cubic is always real, one normal can be drawn, whatever is the position of (ξ, η); if two roots of (82) are equal, we have

$$3y^2 - 4a(\xi - 2a) = 0; \qquad (83)$$

and if we eliminate y from this equation and (82), we have

$$27a\eta^2 = 4(\xi - 2a)^3; \qquad (84)$$

from all points on which curve two normals can be drawn. Thus (84) divides the plane of reference into two districts, from all points in one of which three normals, and from all points in the other of which only one normal can be drawn to the parabola. (84) is called the Evolute of the Parabola.

In a memoir by Professor Steiner of Berlin, and contained in Crelle's Journal, Vol. XLIX. 1855, three geometrical proofs are given of the theorem of this Article. One of them I will insert here. Let P be the point from which the normals are to be drawn. Let the curve be of the nth degree, and let it be moved round in its plane about the point P; then the curve in its new position will intersect that in the old position, in n^2 points, which are either real or imaginary: and when the displacement is infinitesimal, the radii vectores drawn from P to the same points on the curves, when the second is displaced, will at the points of intersection be equal; and thus the line joining these two points will be perpendicular to the radii vectores, and the radius vector will be a normal; and as the number of points at which these circumstances may occur will be n^2, so from P may n^2 normals be drawn to a curve of the nth degree.

230.] The normal is the longest or the shortest line that can be drawn to a plane curve from a given point in its plane.

Let η_0 and ξ_0 be the coördinates to the given point, and x and y the current coordinates to the curve; then, if r is the distance between (η_0, ξ_0) and (x, y),

$$r^2 = (\xi_0 - x)^2 + (\eta_0 - y)^2;$$

$$\therefore \quad r\,dr = 0 = -(\xi_0 - x)\,dx - (\eta_0 - y)\,dy. \tag{85}$$

Let $u = \text{F}(x, y) = c$ be the equation to the curve;

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy = 0; \tag{86}$$

and as (85) and (86) are simultaneously true,

$$\frac{\xi_0 - x}{\left(\frac{d\text{F}}{dx}\right)} = \frac{\eta_0 - y}{\left(\frac{d\text{F}}{dy}\right)}; \tag{87}$$

and if we compare (85) with (20), or (87) with (21), it is manifest that they are respectively identical; and that therefore the longest and shortest lines coincide in direction with the normal.

SECTION 2.—*Asymptotes to plane curves referred to rectangular coordinates.*

231.] A line is said to be an asymptote to a curve, when the curve approaches continually nearer and nearer to it, but is not coincident with it within a finite distance.

From this definition it is plain that there are two classes of asymptotes, rectilinear and curvilinear, which it is convenient to discuss separately.

If the curve has asymptotes which are either the coordinate axes themselves or straight lines parallel to them, they may be determined in the following manner. If $y = \infty$, when $x = 0$, the axis of y is an asymptote; and if $y = 0$, when $x = \infty$, the axis of x is an asymptote: such are the axes of coordinates to the curve, $xy = k^2$.

Again, if $y = \infty$, when $x = a$, a line parallel to the axis of y, at a distance a from it, is an asymptote; and if $x = \infty$, when $y = b$, a line parallel to the axis of x, at a distance b from it, is an asymptote.

Thus suppose the equation to a curve to be

$$xy - ay - bx = 0;$$

then, as it may be put under either of the forms,

$$y = \frac{bx}{x-a}, \quad \text{or} \quad x = \frac{ay}{y-b},$$

$$y = \infty, \text{ when } x = a; \text{ and } x = \infty, \text{ when } y = b;$$

that is, two lines parallel to the coordinate axes are asymptotes; the curve is represented in fig. 51; wherein OA $= a$, OB $= b$.

So of the logarithmic curve, see fig. 32, the axis of x is an asymptote; its equation is

$$y = a^x;$$

$$\therefore \quad y = 0, \text{ when } x = -\infty;$$

therefore OB is an asymptote to the branch AC.

So in the cissoid, equation (12), Art. 194, and fig. 34, $y = \infty$, when $x = 2a$; therefore the ordinate through A is an asymptote to the curve.

And in the tractory, equation (27), Art. 200, and fig. 38, $y = 0$, when $x = \infty$, and therefore the axis of x is an asymptote.

If however a curve has rectilinear asymptotes not parallel to the axes of coordinates, they are to be determined by one or other of the following methods.

232.] Method of determining rectilinear asymptotes by expansion in descending powers of x.

If by any artifice, as by the Binomial Theorem, or by Maclaurin's Theorem, the equation to a curve can be expanded in a series of the form

$$y = a_1 x + a_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \dots\dots; \tag{88}$$

then, as all and every term after the first two, that is, every term which involves a negative power of x, diminishes without limit, and ultimately becomes infinitesimal, when x becomes infinity, the difference between the ordinate to the curve represented in equation (88), and that to the straight line whose equation is

$$y = a_1 x + a_0, \tag{89}$$

is infinitesimal, and the straight line represented by equation (89) is an asymptote to the curve.

And according as the first term after a_0, be it $\frac{b_1}{x}$ or $\frac{b_2}{x^2}\dots$, is positive or negative, so will the ordinate to the curve be greater or less than the ordinate to the asymptote, and the curve will be above or below the asymptote.

The equation (89) is to be constructed in the ordinary way.

And if, finally, the equation to the asymptote is affected with $\pm\sqrt{-}$, it indicates that the asymptote lies out of the plane of reference, and is asymptotic therefore to a branch of a curve similarly placed, and to be drawn according to the methods of Section 2 of the last Chapter.

233.] Ex. 1. To find the equations to the asymptotes of the Cissoid of Diocles.

$$y^2 = \frac{x^3}{2a-x} = -x^2\left(1-\frac{2a}{x}\right)^{-1};$$

$$\therefore\quad y = \pm\sqrt{-}\,x\left(1+\frac{a}{x}+\dots\right);$$

$$\therefore\quad y = \pm\sqrt{-}\,(x+a),$$

are the equations to the asymptotes, and represent two straight lines out of the plane of reference inclined to the axis of x at ±45°, and cutting the axis of x at a distance $-a$ from the origin, which are delineated by the dotted straight lines of fig. 34.

Ex. 2. To determine the asymptotes of the Witch of Agnesi. From the equation (15) of Art. 195, we have

$$y^2 = 4a^2\,\frac{2a-x}{x},$$

$$= 4a^2\left(\frac{2a}{x}-1\right);$$

$$\therefore\quad y = \pm\sqrt{-}\,2a,\ \text{when } x = \infty;$$

$y = \pm 2a\sqrt{-1}$

which equations are those to the asymptotes, and express two straight lines out of the plane of reference, and parallel to the axis of x, at distances $\pm 2a$ from it; see fig. 35.

Ex. 3. To determine the asymptotes of

$$y^2 = x^2\cdot\frac{x^2-1}{x^2+1};$$

$$y = \pm x\left(1-\frac{1}{x^2}\right)^{\frac{1}{2}}\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}},$$

$$= \pm x\left(1-\frac{1}{2x^2}-\frac{1}{8x^4}+\dots\right)\left(1-\frac{1}{2x^2}+\frac{3}{8x^4}+\dots\right),$$

$$= \pm x\left(1-\frac{1}{x^2}+\dots\right);$$

therefore neglecting terms involving negative powers of x, the equations to the asymptotes are $y = \pm x$; and as the next term of the series is negative, it follows that the curve is, in the first quadrant, below the asymptote.

234.] The preceding method does not indicate any general property of equations of curves which have asymptotic straight lines; and frequently the equation does not admit of development in the form (88) with the requisite facility. These defects are supplied by the following process, which is easy of application in all cases.

If a curve approaches nearer and nearer to a definite straight line and meets that line at an infinite distance, it is evident that this line is a rectilinear asymptote and that it is the tangent to the curve at infinity; thus the equation to this particular tangent is the equation to the asymptote; and the equation to it may be deduced from the general equation to the tangent, if the necessary alterations are made. This is the theory of these asymptotes in the general case. If the asymptote is parallel to one of the coordinate axes, say to the axis of x, that we may fix our thoughts; then, if it is at a distance b from it, $y = b$, when $x = \infty$: and if it is parallel to the axis of y at a distance a from it, then $x = a$, when $y = \infty$. If however it is not parallel to either of these axes, the solution of the problem requires the determination of a tangent when $x = y = \infty$.

Let us take the equation to a curve in the general form given in (48), Art. 207; then the equation to the tangent is, see (52), Art. 222,

$$\xi\left(\frac{d\text{F}}{dx}\right) + \eta\left(\frac{d\text{F}}{dy}\right) + u_{n-1} + 2u_{n-2} + \ldots + (n-1)u_1 + nu_0 = 0. \quad (90)$$

When $x = y = \infty$, the terms of the highest dimensions alone may be retained; in which case,

$$\left(\frac{d\text{F}}{dx}\right) = \left(\frac{du_n}{dx}\right), \qquad \left(\frac{d\text{F}}{dy}\right) = \left(\frac{du_n}{dy}\right);$$

and (90) becomes

$$\xi\left(\frac{du_n}{dx}\right) + \eta\left(\frac{du_n}{dy}\right) + u_{n-1} = 0; \quad (91)$$

this is evidently an equation of $n-1$ dimensions in all its terms; and it is the equation to the asymptote when the several terms in it are evaluated for $x = y = \infty$.

As the asymptote is a tangent to the curve, so the tangent of the angle contained between it and the axis of x is $\frac{dy}{dx}$. In this case then

$$\frac{dy}{dx} = -\frac{\left(\frac{du_n}{dx}\right)}{\left(\frac{du_n}{dy}\right)}, \text{ when } x = y = \infty; \tag{92}$$

and the values of the intercepts of the coordinate axes of x and y by this asymptote are respectively

$$-\frac{u_{n-1}}{\left(\frac{du_n}{dx}\right)}, \text{ and } -\frac{u_{n-1}}{\left(\frac{du_n}{dy}\right)}, \text{ when } x = y = \infty. \tag{93}$$

Hence it appears that the asymptote passes through the origin, when $u_{n-1} = 0$; that is, if the equation to the curve is of the nth degree, and has no term of the $(n-1)$th degree, the asymptotes pass through the origin. If on evaluation (92) becomes zero, the asymptote is parallel to the axis of x; and if it $= \infty$, the asymptote is parallel to the axis of y: in either case the intercepts of the coordinate axes are to be determined from (93).

235.] Let us apply the process to some examples.

Ex. 1. The equation to the hyperbola being $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, it is required to find its rectilinear asymptotes.

As this equation is of two dimensions and has no term of one dimension, the asymptotes pass through the origin, which is the centre of the curve. The tangents of the angles at which they are inclined to the axis of x are thus found:

Let the equation to the hyperbola be considered in the form $u_2 + u_0 = 0$, so that

$$\left(\frac{du_2}{dx}\right) = \frac{2x}{a^2}, \qquad \left(\frac{du_2}{dy}\right) = -\frac{2y}{b^2};$$

$$\therefore \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y} = \frac{\infty}{\infty}, \text{ when } x = y = \infty;$$

$$= \frac{b^2\, dx}{a^2\, dy};$$

$$\therefore \quad \frac{dy}{dx} = \pm \frac{b}{a}, \text{ when } x = y = \infty:$$

and the equations to the asymptotes are

$$\frac{x}{a} \pm \frac{y}{b} = 0. \tag{94}$$

Ex. 2. To find the equations to the asymptotes of the curve

$$y^3 + x^3 - ax^2 = 0.$$

In this case from (92),

$$\frac{dy}{dx} = -\frac{x^2}{y^2} = \frac{\infty}{\infty}, \text{ if } x = y = \infty,$$

$$= -\frac{x\,dx}{y\,dy} = \frac{\infty}{\infty}, \text{ - - - - - .}$$

In the next differentiation $\frac{dy}{dx}$ must be considered constant; because the inclination of the asymptote to the axes is the same for all the points infinitesimally consecutive on the curve at an infinite distance; see also Art. 139; and therefore differentiating again,

$$\frac{dy}{dx} = -\frac{dx^2}{dy^2}; \text{ and therefore } \frac{dy}{dx} = -1.$$

Again, from (93) the intercept of the axis of x

$$= \frac{ax^2}{3x^2} = \frac{a}{3};$$

so that the equation to the asymptote is

$$\eta = -\xi + \frac{a}{3}.$$

Ex. 3. To find the asymptotes to the Folium of Descartes, see fig. 63, of which the equation is

$$x^3 + y^3 - 3axy = 0.$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2} = \frac{\infty}{\infty}, \text{ when } x = y = \infty;$$

whence, as in the last example, $\frac{dy}{dx} = -1$.

And from (93) the intercept of the axis of y

$$= \frac{3axy}{3y^2} = a\frac{x}{y} = \frac{\infty}{\infty}, \text{ if } x = y = \infty,$$

$$= a\frac{dx}{dy} = -a, \text{ if } x = y = \infty;$$

so that the equation to the asymptote is

$$\eta = -\xi - a.$$

Ex. 4. If the equation to a curve is

$$x^3 + y^3 + 3a^2x + 3b^2y + c^3 = 0,$$

the equation to the asymptote is

$$x + y = 0.$$

I must observe that as the asymptote is a tangent to a curve at an infinite distance, the general properties of tangents are true also, in their degree, of asymptotes. Thus an asymptote must be considered to have two points common with the curve. Now as a straight line can generally cut a curve of the nth degree in n points, so an asymptote cannot cut it in more than $n-2$ points. Thus an asymptote to a conic cannot cut the conic, at least at a point within a finite distance. An asymptote to a curve of the third order must also cut the curve. Hence also it follows that to a curve of the nth degree not more than $n(n-1)-2$ tangents can be drawn parallel to an asymptote, for in this aspect the asymptote counts for two tangents.

236.] Sometimes by this method, as well as by the former, we arrive at results affected with $\pm\sqrt{-}$, in which case the lines must be drawn in their own planes. And sometimes curves have branches out of the plane of reference, which are asymptotic to straight lines in the plane, as in the following example; and these must be determined in one or other of the methods which have been just explained. Thus if

$$x^4(y-x)^2 = a^2-x^2, \qquad \text{then} \quad y-x = \pm\frac{(a^2-x^2)^{\frac{1}{2}}}{x^2}.$$

When x is infinitely great, the right-hand member of the equation vanishes; and we have $y = x$, which is the equation to a line passing through the origin, and inclined at 45° to the axis of x, and which is asymptotic to two branches of the curve; but, for all values of x not within the limits $\pm a$, the curve lies out of the plane of reference, and therefore the asymptote is that to which these branches are continually approaching. The form of the curve is given in fig. 52, the dotted lines representing the branches in a plane perpendicular to that of the paper.

237.] Two curves may also be asymptotic to each other; for suppose the equation to a given curve, when expanded in descending powers of x, to be

$$y = a_2x^2+a_1x+a_0+\frac{b_1}{x}+\dots; \tag{95}$$

then, if we neglect on the right-hand side of the equation all terms after the first three, which is equivalent to making $x = \infty$, the curve whose equation is

$$y = a_2x^2+a_1x+a_0, \tag{96}$$

and which is a parabola, is asymptotic to the given curve.

Also if in equation (88), Art. 232, we omit all terms except the first three, and multiply through by x, we have the equation to a hyperbola, viz.

$$xy = a_1 x^2 + a_0 x + b_1;$$

and this curve is asymptotic to the given curve, because the difference between the lengths of their ordinates is a quantity which diminishes without limit as x increases without limit. And so again if we take account of the first four terms of the same equation, and neglect all subsequent ones, we shall obtain the equation to a curve which is nearer to the given curve than either the rectilinear asymptote or the hyperbola; and thus by a similar process we may obtain a series of curves more and more asymptotic to a given curve. Thus also we often find curves which have cubical and semicubical parabolas asymptotic to them: as for instance,

$$y^2 = x^2 \{x^2 - a^2\}^{\frac{1}{2}};$$

$$\therefore \quad y = \pm x^{\frac{3}{2}} \left\{1 - \frac{a^2}{x^2}\right\}^{\frac{1}{4}},$$

$$= \pm x^{\frac{3}{2}} \left\{1 - \frac{a^2}{4x^2} - \dots\right\},$$

$$= \pm \left\{x^{\frac{3}{2}} - \frac{a^2}{4x^{\frac{1}{2}}} - \dots\right\};$$

and therefore, neglecting all terms involving negative powers of x, we have

$$y = \pm x^{\frac{3}{2}},$$

which is the equation to a semicubical parabola, the form of which is given in fig. 64, and is asymptotic to the curve.

238.] And I must enter further into the relation between the equation of a curve and those of its asymptotes, whether they are curvilineal or rectilineal, because a general theory will be hereby obtained; and because we shall also have the occasion of explaining, algebraically and geometrically, the points of intersection of a curve with a line, straight or curved, when the intersection takes place at an infinite distance.

Let the equation to the curve be that given in (48), Art. 207; and let us assume α to be the abridging symbol for a linear function of the form $ax + by + c$; so that

$$\alpha = ax + by + c; \qquad (97)$$

and thus $a = 0$ is the equation to a straight line; and a is proportional to the perpendicular distance from the point (x, y) on that line; it will also be convenient to take a as the type of $a_1, a_2, \ldots a_m$, which are other linear functions similar to (97). As heretofore observed, the abscissæ of the points of intersection of a straight line with the curve (48), Art. 207, are the roots of the equation obtained by eliminating y between the equations to the curve and the straight line: and as this equation is generally of the nth degree, it has n roots either real or imaginary. Suppose however that the degree of the equation which determines these abscissæ of the points of intersection is less than n by m units, and yet that no factor involving the variable has been omitted, what explanation does this fact admit of? It is of course due to a relation between the coefficients of the equations, by reason of which the coefficients of the first m terms vanish in the equation resulting from the elimination; and therefore by reason of which m constants become zero. But if a constant $= 0$, see Art. 11, a straight line at an infinite distance is expressed; and therefore if the resulting equation is of $n-m$ dimensions, m of the points of intersection are at an infinite distance. Now as $a = 0$ contains two undetermined constants, we may determine them so that two of its points of intersection with the curve may be at infinity; that is, so that the two highest terms in the resulting equation may vanish; in which case the straight line becomes an asymptote to the curve. Although these results are arrived at in reference to the axis of x yet they are evidently true also with reference to any other line; and therefore we may find the equation to asymptotes in the following way:

239.] Suppose a_1 to be a linear factor of the two highest terms $u_n + u_{n-1}$ in the equation to the curve: so that if v_{n-1} is a function of x and y of $n-1$ dimensions,

$$u_n + u_{n-1} = a_1 \mathrm{v}_{n-1}; \tag{98}$$

thus the equation to the curve becomes

$$a_1 \mathrm{v}_{n-1} + u_{n-2} + \ldots u_2 + u_1 + u_0 = 0; \tag{99}$$

and therefore if $a_1 = 0$ the two highest terms in the equation of the curve disappear, and the line $a_1 = 0$ will intersect the curve in two points at an infinite distance, and thus will be an asymptote to the curve. This result is also evident by the following

$a_1 \overline{U}_{n-1}$
should
any ca
[illegible]
[illegible]
not con
divide to

considerations: in (99) let $x = y = \infty$, then all the terms must be neglected except the first; and thus the direction of the curve at infinity is given by means of the first term of (99). Now this breaks up into two factors $a_1 = 0$ and $\text{U}_{n-1} = 0$, of which the former represents a straight line, and the latter a curve of the $(n-1)$th degree: to each of these therefore is the curve infinitesimally near at infinity; and thus $a_1 = 0$, and $\text{U}_{n-1} = 0$, are the equations to asymptotic lines. Hereby therefore we have not only rectilinear but also curvilinear asymptotes. Thus, for instance, in Art. 235, Ex. 2, the equation $y^3 + x^3 - ax^2 = 0$ may be expressed in the form

$$\left(y+x-\frac{a}{3}\right)\left\{y^2 \overset{-}{+} \left(x-\frac{a}{3}\right)y \overset{+}{-} \left(x-\frac{a}{3}\right)^2\right\} \overset{-}{+} \frac{a^2x}{3} \overset{+}{-} \frac{a^3}{27} = 0;$$

and therefore the asymptotic lines are

$$y+x-\frac{a}{3} = 0, \qquad \text{and} \qquad y^2 + \left(x-\frac{a}{3}\right)y - \left(x-\frac{a}{3}\right)^2 = 0;$$

of which the first is a straight line and the second is a hyperbola.

Similarly, in Ex. 4, the equation may be expressed in the form

$$(x+y)(x^2-xy+y^2) + 3a^2x + 3b^2y + c^3 = 0;$$

and $x+y = 0$ is the equation to a rectilinear asymptote.

Suppose also a_1 and a_2 to be linear functions of the coordinates; and suppose U_{n-2} to be a function of x and y of $n-2$ dimensions; and $u_n + u_{n-1} + u_{n-2}$ to be capable of expression in the form $a_1 a_2 \text{U}_{n-2}$; so that the equation to the curve becomes

$$a_1 a_2 \text{U}_{n-2} + u_{n-3} + \ldots + u_1 + u_0 = 0; \qquad (100)$$

then if we consider the points on the curve where $x = y = \infty$, only the first term of (100) is to be taken account of, and we have

$$a_1 a_2 \text{U}_{n-2} = 0,$$

which is satisfied by $a_1 = 0$, $a_2 = 0$, $\text{U}_{n-2} = 0$; of which the first two equations are those of straight lines, and the last is that of a curve of the $(n-2)$th degree: the curve therefore at infinity breaks up into two straight lines which are asymptotes to it, and into another curve of the $(n-2)$th degree which is also asymptotic. Thus, if the equation to a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

we may put it into the form

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)-1=0;$$

and the lines $\frac{x}{a}-\frac{y}{b}=0$, $\frac{x}{a}+\frac{y}{b}=0$ are two rectilinear asymptotes.

Again, if the first four terms of the general equation are such that

$$u_n+u_{n-1}+u_{n-2}+u_{n-3} = a_1 a_2 a_3\, \text{v}_{n-3}, \tag{101}$$

then the curve has three rectilinear asymptotes, and a curvilinear one whose equation is $\text{v}_{n-3}=0$: it is unnecessary to enter farther into the investigation of special cases. It is however to be observed that all the terms in the equation of lower dimensions may be altered when the equation is expressed in the form (101). Such is the case in the first example given above.

It is also to be observed that, generally, a curve of the nth degree has n rectilinear asymptotes. For if $x=y=\infty$, all the terms of lower dimensions must be neglected, and we have

$$u_n = 0,$$

which is a homogeneous equation of n dimensions in terms of x and y; and which is therefore capable of resolution into n factors of the form $y-m_1x$, $y-m_2x$, ... $y-m_nx$; each of which gives the direction in which the curve is going at infinity; and will become the equations to rectilinear asymptotes, or of lines parallel to them, if $m_1, m_2, \ldots$ are real numbers; if the m's are, some or all, imaginary, the asymptotes refer to branches of the curves in other planes.

SECTION 3.—*The direction of curvature, and points of inflexion.*

240.] The value of $\frac{dy}{dx}$, which we have discussed in the preceding Sections, enables us to determine the inclination to either of the coordinate axes of a curve at any point, but tells us nothing as to *the direction of curvature*, that is, as to whether the curve is *concave* or *convex* towards a certain line or in a given direction; we use these words in their common meaning; and we proceed to discover criteria which will determine such direction. To simplify the formulæ we shall take x to be an equicrescent variable.

$\frac{d^2y}{dx^2} = \frac{d \tan\tau}{dx}$

$= \frac{d \tan\tau}{d\tau} \cdot \frac{d\tau}{dx}$

from which follows &c.

Since under this supposition $\frac{d^2y}{dx^2} = \frac{d.\frac{dy}{dx}}{dx} = \frac{d.\tan\tau}{dx}$, see equation (24), Art. 218; and since $\frac{d.\tan\tau}{d\tau} = (\sec\tau)^2$, which is always positive; it follows that if $\frac{d^2y}{dx^2}$ is positive, τ and x are increasing and decreasing simultaneously; and if $\frac{d^2y}{dx^2}$ is negative, as x increases, τ decreases, and *vice versâ.*

Now from the geometry it is plain, that if τ and x simultaneously increase and decrease, the form of the curve must be such as that of fig. 53, that is, the curve must be *convex downwards;* and if as x increases τ decreases, or *vice versâ,* the form must be such as that of fig. 54, that is, the curve must be *concave downwards.* Hence we have the following criteria of the direction of curvature.

If $\frac{d^2y}{dx^2}$ is positive, the curve is convex downwards; and if $\frac{d^2y}{dx^2}$ is negative, the curve is concave downwards.

Suppose that at a certain point $\frac{d^2y}{dx^2}$ changes its sign, which in algebraical curves it can do only by passing through 0 or ∞, the direction of curvature changes; if the change of sign is from + to −, then the curvature, having been convex downwards, becomes concave; if the change is from − to +, the reverse is the case. A point where such a change of curvature takes place is called *a point of inflexion.* To determine its position we must equate $\frac{d^2y}{dx^2}$ to 0, and to ∞; and if at the corresponding critical value there is a change of sign, then such a point is a point of inflexion; fig. 55, (γ) and (δ) of fig. 48, fig. 14, and fig. 15 illustrate such points of inflexion.

This is also evident from the following considerations.

241.] Let $y = f(x)$ be the equation to a curve; and suppose that we consider it not only at the point (x, y), but also at another point $(x+h,\ y+k)$, so that

$$y + k = f(x+h); \tag{102}$$

then, expanding $f(x+h)$ to three terms by Taylor's Theorem,

$$y+k = f(x) + f'(x)\frac{h}{1} + \frac{h^2}{1.2} f''(x+\theta h). \tag{103}$$

If a tangent is drawn to the curve at the point (x, y), its equation is

$$\eta = y + \frac{dy}{dx}(\xi - x); \tag{104}$$

and therefore its ordinate, when ξ is $x+h$, is

$$\eta = y + \frac{dy}{dx} h; \tag{105}$$

in which, replacing y by $f(x)$, and $\frac{dy}{dx}$ by $f'(x)$, and subtracting (105) from (103), we obtain the difference between the ordinates to the curve and to the tangent corresponding to the point $x+h$, and we have

$$y+k-\eta = \frac{h^2}{1.2} f''(x+\theta h); \tag{106}$$

and taking h to be infinitesimal, that is, considering only the point in the curve which is next to $(x+dx, y+dy)$, whereby $f''(x+\theta h)$ becomes $f''(x)$, that is, $\frac{d^2y}{dx^2}$, we have

$$y+k-\eta = \frac{d^2y}{dx^2}\frac{h^2}{1.2}. \tag{107}$$

And therefore if $\frac{d^2y}{dx^2}$ is positive, the ordinate to the curve is greater than the ordinate to the tangent, and the curve lies above the tangent, whatever is the sign of h; that is, the curve is convex downwards, as in fig. 53; but if $\frac{d^2y}{dx^2}$ is negative, contrary results follow, and the curve is concave downwards, as in fig. 54.

If therefore at any point the curve passes through the tangent so as to be above it on one side of the point of contact and section, and below it on the other, then $y+k-\eta$ must change sign as h changes sign, which can only be the case when $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3}$ is a finite quantity; or in general, when the term in the expansion of $f(x+h)$, which gives sign to the whole, is one involving an odd power of h, in which case the curve will pass from below the tangent to above it, or *vice versâ*, in one or other of the ways indicated in fig. 55. It is plain that, at a point of inflexion, $\frac{dy}{dx}$, and therefore the angle τ, attains to a maximum or minimum, and is in fact at the point stationary; that is, τ is the same for three consecutive points on the curve: the corresponding tangent is called a stationary tangent.

The curvature of a curve towards the right or the left may be determined in a similar manner by the value and sign of $\frac{d^2x}{dy^2}$.

242.] Examples illustrative of the foregoing theory.

Ex. 1. To determine the direction of curvature of the curve whose equation is

$$y = \frac{(x-1)(x-3)}{x-2}.$$

$$\frac{d^2y}{dx^2} = -\frac{2}{(x-2)^3};$$

that is, $\frac{d^2y}{dx^2}$ is positive or negative according as x is less or greater than 2; and therefore the curve is convex downwards for all values of x less than 2, and concave downwards for all values of x greater than 2; and when $x = 2$, there is a point of inflexion.

Ex. 2. To determine the direction of curvature of the Witch of Agnesi.

$$y = 2a\left(\frac{2a-x}{x}\right)^{\frac{1}{2}}; \qquad \frac{d^2y}{dx^2} = \frac{2a^2(3a-2x)}{x(2ax-x^2)^{\frac{3}{2}}};$$

therefore the upper branch of the curve in the plane of reference is convex downwards for all values of x between 0 and $x = \frac{3a}{2}$; and when x is greater than $\frac{3a}{2}$, $\frac{d^2y}{dx^2}$ is negative, and the curve is concave downwards; at the point therefore, $x = \frac{3a}{2}$, there is a point of inflexion. This investigation explains the form of the curve as drawn in fig. 35.

Ex. 3. To prove that the equitangential curve, see Art. 200, is always convex downwards.

$$\frac{dy}{dx} = -\frac{y}{(a^2-y^2)^{\frac{1}{2}}};$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{y^2}{a^2-y^2}, \qquad 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = \frac{2a^2y}{(a^2-y^2)^2}\cdot\frac{dy}{dx},$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{a^2y}{(a^2-y^2)^2},$$

and therefore, as y is always positive, see fig. 38, $\frac{d^2y}{dx^2}$ is always positive; and the curve is convex downwards.

Ex. 4. To determine the point of inflexion of the Companion to the Cycloid.

$$y = a\theta, \qquad \therefore \quad dy = a\,d\theta;$$

$$x = a(1-\cos\theta), \qquad dx = a\sin\theta\,d\theta;$$

$$\frac{dy}{dx} = \operatorname{cosec}\theta, \qquad \frac{d^2y}{dx^2} = -\frac{\cos\theta}{a(\sin\theta)^3};$$

for all values of θ between 0 and $\frac{\pi}{2}$, $\frac{d^2y}{dx^2}$ is negative, and the curve is concave downwards; and for all values of θ between $\frac{\pi}{2}$ and π, the curve is convex downwards; and when $\theta = \frac{\pi}{2}$, that is, when $y = \frac{\pi a}{2}$, and $x = a$, there is a point of inflexion; see fig. 41.

In a similar manner it may be shewn that the logarithmic curve and the catenary are both convex downwards.

243.] And now let us consider the results of the preceding Articles from a geometrical point of view; and with this object let us examine fig. 56; wherein a curve is drawn, the infinitesimal arcs of which are assumed to be infinitesimal straight lines, and which in the figure are infinitely magnified. The curve is imagined and placed so that $\frac{d^2y}{dx^2}$ may be positive.

Let $y = f(x)$ be the equation to the curve, OK $= x$, KP $= y$, and let us consider x to be equicrescent, so that KL $=$ LM $=$ MN $= \ldots\ldots = dx$, and $d^2x = d^3x = \ldots\ldots = 0$; and let P, Q, R, S be points on the curve corresponding to the successive values of x; and if we conceive the successive elements of the curve, PQ, QR, RS, to be infinitely magnified, then they are such straight lines as we have imagined ds to be in Art. 218. In the same manner then as in Art. 68,

$$f(x) = y = \text{KP},$$
$$f(x+dx) = y+dy = \text{LQ},$$
$$f(x+2dx) = y+2dy+d^2y = \text{MR},$$
$$f(x+3dx) = y+3dy+3d^2y+d^3y = \text{NS},$$

and so on; whence, by subtraction, we have

$$dy = \text{LQ}-\text{KP} = \text{QU} = \text{XV} = \text{TW};$$

and $\text{MR} = \text{MZ}+\text{RZ}$: but $\text{MZ} = \text{MV}+2\text{VX} = y+2dy$;

$\therefore$ substituting for MR from above, we have

$$y+2dy+d^2y = y+2dy+\text{RZ},$$
$$\therefore\ d^2y = \text{RZ}.$$

As we have not deduced any geometrical properties from d^3y, it is unnecessary to do more than to shew what line is represented by it. From above we have

$$\text{NS} = y+3dy+3d^2y+d^3y:$$

but $\text{NW} = y,\ \text{WG} = 3dy,\quad \text{GE} = 2d^2y,$

$$\therefore\ \text{SE} = d^2y+d^3y,$$
$$= \text{RZ}+d^3y;$$
$$\therefore\ d^3y = \text{SE}-\text{RZ}.$$

Hence it is manifest that $\frac{dy}{dx} = \frac{\text{QU}}{\text{PU}} = \tan \text{QPU} = \tan \text{PJO}$ = the trigonometrical tangent of the angle made with axis of x by the tangent to the curve; and therefore if at any point on a curve $\frac{dy}{dx} = 0$, y does not increase as we pass from one point to the next consecutive point, and therefore the element of the curve PQ is along the line PU, and is parallel to the axis of x. Similarly, whenever $\frac{dy}{dx} = \infty$, the element PQ is perpendicular to PU, and therefore parallel to the axis of y.

And since $d^2y = \text{RZ}$, it is plain that d^2y represents the deflexion of the curve from the tangent line: and therefore if $d^2y = 0$, three consecutive points are in the same straight line, and the curve has for those three points become straight; and if d^2y is positive, the line RZ is to be measured up from the tangent, and the curve lies above the tangent: but if d^2y is negative, it is to be measured downwards, and the curve lies below the tangent; if therefore d^2y is positive on both sides of the point P, the curve is convex downwards, and is such as is drawn in fig. 53, and if d^2y is negative on both sides of the point (x, y), then the curve is concave downwards as in fig. 54; and if d^2y changes its sign at the point by passing through 0 or ∞, the curve is above the tangent on one side of the point, and below it on the other, and therefore there is a point of inflexion.

Hence we learn the geometrical meaning of the process of differentiation; it implies a passage from one point of a curve to the next consecutive point; and, as often as we differentiate, we pass to successive points, and obtain expressions which represent deflexions from straight lines, and so on.

Thus, by means of one differentiation, we consider the curve with respect to two points on it; by two differentiations we consider the curve at three points, and so on. More will be said hereafter on the properties of curves under this mode of considering them; and especially in Chapters XII and XIII with respect to several successive points being common to two or more curves.

244.] Let us however consider the direction of curvature and the conditions of points of inflexion when the equation to the curve is given in the implicit form, and in the more general forms, (48) and (49), of Arts. 207 and 208.

First, let the equation to the curve be given in the form (48) of Art. 207; viz.

$$\mathrm{F}(x, y) = u_n + u_{n-1} + \dots + u_1 + u_0 = 0. \tag{108}$$

then by equation (122), Art. 84, x being equicrescent,

$$\frac{d^2y}{dx^2} = \frac{-\left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d\mathrm{F}}{dy}\right)^2 + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)\left(\frac{d\mathrm{F}}{dx}\right)\left(\frac{d\mathrm{F}}{dy}\right) - \left(\frac{d^2\mathrm{F}}{dy^2}\right)\left(\frac{d\mathrm{F}}{dx}\right)^2}{\left(\frac{d\mathrm{F}}{dy}\right)^3}; \tag{109}$$

and therefore at all points at which the right-hand member of this equation is positive, the curve is convex downwards; and at all points for which it is negative, the curve is concave downwards. And if at any point on the curve

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d\mathrm{F}}{dy}\right)^2 - 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)\left(\frac{d\mathrm{F}}{dx}\right)\left(\frac{d\mathrm{F}}{dy}\right) + \left(\frac{d^2\mathrm{F}}{dy^2}\right)\left(\frac{d\mathrm{F}}{dx}\right)^2 = 0, \tag{110}$$

and changes sign, and if at the same point the curve is not parallel to the axis of y so that $\left(\frac{d\mathrm{F}}{dy}\right)$ does not vanish, then there is a point of inflexion.

The condition therefore for a point of inflexion is primarily (110). Now this equation is in its present form evidently of $3n-4$ dimensions in terms of x and y: it expresses therefore a curve of that order: the intersections of which with (108) are the points of inflexion: and, as (108) is of n dimensions, it follows that the number of points of inflexion on a curve of the nth degree cannot exceed $n(3n-4)$.

245.] And we can shew that $2n$ of these are at an infinite distance, so that the number of points of inflexion on a curve within a finite distance does not exceed $3n(n-2)$. Let $\mathrm{U}_n = 0$ be the equation of all the asymptotes of (108) determined by the process of Art. 239; then if $\mathrm{U}_{n-2}, \mathrm{U}_{n-3}, \dots \mathrm{U}_0$ are homogeneous functions of $n-2, n-3, \dots 0$ dimensions respectively, equation (108) may be expressed in the form

$$\mathrm{F}(x,y) = \mathrm{U}_n + \mathrm{U}_{n-2} + \dots + \mathrm{U}_1 + \mathrm{U}_0 = 0; \tag{111}$$

the terms of $n-1$ dimensions being contained in U_n.

Let us take the equation in this form to be the subject of the operations which are indicated in (110). As $\mathrm{U}_n = 0$ is the equation of all the rectilinear asymptotes of the curve, U_n is the product of n linear factors, each of which is of the form $ax+by+c=0$; so that

$$\mathrm{U}_n = (a_1x+b_1y+c_1)(a_2x+b_2y+c_2)\dots(a_nx+b_ny+c_n); \tag{112}$$

for the sake of brevity let us express this equation in the form

$$\mathrm{U}_n = \mathrm{a}_1\mathrm{P}, \tag{113}$$

where $\mathrm{a}_1 = a_1x+b_1y+c_1$, and P is the product of the other $n-1$ linear factors. Now if we take of this the x- and y-partial derived-functions,

$$\left(\frac{d^2\mathrm{U}_n}{dx^2}\right) = 2a_1\left(\frac{d\mathrm{P}}{dx}\right) + \mathrm{a}_1\left(\frac{d^2\mathrm{P}}{dx^2}\right),$$

$$\left(\frac{d^2\mathrm{U}_n}{dx\,dy}\right) = a_1\left(\frac{d\mathrm{P}}{dy}\right) + b_1\left(\frac{d\mathrm{P}}{dx}\right) + \mathrm{a}_1\left(\frac{d^2\mathrm{P}}{dx\,dy}\right),$$

$$\left(\frac{d^2\mathrm{U}_n}{dy^2}\right) = 2b_1\left(\frac{d\mathrm{P}}{dy}\right) + \mathrm{a}_1\left(\frac{d^2\mathrm{P}}{dy^2}\right);$$

$$\therefore\ \left(\frac{d^2\mathrm{U}_n}{dx^2}\right)\left(\frac{d\mathrm{U}_n}{dy}\right)^2 - 2\left(\frac{d^2\mathrm{U}_n}{dx\,dy}\right)\left(\frac{d\mathrm{U}_n}{dx}\right)\left(\frac{d\mathrm{U}_n}{dy}\right) + \left(\frac{d^2\mathrm{U}_n}{dy^2}\right)\left(\frac{d\mathrm{U}_n}{dx}\right) = \mathrm{a}_1\mathrm{Q},$$

where Q is the symbol for a quantity which it is unnecessary to express at length; but a_1 is a factor of the result; and for a similar reason every other linear factor of the right-hand member of (112) will enter into the result; and (110) will take the form

$$\text{U}_n \text{V}_{2n-4} + \text{V}_{3n-6} = 0; \qquad (114)$$

where V_{2n-4} and V_{3n-6} are two functions of x and y of $2n-4$ and $3n-6$ dimensions respectively. Now the points of inflexion are the points common to the two curves whose equations are (111) and (114); and if we eliminate U_n between these two, the resulting equation will be of $3n-6$ dimensions; that is, the number of points of intersection of this curve with the original curve will be only $3n(n-2)$, and this will be the number of the points of inflexion at a finite distance. The $2n$ other points of inflexion are at an infinite distance; for $\text{U}_n = 0$ is the equation of the n rectilinear asymptotes of the curves (108) or (111) and (114): both curves therefore have the same n asymptotes, and therefore there are $2n$ points at infinity common to the two curves. Hence it appears that in conics there is no point of inflexion at a finite distance, but there may be four points of inflexion at infinity; of these we have instances in the hyperbola, where the asymptotes are intersected by the infinite branches. Again, curves of the third order may have fifteen points of inflexion, of which six are at infinity, and of the remaining nine, one at least must be real.

246.] As the curve whose equation is (110) has many important properties in reference to not only points of inflexion but also other singular points on the original curve, so it has been found necessary to give it a particular name: and as its use was first indicated by M. Otto Hesse of Kœnigsberg*, so it has been called by Mr. Sylvester the Hessian of the curve (108); we shall refer to it hereafter by that name.

If the equation to the original curve is expressed in terms of trilinear coordinates, in the form (49), Art 207, the equation to the tangent is, see Art. 216,

$$\xi \left(\frac{d\text{F}}{dx}\right) + \eta \left(\frac{d\text{F}}{dy}\right) + \zeta \left(\frac{d\text{F}}{dz}\right) = 0. \qquad (115)$$

At a point of inflexion the position of the tangent does not change, as we pass from the second to the third consecutive point on the curve; that is, the tangent at a point of inflexion is stationary; therefore the ratios $\xi : \zeta$ and $\eta : \zeta$ do not vary, if x, y, z vary; let us take the x-, y-, z-partial derived-functions of (115) and we have

---

* See Crelle, Vol. XXVIII, p. 104.

$$\left.\begin{aligned} \xi\left(\frac{d^2\mathrm{F}}{dx^2}\right) + \eta\left(\frac{d^2\mathrm{F}}{dy\,dx}\right) + \zeta\left(\frac{d^2\mathrm{F}}{dz\,dx}\right) &= 0, \\ \xi\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) + \eta\left(\frac{d^2\mathrm{F}}{dy^2}\right) + \zeta\left(\frac{d^2\mathrm{F}}{dz\,dy}\right) &= 0, \\ \xi\left(\frac{d^2\mathrm{F}}{dx\,dz}\right) + \eta\left(\frac{d^2\mathrm{F}}{dy\,dz}\right) + \zeta\left(\frac{d^2\mathrm{F}}{dz^2}\right) &= 0 \end{aligned}\right\}; \tag{116}$$

whence eliminating ξ, η, ζ, we have

$$\begin{aligned} &\left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d^2\mathrm{F}}{dy\,dz}\right)^2 + \left(\frac{d^2\mathrm{F}}{dy^2}\right)\left(\frac{d^2\mathrm{F}}{dz\,dx}\right)^2 + \left(\frac{d^2\mathrm{F}}{dz^2}\right)\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)^2 \\ &\quad - \left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d^2\mathrm{F}}{dy^2}\right)\left(\frac{d^2\mathrm{F}}{dz^2}\right) - 2\left(\frac{d^2\mathrm{F}}{dy\,dz}\right)\left(\frac{d^2\mathrm{F}}{dz\,dx}\right)\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) = 0; \end{aligned} \tag{117}$$

and this is the Hessian of $\mathrm{F}(x, y, z) = 0$. As to its dimensions it is to be observed that as each of the second partial derived functions is of the $(n-2)$th degree, so is the equation of the $3(n-2)$th degree; and the Hessian curve therefore has $3n(n-2)$ points common with the original curve: hence we infer that $3n(n-2)$ is the number of points on a curve at which the tangent is stationary.

247.] Stationary tangents are also a particular form of certain straight lines which touch a curve at many different points. Such lines are called generally Multiple Tangents; of which particular species are double, triple, ... tangents, which are so named according as they touch the curve at two, three, ... points. Let us consider the most simple case in which a curve exhibits this property; that, viz. in which the axis of x is a double tangent. In the equation to the curve let $y = 0$; then, if the curve is of the nth degree in terms of x, we have an expression of the form,

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots \pm p_{n-1} x \mp p_n = 0; \tag{118}$$

which may be resolved into its factors; and we have

$$(x-a_1)(x-a_2)\dots(x-a_n) = 0. \tag{119}$$

If all the roots of this equation are real and unequal the curve cuts the axis of x in n different points, and the curve is such as that drawn in fig. 23. But let us suppose two of these roots to be equal; then two of the points of section with the axis of x coincide, and the axis of x touches the curve at the point. Let two pairs of roots be equal; that is, suppose $a_1 = a_2$, and $a_3 = a_4$, so that (82) becomes

$$(x-a_1)^2(x-a_3)^2(x-a_5)\dots(x-a_n) = 0; \qquad (120)$$

then the axis of x touches the curve at $x = a_1$, and at $x = a_3$; and we have a double tangent. Similarly, if three pairs of roots are equal, the axis of x is a triple tangent: and similarly as many pairs of equal roots as there are, such is the order of the multiplicity of the tangent. It is evident that a double tangent cannot occur in a curve, the order of which is lower than the fourth: and a triple tangent cannot be in a curve whose order is lower than the sixth; and so on for multiple tangents of a higher order.

Suppose, however, three roots of (118) or (119) to be equal; say $a_1 = a_2 = a_3$; then

$$(x-a_1)^3(x-a_4)\dots(x-a_n) = 0; \qquad (121)$$

and thus the axis of x meets the curve in three consecutive points; and thus two consecutive tangents, those, viz. which pass through the first and second, and the second and third points become coincident; for this reason the tangent to the curve at such a point is called a stationary tangent, and the point is evidently a point of inflexion; for if, in fig. 23, A_1, A_2, A_3 become consecutive points, the intermediate points of maximum and minimum ordinates will become coincident with them; the curve will have two tangents coincident with the axis of x, and the curve having been below the axis of x will intersect it, and pass to the upper side of it. It is evident also that under these circumstances $\frac{dy}{dx}$ becomes stationary, and therefore $\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$.

As a straight line cannot generally, omitting infinite values, cut in three points a curve of any degree lower than the 3rd, so points of inflexion do not generally exist on curves of degrees lower than the third. I say generally, because the hyperbola, which is of the second degree, has a rectilinear asymptote, with which the curve at infinity coincides, and in this respect more than two points of the curve are on the tangent, and there is a point of inflexion.

If four roots of (118) are equal then we have

$$(x-a_1)^4(x-a_5)\dots(x-a_n) = 0: \tag{122}$$

and thus the axis of x meets the curve in four consecutive points, and three tangents become coincident. A point at which these circumstances occur is called a point of undulation. Into this subject however it is unnecessary to enter further, as the principle of the preceding inquiry can be easily extended to any other number of points. I may observe however that if the number of consecutive points common to the curve and the tangent is even, the curve is on the same side of the tangent on both sides of the point of contact, and there is no change of curvature; whereas, if the number of points which are common to the tangent and the curve is odd, the curvature changes, and the curve, having been on one side of the tangent, intersects it, and comes to the other; that is, the curve having been convex, say, downwards, becomes concave, or *vice versâ*.

SECTION 4.—*On multiple points of plane curves.*

248.] Thus far we have considered the geometrical properties which belong to the first and second derived-functions of an equation to a curve when they have determinate values; but suppose that at any point on a curve, $\frac{dy}{dx}$ assumes the indeterminate form $\frac{0}{0}$, a question occurs, and which has to be inquired into, What is the meaning of such indeterminateness? In the following discussion we shall find it most convenient to use the implicit form of the equation to the curve; viz.

$$u = \mathrm{F}(x, y) = c; \tag{123}$$

whence we have

$$\frac{dy}{dx} = -\frac{\left(\frac{d\mathrm{F}}{dx}\right)}{\left(\frac{d\mathrm{F}}{dy}\right)}. \tag{124}$$

Suppose that at a certain point on the curve $\frac{dy}{dx} = \frac{0}{0}$; the conditions of which are that $\left(\frac{d\mathrm{F}}{dx}\right) = 0$, $\left(\frac{d\mathrm{F}}{dy}\right) = 0$; and let the expression be evaluated according to the method of Art. 139; then, see (45) of that Article, we have

$$\frac{dy}{dx} = -\frac{\left(\frac{d^2F}{dx^2}\right)dx + \left(\frac{d^2F}{dx\,dy}\right)dy}{\left(\frac{d^2F}{dy^2}\right)dy + \left(\frac{d^2F}{dx\,dy}\right)dx},$$

$$= -\frac{\left(\frac{d^2F}{dx^2}\right) + \left(\frac{d^2F}{dx\,dy}\right)\frac{dy}{dx}}{\left(\frac{d^2F}{dy^2}\right)\frac{dy}{dx} + \left(\frac{d^2F}{dx\,dy}\right)}. \qquad (125)$$

And suppose that this quantity does not become $\frac{0}{0}$ at the point in question; that is, suppose that all the second partial differential coefficients do not vanish or become infinite; then, multiplying and reducing, we have

$$\left(\frac{d^2F}{dy^2}\right)\frac{dy^2}{dx^2} + 2\left(\frac{d^2F}{dx\,dy}\right)\frac{dy}{dx} + \left(\frac{d^2F}{dx^2}\right) = 0, \qquad (126)$$

which is a quadratic in $\frac{dy}{dx}$; and gives therefore two values for $\frac{dy}{dx}$; and thereby shews that two branches of the curve pass through the point. This is called a *double point*, admitting of several varieties, according as the roots of (126) are real and unequal, real and equal, or imaginary; and according as the curve extends or not in the plane of reference on both sides of the point in question. Now the roots of (126) are

$$\left.\begin{array}{l}\text{real and unequal,}\\ \text{real and equal,}\\ \text{imaginary,}\end{array}\right\}\text{ according as }\left(\frac{d^2F}{dy^2}\right)\left(\frac{d^2F}{dx^2}\right)\text{ is }\left\{\begin{array}{c}<\\=\\>\end{array}\right\}\left(\frac{d^2F}{dx\,dy}\right)^2. \qquad (127)$$

Let us first consider the case of the two roots being unequal; then we have,

$$\frac{dy}{dx} = \frac{-\left(\frac{d^2F}{dx\,dy}\right) \pm \left\{\left(\frac{d^2F}{dx\,dy}\right)^2 - \left(\frac{d^2F}{dx^2}\right)\left(\frac{d^2F}{dy^2}\right)\right\}^{\frac{1}{2}}}{\left(\frac{d^2F}{dy^2}\right)}. \qquad (128)$$

Now although in algebraical expressions, so far as our knowledge goes, a quantity does not pass from + to − or *vice versâ* without passing through zero or infinity, yet we may be needlessly restricting the geometrical properties of equations, and our means of interpreting them, if we assume that in all equations, transcendental as well as algebraical, no change of sign can take place unless the quantity passes through zero or infinity. I shall suppose therefore that the sign of $\left(\frac{d^2F}{dx\,dy}\right)^2 - \left(\frac{d^2F}{dx^2}\right)\left(\frac{d^2F}{dy^2}\right)$ may be changed and yet that the quantity may not become zero or infinity; that is, that the two values of $\frac{dy}{dx}$ having been real may become imaginary, and *vice versâ*, without having become equal.

If the two values of $\frac{dy}{dx}$ are real and unequal, and are so on both sides of the point in question, then the curve extends in the plane of reference on both sides, as at P_0 in fig. 57, and has the same tangents, and the point is called a *real double point*. If the two values of $\frac{dy}{dx}$ are unequal, and are real on one side of the point in question and imaginary on the other, the point is such as that indicated in fig. 58: the curve is in the plane of reference on one side, and in another plane on the other, and the tangents are similarly so; such a point is called a *salient point:* evidently no algebraical equation can express a curve with a salient point, because the curve will be discontinuous at the point. If the two values of $\frac{dy}{dx}$ are imaginary on both sides of the point, that is, if the two roots of (126) are imaginary, then the curve is out of the plane of reference on both sides of the point, and thus two branches of the curve, both of which are out of the plane of reference, pierce the plane at the point in two different directions, and the point is called a *conjugate point.*

Again, if two roots of (126) are equal, we have

$$\frac{dy}{dx} = -\left\{\frac{\left(\frac{d^2F}{dx^2}\right)}{\left(\frac{d^2F}{dy^2}\right)}\right\}^{\frac{1}{2}}; \tag{129}$$

and thus two branches pass through the point and have the same tangent.

If these branches are in the plane of reference on both sides of the point, the curve is such as one or the other of those delineated in fig. 59, and the points where the curve meets the tangent are called points of *osculation,* and by French writers *points d'embrassement*; and if they are in the plane of reference on one side of the point, and on the other side pass out of it, then the curve at the point is such as one or other of those drawn in fig. 60, where the dotted lines indicate the course of the curve out of the plane of the paper, and the points are called *cusps*, and by French writers *points de rebroussement*. That in the fig. 60, marked (a), is called a *ceratoid* cusp, or a cusp of the first species, in which the two branches touch the common tangent on opposite sides of it; that marked (β) is called a *ramphoid cusp*, or cusp of the second species, in which the two branches touch the common tangent on the same side. Cusps in other positions are shewn in fig. 61.

These several subordinate varieties of double points must be distinguished by examining the form and nature of the equation to the curve, and of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, when $x = x_0 \pm h$, $y = y_0 \pm k$, h and k being taken very small, and (x_0, y_0) being the point of the curve in question; as, for example, in fig. 60, if when $x = x_0 + h$, $\frac{d^2y}{dx^2}$ is positive for one and negative for the other branch of the curve, and when $x = x_0 - h$, $\frac{dy}{dx}$ is affected with $\sqrt{-}$, the curve is of the form drawn in fig. (a); but if $\frac{d^2y}{dx^2}$ is positive for both branches of the curve, and the curve is out of the plane of the paper when x is less than x_0, then it is such as that delineated in fig. (β).

249.] Of these double points we shall give instances, and apply to particular cases the principles of the foregoing method, without adapting them directly to the general forms.

Ex. 1. Examine the nature of the point at the origin of the lemniscata whose equation is, see Art. 197 and fig. 37,

$$(x^2+y^2)^2 = a^2(x^2-y^2).$$

Differentiating we have

$$4(x^2+y^2)(x\,dx+y\,dy) = 2a^2(x\,dx-y\,dy);$$

$$\therefore \quad \frac{dy}{dx} = \frac{a^2x-2x(x^2+y^2)}{a^2y+2y(x^2+y^2)} = \frac{0}{0}, \text{ when } x=0, \text{ and } y=0,$$

$$= \frac{a^2\,dx - \dots}{a^2\,dy + \dots} = \frac{dx}{dy}, \quad - \; - \; - \; - \; - \; .$$

$$\therefore \quad \frac{dy^2}{dx^2} = 1, \qquad \frac{dy}{dx} = \pm 1;$$

that is, two branches of the curve pass through the origin, cutting the axis of x at angles respectively of 45° and 135°, and which are in the plane of reference on both sides of the point; and therefore all the characteristics of a true double point are satisfied.

Ex. 2. Determine the nature of the point at the origin of the curve, $y^2 = x^2(1-x^2)$.

$$\therefore \quad \frac{dy}{dx} = \frac{x-2x^3}{y} = \frac{0}{0}, \text{ at the origin,}$$

$$= \frac{(1-6x^2)\,dx}{dy} = \frac{dx}{dy}; \text{ - - -}$$

$$\therefore \quad \frac{dy^2}{dx^2} = 1, \qquad \frac{dy}{dx} = \pm 1;$$

and two branches of the curve pass through the origin, which are inclined to the axis of x at angles respectively of 45° and 135°; and there is at the origin a true double point; see fig. 62.

Ex. 3. Determine the nature of the point at the origin of the Folium of Descartes, the equation to which is

$$x^3 - 3axy + y^3 = 0.$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} = \frac{0}{0}, \text{ at the origin,}$$

$$= \frac{a\,dy - 2x\,dx}{2y\,dy - a\,dx} = -\frac{dy}{dx}; \text{ - - - - - - -;}$$

therefore $\frac{dy}{dx} = 0$, and $\frac{dy}{dx} = \infty$; and there are two branches passing through the origin, and touching respectively the axes of x and y; see fig. 63.

Ex. 4. Determine the nature of the point at the origin of the curve whose equation is $y^2(a^2-x^2) = x^4$.

$$\frac{dy}{dx} = \frac{2x^3 + xy^2}{y(a^2-x^2)} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{(6x^2+y^2)\,dx + 2xy\,dy}{(a^2-x^2)\,dy - 2xy\,dx} = \frac{0\,dx}{a^2\,dy}, \text{ - - - -}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{0}{a^2}; \qquad \therefore \quad \frac{dy}{dx} = \pm 0;$$

that is, two branches of the curve pass through the origin and touch the axis of x on different sides of it, and are both in the plane of reference.

If the equation to the curve were $y^2(x^2-a^2) = x^4$, the two branches which pass through the origin would touch the axis of x, but both would be out of the plane of reference; see fig. 77.

Ex. 5. Discuss the nature of the point at the origin of the curve whose equation is $y^2 = ax^3$.

$$\frac{dy}{dx} = \frac{3ax^2}{2y} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{6ax\,dx}{2\,dy};$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = 3ax; \qquad \frac{dy}{dx} = \pm(3ax)^{\frac{1}{2}} = 0, \text{ if } x = 0;$$

and is affected with $\pm$, when x is positive, but with $\pm\sqrt{-}$, if x is negative: hence at the origin there is a cusp, both branches of which touch the axis of x, and the curve is out of the plane of the paper when x is negative. The curve is, on account of the form of its equation, called the Semicubical Parabola.

Also since
$$\frac{d^2y}{dx^2} = \pm\frac{3a^{\frac{1}{2}}}{4x^{\frac{1}{2}}},$$

which is positive or negative according as the branch of the curve is above or below the axis of x, the cusp is of the first species; see fig. 64.

Ex. 6. Discuss the nature of the point at the origin of the curve $y^3 = ax^2-x^3$.

$$\frac{dy}{dx} = \frac{2ax-3x^2}{3y^2} = \frac{0}{0}, \text{ at the origin,}$$

$$= \frac{(2a-6x)\,dx}{6y\,dy};$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{a}{3y}, \qquad \frac{dy}{dx} = \pm\left(\frac{a}{3y}\right)^{\frac{1}{2}} = \pm\infty, \text{ if } y = 0.$$

Hence the curve has two branches at the origin touching the axis of y, but as $\frac{dy}{dx}$ is affected with $\pm\sqrt{-}$, when y is negative, it shews that at the origin there is a cusp of the first species, and such as is drawn in fig. 65.

Ex. 7. The equation to a curve is $y^2 = x(x+a)^2$; determine the nature of the point where $x = -a$, and $y = 0$.

$$\frac{dy}{dx} = \frac{3x^2+4ax+a^2}{2y} = \frac{0}{0}, \text{ when } x = -a, \text{ and } y = 0,$$

$$= \frac{(6x+4a)\,dx}{2\,dy};$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = -a, \qquad \frac{dy}{dx} = \pm\sqrt{-a};$$

that is, there is a conjugate point, both branches of the curve being out of the plane of the paper, and piercing it at the point; and their directions making, with the axis of x, angles whose tangents are $\pm\sqrt{a}$.

250.] The equations of many curves admit of being reduced to an explicit form, which is well adapted to exhibit the peculiarities of cusps of both species.

Suppose that the equation to a curve can be put into the form

$$y = f(x) \pm \phi(x), \tag{130}$$

of which $f(x)$ is possible for all values of x through which we consider it, and $\phi(x)$ is possible for some and impossible for others; and, to fix our thoughts, suppose that $\phi(x)$ is imaginary or real, according as x is less or greater than a. The curve whose equation is $y = f(x)$ is aptly called the *diametral* curve of (130), the ordinates of (130) being equal to $f(x)$ increased and diminished by the same quantity, viz. $\phi(x)$. Then, if $\phi(x)$ is such that $\phi(a) = 0$ and $\phi'(a) = 0$, the curve (130) has, when $x = a$, a common point, and is coincident in direction, with $y = f(x)$; but as two branches unite at the point, and are distinct when $x = a+h$, and affected with $\pm\sqrt{-}$, when $x = a-h$, we have a cusp of the first or second species, according as the curvature is turned in opposite or in the same directions. The following examples will illustrate the method.

Ex. 1.
$$(y-a-x)^2 = (x-c)^5.$$
$$y = a+x \pm (x-c)^{\frac{5}{2}},$$
$$\frac{dy}{dx} = 1 \pm \frac{5}{2}(x-c)^{\frac{3}{2}},$$
$$\frac{d^2y}{dx^2} = \pm\frac{15}{4}(x-c)^{\frac{1}{2}}.$$

Hence the diametral line is that whose equation is $y = a+x$, and as y is affected with $\pm\sqrt{-}$, for all values of x less than c, it follows that the curve is in or out of the plane of reference, according as x is greater or less than c; and as $\frac{dy}{dx} = 1$ when $x = c$, it is plain that at that point both branches touch a common tangent, one being above and the other below it; and as $\frac{d^2y}{dx^2}$ is $+$ when x is greater than c, the cusp is of the first species; see fig. 66.

Ex. 2.
$$(y-x^2)^2 = ax^5;$$
$$y = x^2 \pm a^{\frac{1}{2}} x^{\frac{5}{2}},$$
$$\frac{dy}{dx} = 2x \pm \frac{5a^{\frac{1}{2}}}{2} x^{\frac{3}{2}},$$
$$\frac{d^2y}{dx^2} = 2 \pm \frac{15a^{\frac{1}{2}}}{4} x^{\frac{1}{2}}.$$

From the above equations it appears that the diametral curve is the parabola whose equation is $y = x^2$; that the curve is in the plane of reference on the positive side of the axis of y, and out of it on the negative side; that there is a cusp of the second species at the origin, since $\frac{d^2y}{dx^2}$ is positive for both branches; see fig. 67.

Ex. 3.
$$y = a + bx + cx^2 \pm x^{\frac{3}{2}}.$$
$$\frac{dy}{dx} = b + 2cx \pm \frac{3}{2} x^{\frac{1}{2}},$$
$$\frac{d^2y}{dx^2} = 2c \pm \frac{3}{4x^{\frac{1}{2}}}.$$

The diametral curve is plainly a parabola, and the curve is in or out of the plane of reference, according as x is positive or negative; there is a cusp of the first species at the point where the diameter cuts the axis of y, and the tangent at the point makes, with the axis of x, an angle whose tangent is b; see fig. 68.

251.] For a double point on a curve we have the two following conditions,
$$\left(\frac{dF}{dx}\right) = 0, \text{ and } \left(\frac{dF}{dy}\right) = 0, \qquad (131)$$
at the point. Let us suppose the equation to the curve to be

of the *n*th degree, and of the form (45), Art. 207; then each of the equations (131) is of $n-1$ dimensions, and therefore represents a curve of that degree; and if the points of intersection of these two curves are on the original curve, they are the double points of that curve: hereby we have a limit to the number of double points which can be on a curve. The two curves (131) cannot intersect in more than $(n-1)^2$ points, and therefore the number of double points cannot be greater than this; it may however be less because all the points of intersection may not be on the original curve: thus, in a conic for example, $\left(\frac{dF}{dx}\right) = 0$, and $\left(\frac{dF}{dy}\right) = 0$, are the equations to two diameters, and they intersect in the centre which is not on the curve, unless the conic can be broken up into two straight lines; in which case the centre is a double point. Thus the point of intersection of the two equations (131) in the case of a conic is not ordinarily a double point.

As two branches of a curve intersect at a double point, so a double point is two coincident points; and therefore if a straight line passes through a double point, this passage is equivalent to its passing through two points of a curve: hence a proper curve of the third degree cannot have more than one double point, because, if it had two, a straight line might be drawn through both, and would thus pass through four points on the curve of the third order, and this is impossible for a proper curve of that order, see Art. 210; for a proper curve, I say; because it is possible, if the curve can be broken up into a straight line and a conic, since then the two points where the line cuts the curve are double. Similarly a proper curve of the fourth order cannot have four double points; because if it had, through these and any fifth point on the curve a conic might be drawn, and as each double point is equivalent to two points, this conic would pass through nine points on the curve; and thus a conic and a curve of the fourth order would intersect in more than eight points, which is impossible, see Article 210. And in general a curve of the *n*th degree cannot have more than $\frac{(n-1)(n-2)}{2}$ double points*; for suppose it to have one more: then through these

---

* See Salmon, Higher Plane Curves, page 31; and Annales de Mathematiques, Tome X, p. 95, Paris, 1851.

$\frac{(n-1)(n-2)}{2}+1$ double points and through $n-3$ other points of the curve, a curve of the $(n-2)$th order may be drawn; p. 321 eq. (47) and this must be considered to meet the original curve in $(n-1)(n-2)+2+n-3$, that is, in $n(n-2)+1$ points, which is impossible if the given curve is a proper curve.

In Art. 223 it has been shown that generally $n(n-1)$ tangents can be drawn to a curve of the nth degree from a given point: in the proof of this theorem it is assumed that every line passing through two consecutive points on the curve is a tangent; a line however passing through a double point passes through two consecutive points on the curve and in two different ways, viz. through the point common to both branches and the consecutive point on each branch; and yet is not in either way a tangent in the meaning in which the word is commonly used; so that the preceding number of tangents must be diminished by 2 for every double point on the curve. If therefore a curve has δ double points, the number of tangents which can be drawn to it from a given point is $n(n-1)-2\delta$; and this is therefore the class of the curve.

252.] The number of cusps on a curve of the nth degree cannot be greater than $2n(n-2)$; and all these are on a curve of the $2(n-2)$th order. For the condition of a cusp on a curve is, see (127), the equation

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d^2\mathrm{F}}{dy^2}\right)-\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)^2=0; \tag{132}$$

which is manifestly of $2(n-2)$ dimensions; all cusps therefore are found on a curve of this order: and as the points of intersection of this curve with the original curve cannot be more than $2n(n-2)$, so the number of cusps on a curve of the nth order cannot be greater than $2n(n-2)$. Indeed, as a cusp is a double point, the number cannot be greater than $\frac{(n-1)(n-2)}{2}$, as is shewn in the preceding Article.

And for every cusp on a curve, the number of tangents which can be drawn to the curve from a given point must be diminished by 3. Because at a cusp three points of the curve become coincident; and as these may be taken in pairs in three different ways, so may three lines be drawn passing through these several pairs; and as these lines are not tangents in the ordinary meaning of the word, the number of tangents which can be drawn to the curve from a given point is to be diminished by three for each cusp.

Hence if a curve has κ cusps, the number of tangents which can be drawn to it from a given point, and therefore the class of the curve, is

$$n(n-1)-3\kappa. \tag{133}$$

If therefore a curve has δ double points, and κ cusps, and if its class is m; then

$$m = n(n-1)-2\delta-3\kappa. \tag{134}$$

253.] From (126) it appears that the equation which gives the directions of the tangents at a double point is

$$\left(\frac{d^2\mathrm{F}}{dy^2}\right)\frac{dy^2}{dx^2}+2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)\frac{dy}{dx}+\left(\frac{d^2\mathrm{F}}{dx^2}\right)=0; \tag{135}$$

hence substituting $\frac{dy}{dx}$ from (124), we have

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d\mathrm{F}}{dy}\right)^2-2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)\left(\frac{d\mathrm{F}}{dx}\right)\left(\frac{d\mathrm{F}}{dy}\right)+\left(\frac{d^2\mathrm{F}}{dy^2}\right)\left(\frac{d\mathrm{F}}{dx}\right)^2=0, \tag{136}$$

which is the Hessian of the curve in terms of x and y, see Art. 246, and is the same equation as that which gives the points of inflexion; see Art. 244. Hence it appears that the Hessian passes through all the double points of a curve as well as through all its points of inflexion.

*Now the relation of a curve to its Hessian deserves closer consideration. In the first place, a double point on a curve is also a double point on the Hessian; and the tangents to the two are identical at the point. This may be shewn most easily if we take the double point to be the origin and the tangents at it to be the coordinate axes; in which case the equation (48), Art. 207, takes the form

$$b_2xy+u_3+u_4+\ldots+u_n=\mathrm{F}(x,y)=0; \tag{137}$$

and substituting in (136) and taking only the terms involving the lowest powers of x and y, the Hessian becomes

$$-2b_2^3xy+\text{terms of three and higher dimensions}=0; \tag{138}$$

which evidently passes through the origin, and at it touches the axes of x and y. Thus the original curve and its Hessian have the same tangents at the common double point. But when two

---

* See an Article by Mr. Cayley in Crelle's Journal, Vol. XXXIV, p. 43.

curves have a common double point, this is equivalent to four points of intersection being coincident; and if the tangents to the branches are also common, this is equivalent to two more points of intersection being coincident; and therefore a double point with tangents common to the curve and its Hessian is equivalent to six coincident points of intersection. Now the number of points of inflexion is $3n(n-2)$, being the number of points of intersection at a finite distance of a curve and its Hessian; and by as many as these points are diminished, by so many is the number of the points of inflexion diminished. Therefore if a curve has δ double points, the number of its points of inflexion cannot be more than

$$3n(n-2)-6\delta. \tag{139}$$

Also again, if the original curve has a cusp, three branches of the Hessian will pass through the point, the tangents of two of which will coincide with the tangent of the original curve at the cusp. Let us take the origin at the cusp, and let the tangent be the axis of y: in which case (48), Art. 207, takes the form,

$$a_2 x^2+u_3+u_4+\ldots+u_n = \mathrm{F}(x,y) = 0; \tag{140}$$

and substituting in the Hessian, and retaining only the terms involving the lowest powers of the variables x and y, the Hessian becomes

$$4a_2^2 x^2\left(\frac{d^2u_3}{dy^2}\right) + \text{terms of four and higher dimensions} = 0. \tag{141}$$

Hence the Hessian has three branches passing through the origin; to two of which the axis of y is the tangent, and to the third the tangent is $\left(\frac{d^2u_3}{dy^2}\right) = 2c_3x+6d_3y = 0$. But when two curves have a common point, through which two branches of one and three branches of the other pass, this point is equivalent to six coincident points of intersection; and as two branches of one have the same tangent as two branches of the other, two more points are common to the two curves; so that this cusp common to the curve and its Hessian is equivalent to eight coincident points of intersection. Hence if a curve has δ double points, and κ cusps, and if i is the number of its points of inflexion,

$$i = 3n(n-2)-6\delta-8\kappa. \tag{142}$$

I must in conclusion observe, that M. Hesse has shewn, Crelle, Vol. XXVIII, that in curves of the third order, the points of inflexion on the original curve are also points of inflexion on the Hessian: but in other curves the points of inflexion are not generally points of inflexion of the Hessian.

254.] Lastly, let us consider the subject of double points in curves whose equations are given in terms of three variables x, y, z, and in the form (49), Art. 208. Now the equation to the tangent, see (18), Art. 216, is

$$\xi\left(\frac{d\mathrm{F}}{dx}\right)+\eta\left(\frac{d\mathrm{F}}{dy}\right)+\zeta\left(\frac{d\mathrm{F}}{dz}\right)=0; \tag{143}$$

and therefore if at any point on the curve this equation is identically satisfied, that is, if $\left(\frac{d\mathrm{F}}{dx}\right)=\left(\frac{d\mathrm{F}}{dy}\right)=\left(\frac{d\mathrm{F}}{dz}\right)=0$, the direction of the tangent at that point is indeterminate; and to determine it we must pass to the next consecutive point on the curve: that is, in other words, we must take the x-, the y-, and the z-partial differentials of (143): hence we have the three equations (116), Art. 246; and hence arises the condition,

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d^2\mathrm{F}}{dy\,dz}\right)^2+\left(\frac{d^2\mathrm{F}}{dy^2}\right)\left(\frac{d^2\mathrm{F}}{dz\,dx}\right)^2+\left(\frac{d^2\mathrm{F}}{dz^2}\right)\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)^2$$
$$-\left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d^2\mathrm{F}}{dy^2}\right)\left(\frac{d^2\mathrm{F}}{dz^2}\right)-2\left(\frac{d^2\mathrm{F}}{dy\,dz}\right)\left(\frac{d^2\mathrm{F}}{dz\,dx}\right)\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)=0; \tag{144}$$

which is the equation to the Hessian in terms of three variables. And this is the condition that a curve should have a double point. Let us apply it to a conic; that is, investigate the circumstances under which a conic has a double point.

Let the equation to the conic be

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{E}yz+2\mathrm{G}zx+2\mathrm{H}xy=0; \tag{145}$$

then (144) becomes

$$\mathrm{AE}^2+\mathrm{BG}^2+\mathrm{CH}^2-\mathrm{ABC}-2\mathrm{EGH}=0, \tag{146}$$

which is the well-known condition that (145) should break up into two straight lines; and at the point of intersection of these two lines we have the characteristics of a double point. Therefore a proper curve of the second degree does not admit of a double point.

255.] Let us now return to equation (125), Art. 248, and suppose that at the point under discussion, not only $\left(\frac{dF}{dx}\right) = 0$, and $\left(\frac{dF}{dy}\right) = 0$, but also

$$\left(\frac{d^2F}{dx^2}\right) = 0, \quad \left(\frac{d^2F}{dx\,dy}\right) = 0, \quad \left(\frac{d^2F}{dy^2}\right) = 0; \tag{147}$$

then the value of $\frac{dy}{dx}$ again assumes the form $\frac{0}{0}$, and the numerator and denominator of it must be differentiated; in which operation however it is to be borne in mind, see Art. 139, that $\frac{dy}{dx}$ does not vary with x and y near to the point, and is therefore to be considered constant; the true meaning and effect of these successive differentiations being as follows. Several branches of the curve have certain consecutive points in common, and certain elements in common; whilst therefore we are considering the curve, as to its continuation at one or more of these common points, it is indeterminate to which branch of the curve the points and elements belong, and therefore we must pass on from these common points to those contiguous ones which are on different branches; in which case the tangent lines drawn through these become separate for each branch, and the direction of each thereby becomes determined. Let the reader try to draw for himself an infinitely magnified diagram of such points and curves in the same manner as we have drawn fig. 56.

Differentiating therefore the numerator and denominator of the right-hand member of equation (125), and dividing through by dx.

$$\frac{dy}{dx} = -\frac{\left(\frac{d^3F}{dx^3}\right) + 2\left(\frac{d^3F}{dx^2\,dy}\right)\frac{dy}{dx} + \left(\frac{d^3F}{dx\,dy^2}\right)\frac{dy^2}{dx^2}}{\left(\frac{d^3F}{dy^3}\right)\frac{dy^2}{dx^2} + 2\left(\frac{d^3F}{dx\,dy^2}\right)\frac{dy}{dx} + \left(\frac{d^3F}{dx^2\,dy}\right)}; \tag{148}$$

whence, multiplying and reducing,

$$\left(\frac{d^3F}{dy^3}\right)\frac{dy^3}{dx^3} + 3\left(\frac{d^3F}{dx\,dy^2}\right)\frac{dy^2}{dx^2} + 3\left(\frac{d^3F}{dx^2\,dy}\right)\frac{dy}{dx} + \left(\frac{d^3F}{dx^3}\right) = 0; \tag{149}$$

which is a cubic equation in $\frac{dy}{dx}$, and therefore has three roots; shewing that three branches of the curve pass through the point, which is accordingly called a *triple* point; the three branches being all in the plane of reference, or one in and two out of the plane, according as the roots of (149) are all real, or one real and two impossible.

As the criteria of this division lead to a long and complicated expression, it is needless to investigate it; and, moreover, as the determination of the several values of $\frac{dy}{dx}$, corresponding to the several branches of the curve, is not difficult, we shall only add an example.

Ex. 1. To determine the nature of the point at the origin of the curve whose equation is $x^4 - ayx^2 + by^3 = 0$.

$$\frac{dy}{dx} = \frac{4x^3 - 2axy}{ax^2 - 3by^2} = \frac{0}{0}, \text{ at the origin,}$$

$$= \frac{(12x^2 - 2ay)\,dx - 2ax\,dy}{2ax\,dx - 6by\,dy} = \frac{0}{0}, \; - \; - \; - \; -,$$

$$= \frac{24x\,dx^2 - 4a\,dy\,dx}{2a\,dx^2 - 6b\,dy^2} = \frac{-4a\,dy\,dx}{2a\,dx^2 - 6b\,dy^2}, \text{ when } x = y = 0;$$

$$\therefore \quad 6a\frac{dy}{dx} - 6b\frac{dy^3}{dx^3} = 0:$$

whence $$\frac{dy}{dx} = 0, \text{ and } \frac{dy}{dx} = \pm\left(\frac{a}{b}\right)^{\frac{1}{2}};$$

and therefore at the origin there is a triple point, as three branches of the curve pass through it, of which one touches the axis of x, and the other two are inclined to it at angles whose tangents are $\pm\left(\frac{a}{b}\right)^{\frac{1}{2}}$; see fig. 69.

256.] Similarly if all the third partial differential coefficients vanish at the point under discussion, we must differentiate again the numerator and denominator of the right-hand member of (148); by which means we shall obtain a biquadratic expression in $\frac{dy}{dx}$, indicating that four branches of the curve pass through the point, which is therefore called a *quadruple* point.

257.] Such is the general theory of multiple points; of which the analytical character is the vanishing at the point of successive partial differential coefficients of the implicit equation to the curve. That such must vanish, if many branches pass through the same point, may thus be shewn *a priori*.

Let a curve be such that, when $x = a \pm h$, y has many values, or, to borrow language from the theory of equations, the equation formed in powers of y has many unequal roots; but when $x = a$ several of these values of y become equal, say $y = b$; then, in this case, as many roots as become equal which before were unequal, so many branches of the curve pass through the point; and thus in the equation many equal factors will be multiplied together, which produce a factor of the form $(y-b)^n$. By similar reasoning we may prove that at such a point many factors involving x, which at other points are unequal, become equal; and we have a factor of the form $(x-a)^m$, m and n being some numerical quantities greater than 1. Now since differentiation diminishes the exponent of such a quantity only by unity, it is plain that $\left(\frac{dF}{dx}\right)$ will, at the point in question, have a factor of the form $(x-a)^{m-1}$, and therefore will $= 0$. Similarly $\left(\frac{dF}{dy}\right)$ will have a factor of the form $(y-b)^{n-1}$, and will $= 0$ also; and according to the numerical magnitudes of m and n, will be the number of branches passing through the point, and the number of successive partial differential coefficients which $= 0$, for the values $x = a$, $y = b$.

SECTION 5.—*On tracing curves by means of their equations.*

258.] As this treatise is intended in a great measure for didactic purposes, I shall insert at some length an account of the most simple processes by which we can delineate a curve expressed by a given equation: and herein we shall introduce simultaneously and combine the methods which have been in the preceding sections separately investigated for the discovery of singular points of curves. The object of this inquiry is twofold: to give expertness, firstly, in the analysis of an equation and of its derived functions; and, secondly, in the translation of these circumstances into their corresponding geometrical forms. *All* curves however we cannot trace, any more than find the equations to all figures of a character however complicated; the problem is as general as the solution of *all* equations; and therefore what follows is to be taken as an explanation, and as a specimen, of the means we possess of discussing some few simple cases which are for the most part algebraical.

1) If the equation admits of being simplified by a change of origin, or by turning the axes through any angle, or by transforming the equation into its equivalent in terms of polar coordinates, let such a change be effected before we begin the analysis. Thus, for instance, the equation $x^2-2ax+y^2+2by = 0$, admits of being discussed more easily when for x we write $x+a$, and for y, $y-b$: whereby the result becomes $x^2+y^2 = a^2+b^2$. Similarly the curve whose equation is $(x^2+y^2)^{\frac{1}{2}} = a\tan^{-1}\left(\frac{y}{x}\right)$ is more easily traced when it is put in its equivalent polar form, $r = a\theta$. The means of tracing polar curves will be discussed in the next chapter.

2) If the equation to a curve admits of being put in the form

$$y = f(x) \pm \phi(x),$$

in which case, as before observed, $y = f(x)$ is the equation to a curve diametral to the curve to be traced, it is most convenient first to trace the diametral curve, and then to increase and diminish its ordinates by the quantity $\phi(x)$ corresponding to the several values of the abscissa to the curve $y = f(x)$.

Thus, for instance, in the discussion of the general equation of the second degree,

$$ay^2+bxy+cx^2+dy+ex+f = 0, \tag{150}$$

let the equation be solved for y; whence we have

$$y = -\frac{b}{2a}x-\frac{d}{2a}\pm\frac{1}{2a}\{(b^2-4ac)x^2+2(bd-2ae)x+d^2-4af\}^{\frac{1}{2}}; \tag{151}$$

$y = -\frac{b}{2a}x-\frac{d}{2a}$, is the equation to a straight line, and therefore the ordinate to the curve is the ordinate to a straight line increased and diminished by equal quantities; the most convenient method therefore of tracing the curve is first to construct the straight line, and then to add to and subtract from its ordinate such a quantity as arises from an examination of the latter part of (151).

3) Let the equation to the curve, if possible, be put in the explicit form $y = f(x)$; and let all the points be determined

at which the curve meets the coordinate axes, by finding the values of x which render $y = 0$, and the values of y which render $x = 0$; and let the change or continuation of sign be examined in order to determine whether the curve passes from below to above the axis of x, or *vice versâ*; and whether it passes from the left to the right of the axis of y, or *vice versâ;* and whether it touches the axes; and if it cuts the axis, let the value of $\frac{dy}{dx}$ be examined at the point of section, so as to determine the angle at which it cuts. And if, for all values of x from $+\infty$ to $-\infty$, y is unaffected with $\pm\sqrt{-}$, the curve extends infinitely in both directions in the plane of the paper; but if at any point, say $x = a$, $y = b$, the equation is such as on one side of that point to be affected with $\pm$, and on the other side with $\pm\sqrt{-}$, then at that point the curve leaves the plane of the paper. Suppose that at such a point there is only one branch of the curve, so that the symbol of "impossibility" does not arise from the extraction of the square root of a negative number, then there is what is by French writers termed a *point d'arrêt;* or, as we may conveniently call it, a point of abrupt termination; and the branch has only one tangent. Such however can only arise from a discontinuous function, or from such functions as those for which Maclaurin's Theorem fails. Thus, if the equation is $y = x^2 \log x$, $y = 0$, when $x = 0$; by virtue of Ex. 2, Art. 126; also $\frac{dy}{dx} = 0$, when $x = 0$; hence the curve passes though the origin, and touches the axis of x, and is in the plane of the paper on the positive side of the axis of y; but as the logarithms of negative numbers are, see Art. 67, affected with $\pm\sqrt{-}$, the curve is out of the plane of reference on the negative side of the same axis; and therefore there is at the origin a point of abrupt termination. The above curve is traced in fig. 70 as far as it exists in the plane of the paper; $\text{AO} = 1$.

If however at the point where the ordinate becomes affected with $\pm\sqrt{-}$, two branches pass into another plane, there is either a cusp or a salient point, according as the two branches have the same or different tangents. The distinctive characters of these points depend on the corresponding value or values of $\frac{dy}{dx}$. And if the equation to the curve is satisfied by $x = a$, $y = b$, but if when x is increased or decreased by a quantity, however small, y is affected with $\pm\sqrt{-}$, then at such a point the curve, which lies in some other plane, pierces the plane of reference; and the point is a conjugate or isolated point; and of course one or two or more branches of a curve may pass through such a point: as for instance if the equation to a straight line is $y-b = (-)^{\frac{1}{2}}(x-a)$, the equation is satisfied by $x = a$, $y = b$, which indicates a point in the plane of reference; but every other point of the line is in the plane passing through the line BD, see fig. 71, OA $= a$, AB $= b$, and perpendicular to the plane of the paper.

When two branches of the curve simultaneously pierce the plane of the paper, the two roots of (126), Art. 248, are impossible, as is the case in Ex. 8, which is traced below in Art. 260. And a curve may have any number of such conjugate points, by continually passing through the plane of the paper, such as in the subjoined example:

$$y = ax^2 \pm (bx)^{\frac{1}{2}} \sin x.$$

The curve is traced in fig. 72, the dotted line indicating the branches in a plane perpendicular to that of the paper. $y = ax^2$, which represents the diametral curve, is a parabola, B′OB, drawn in the figure; and the ordinate to the curve is periodically reduced to its ordinate when $x = 0$, or $= \pi$, or $= 2\pi$, ... or $=$ any multiple of π; but when x is negative, the part of the ordinate to be added to or subtracted from the ordinate of the parabola is affected with $(-)^{\frac{1}{2}}$, except at the points where $x =$ some multiple of π, at which the branches of the curve pierce the plane of reference: and thus it continues *ad infinitum*, the curve itself being continuous, but there being a series of discontinuous points, if we consider only those points which the plane of the paper contains.

Curves such as the last have been called "Courbes Pointillés*;" which name however has been given by writers who discard the mode of interpretation of the symbol of impossibility which we have employed in this Treatise, and are therefore obliged to allow that algebraical expressions admit of discontinuous *geometrical* interpretation: a result surely at

---

[*] See page 382 of a "Treatise on the Differential Calculus," by Augustus De Morgan, M.A. Baldwin and Cradock, London, 1842.

variance with the *algebraical* nature of such functions, which admit of differentiation, and thereby indicate that they fulfil the law of continuity.

4) On the method of determining the simultaneous increase and decrease of x and y nothing more need be said; but we must be careful to investigate the points at which $\frac{dy}{dx} = 0$, and $= \infty$, and to observe whether or not there is a change of sign, as such is the criterion of maxima and minima. With this object we shall equate $\frac{dy}{dx}$ to 0 and ∞, and examine the course the curve takes at these critical points.

5) In regard to asymptotes, and the course of the curve with respect to them, we must examine the finite values of x for which y is infinite; and the values of y for which x is infinite, as such will be asymptotes parallel to the axes of y and x respectively; and by investigating whether $\frac{dy}{dx}$ changes sign or not for these asymptotic values, we shall determine whether the infinite ordinate is a maximum or a minimum; that is, whether it returns, or whether it continues round the circle of infinite radius, such as we described in the last chapter; and which of the forms delineated in figs. 24, 25, 26, 51 the curve takes. We must also be careful to determine whether there are rectilineal asymptotes inclined at oblique angles to the axes of coordinates, general rules for the discovery of which have been given in Art. 234; and whether the curve is above them or below them. It may happen that two distinct branches of a curve will approach the same asymptote. Sometimes also a curve will cut its asymptote; as e.g. if $y = a\frac{\sin x}{x}$, the axis of x is an asymptote, and the curve cuts it whenever $x =$ an integral multiple of π.

6) The general character of a curve with regard to the curvature of it in a particular direction and its points of inflexion has been sufficiently discussed in Section 3 of the present Chapter. Practically however $\frac{d^2y}{dx^2}$ is of little use in enabling us to trace a curve, unless it assumes a simple and explicit form; and should also at any point of the curve $\frac{d^2y}{dx^2} = 0$, and

not change its sign, we may conclude that more elements of the curve than two, and therefore more points of the curve than three, are in one and the same straight line.

7) There is nothing more to be added on the theory of multiple points and their varieties.

8) And generally it is of little use to examine the values of x and y, except at such critical points as we have above described; and except when $x = \infty$, and $y = \infty$, in order that we may determine the course the curve is taking at such distances from the axes.

259.] Thus, in the discussion of any particular equation representing a plane curve, the method indicated by the following rules is the most convenient to adopt:

I. Reduce the equation if possible to the explicit form, and simplify it, as far as may be, by means of a change of origin, or by a transformation into polar coordinates.

II. Discover, arrange, and tabulate with their proper signs, all the critical values of y and x, both in and out of the plane of reference.

III. Discuss and tabulate the critical values of $\frac{dy}{dx}$, as e.g. determine the angles at which the curve cuts the axes, the maximum and minimum ordinates, &c.

IV. Find the equations to the asymptotes, and determine whether the curve is above or below them.

V. Find, if it is possible in a convenient form, $\frac{d^2y}{dx^2}$; thence determine the direction of curvature, and the points of inflexion.

VI. If at any point $\frac{dy}{dx} = \frac{0}{0}$, evaluate the quantity, and determine the several double, triple points, &c.

260.] Examples illustrative of the preceding theory.

Ex. 1. Trace the curve whose equation is $y^2 = a^2x^3$.

$$\therefore \quad y = \pm a x^{\frac{3}{2}}; \tag{152}$$

$$\frac{dy}{dx} = \frac{3a^2x^2}{2y}, \text{ also } \frac{dy}{dx} = \pm \frac{3ax^{\frac{1}{2}}}{2};$$

$$\frac{d^2y}{dx^2} = \pm \frac{3a}{4x^{\frac{1}{2}}}.$$

From (152) it is plain that the curve is symmetrical with respect to the axis of x; and since the curve passes through the origin, the former value of $\frac{dy}{dx}$ at that point assumes an indeterminate form, the value of which, as before shewn in Ex. 5, Art. 249, is such as to give a cusp, both branches of which touch the axis of x, and which are in the plane of the paper on the positive side of the axis of y, and out of it on the negative side. The same thing is also apparent from the second value of $\frac{dy}{dx}$, which $= 0$ at the origin, and is affected with $\pm\sqrt{-}$ when x is negative, and with $\pm$ when x is positive; also $\frac{d^2y}{dx^2}$, being affected with $\pm$, shews that one branch of the curve is convex, and the other concave, downwards.

Hence we may tabulate as follows:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$\pm\sqrt{-},\ 0,\ \pm$	$\pm\sqrt{-},\ 0,\ \pm$	$\pm\sqrt{-},\ \infty,\ \pm$
2	$+\infty$	$\pm\infty$	$\pm\infty$	$\pm$
3	$-\infty$	$\pm\sqrt{-}\,\infty$	$\pm\sqrt{-}\,\infty$	$\pm\sqrt{-}$

From 1 it appears that the curve passes through the origin, with two branches, both touching the axis of x, one of which is convex, and the other is concave, downwards, and which are out of the plane of the paper on the negative side of the axis of x, and are in it on the positive side. From 2 and 3 it appears, that as x increases, whether positively or negatively, y increases also, and since $\frac{dy}{dx}$ approximates to ∞, that the curve approaches to parallelism with the axis of y; the only critical value is $x = 0$; the curve is drawn in fig. 64.

Ex. 2. Discuss the curve

$$y = x^3-2x^2-5x+6 = (x+2)(x-1)(x-3);$$

$$\frac{dy}{dx} = 3x^2-4x-5 = 3\left(x-\frac{2+\sqrt{19}}{3}\right)\left(x-\frac{2-\sqrt{19}}{3}\right);$$

$$\frac{d^2y}{dx^2} = 6x-4 = 6\left(x-\frac{2}{3}\right).$$

As the equation does not admit of expansion in descending powers of x of a form such that the highest positive power of x may be unity, it follows that the curve has no rectilineal asymptote. The table of the critical values is as follows.

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	-2	$-, 0, +$	$+15$	$-$
2	0	$+6$	-5	$-$
3	1	$+, 0, -$	-6	$+$
4	3	$-, 0, +$	$+10$	$+$
5	$\frac{2+\sqrt{19}}{3}$	$-$	$-, 0, +$	$+$
6	$\frac{2-\sqrt{19}}{3}$	$+$	$+, 0, -$	$-$
7	$\frac{2}{3}$	$+$	$-$	$-, 0, +$
8	$+\infty$	$+\infty$	$+\infty$	$+$
9	$-\infty$	$-\infty$	$+\infty$	$-$

On examination of which table it appears that, when $x = -\infty$, the curve is at an infinite distance below the axis of x, approaching to parallelism with the axis of y, and being concave downwards; whence it cuts the axis of x, when $x = -2$, as shewn by 1, at a large acute angle, and the ordinate attains a maximum at the value of x given by 6; whence the ordinate decreases, cutting the axis of y at a distance $+6$ from the origin, and being concave downwards until $x = \frac{2}{3}$, at which point, as shewn by 7, there is a point of inflexion; and the curve being convex downwards cuts the axis of x, when $x = 1$, and decreases until x is equal to the value given in 5, at which point there is a minimum ordinate: after which the ordinate again increases, cuts the axis of x when $x = 3$, and goes off to an infinite distance approaching to parallelism with the axis of y, and being convex downwards. The curve is drawn in fig. 73, OA $= 2$, OB $= 1$, OC $= 3$; OE $= \frac{2-\sqrt{19}}{3}$, OD $= \frac{2+\sqrt{19}}{3}$, OF $= \frac{2}{3}$.

Ex. 3. Discuss the curve whose equation is

$$\frac{x^3}{a^3} + \frac{y^3}{b^3} = 1.$$

$$y = \frac{b}{a}(a^3-x^3)^{\frac{1}{3}};$$

$$\frac{dy}{dx} = -\frac{b}{a}\frac{x^2}{(a^3-x^3)^{\frac{2}{3}}}; \quad \text{also} \quad \frac{dy}{dx} = -\frac{b^3}{a^3}\frac{x^2}{y^2};$$

$$\frac{d^2y}{dx^2} = \frac{-2a^2bx}{(a^3-x^3)^{\frac{5}{3}}}.$$

Also to find the asymptote,

$$y = -\frac{b}{a}x\left(1-\frac{a^3}{x^3}\right)^{\frac{1}{3}},$$

$$= -\frac{b}{a}x\left\{1-\frac{a^3}{3x^3}-\dots\right\};$$

therefore the equation to the asymptote is

$$y = -\frac{b}{a}x;$$

and as the next term of the expansion is positive, the curve lies above the asymptote.

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	b	$-, 0, -$	$+, 0, -$
2	a	$+, 0, -$	$-, \infty, -$	$-, \infty, +$
3	$+\infty$	$-\infty$	$-\frac{b}{a}$	$+$
4	$-\infty$	$+\infty$	$-\frac{b}{a}$	$+$

An inspection of which table, together with the equation to the asymptote, shews that the curve is such as that drawn in fig. 74; OA $= a$, OB $= b$, with points of inflexion at A and B.

Ex. 4. Discuss the curve whose equation is

$$y = \frac{x}{1+x^2}.$$

$$\frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2}; \qquad \frac{d^2y}{dx^2} = \frac{2x(x^2-3)}{(1+x^2)^3}.$$

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$-, 0, +$	1	$+, 0, -$
2	$+1$	$\frac{1}{2}$	$+, 0, -$	$-$
3	-1	$-\frac{1}{2}$	$-, 0, +$	$+$
4	$+\sqrt{3}$	$+$	$-$	$-, 0, +$
5	$-\sqrt{3}$	$-$	$-$	$-, 0, +$
6	$+\infty$	$+0$	0	$+$
7	$-\infty$	-0	0	$-$

An inspection of which table shews, that the curve is such as is drawn in fig. 75. For when $x = -\infty$, $y = -0$, that is, the curve lies immediately below the axis of x, which is an asymptote to it; and the curve recedes further from it on the negative side, and when $x = -\sqrt{3} =$ OB$'$, there is a point of inflexion; for the curve, having been concave, becomes convex downwards; and when $x = -1 =$ OA$'$, the curve is at its greatest distance below the axis of x, and there is a minimum ordinate; after which the curve approaches to the axis of x, and passes through the origin, cutting the axis of x at 45°, and goes through the same phases as on the negative side, except that it is above, instead of below, the axis of x: and as the sign of y changes from $+0$ to -0, when x changes from $+\infty$ to $-\infty$, so do the two branches which are asymptotic to the axis of x unite, and the curve is continuous from $+\infty$ to $-\infty$.

Ex. 5. Discuss the curve whose equation is

$$y^3 = 2ax^2 - x^3.$$

$$y = x^{\frac{2}{3}}(2a-x)^{\frac{1}{3}};$$

$$\frac{dy}{dx} = \frac{4a-3x}{3x^{\frac{1}{3}}(2a-x)^{\frac{2}{3}}}; \text{ also } \frac{dy}{dx} = \frac{4ax-3x^2}{3y^2};$$

$$\frac{d^2y}{dx^2} = \frac{-8a^2}{9x^{\frac{4}{3}}(2a-x)^{\frac{5}{3}}}.$$

To find the equation to the asymptote,

$$y = -x\left(1-\frac{2a}{x}\right)^{\frac{1}{3}},$$

$$= -x\left\{1-\frac{2a}{3x}-\dots\right\};$$

therefore $y = -x + \frac{2a}{3}$, is the equation to the asymptote, and, as the next term of the expansion is positive, the curve lies above the asymptote.

Also since $x = 0$, if $y = 0$, and as in this case $\frac{dy}{dx} = \frac{0}{0}$, it must be evaluated;

$$\frac{dy}{dx} = \frac{4ax-3x^2}{3y^2} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{(4a-6x)\,dx}{6y\,dy};$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{2a}{3y} = \infty, \text{ when } x = y = 0;$$

$$\therefore \frac{dy}{dx} = \pm\left(\frac{2a}{3y}\right)^{\frac{1}{2}} = \pm\infty, \text{ when } y = 0.$$

Therefore there are at the origin two branches of the curve touching the axis of y; and the value of $\frac{dy}{dx}$ shews that, if y is negative, $\frac{dy}{dx}$ is affected with $\pm\sqrt{-}$, and therefore the origin is a cusp of the first species.

The existence also of such a cusp is manifest from the explicit value of $\frac{dy}{dx}$; hence the table of values is as follows:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$+, 0, +$	$\frac{0}{0} = \pm\infty$	$-$
2	$2a$	$+, 0, -$	$-, \infty, -$	$-, \infty, +$
3	$\frac{4a}{3}$	$+$	$+, 0, -$	$-$
4	$+\infty$	$-\infty$	-1	$+, 0, -$
5	$-\infty$	$+\infty$	-1	$-, 0, +$

Hence, and from the asymptote, the curve is that delineated in fig. 76, in which OA $= 2a$, OB $= \frac{4a}{3}$, OC $= \frac{2a}{3}$. For it appears from 1 that the curve passes through the origin, which is a cusp of the first species, the two branches touching the axis of y, and above the axis of x, both branches being concave downwards; and the curve, having been above the axis of x, from $x = 0$ to $x = 2a$, at this last point cuts the axis of x at right angles, and changes its curvature: for having been concave, it becomes convex downwards. 3 shews that the curve has a maximum ordinate when $x = \frac{4a}{3}$; the curve approaches to the asymptote whose equation is $y = -x + \frac{2a}{3}$, which, as is shewn by 4 and 5, it cuts and touches when $x = \infty$, where is a point of inflexion, and thus the two asymptotic branches unite. We have traced the curve only in the plane of reference, as we have not discussed the geometrical meaning of the cube roots of $+$.

Ex. 6. Discuss the Cissoid of Diocles, the equation to which is

$$y^2 = \frac{x^3}{2a-x}.$$

$$\therefore \quad y = \pm \frac{x^{\frac{3}{2}}}{(2a-x)^{\frac{1}{2}}}.$$

As y is affected with $\pm$, the curve is symmetrical with respect to the axis of x.

$$\frac{dy}{dx} = \pm \frac{x^{\frac{1}{2}}(3a-x)}{(2a-x)^{\frac{3}{2}}}; \quad \text{also} \quad \frac{dy}{dx} = \frac{x^2(3a-x)}{y(2a-x)^2};$$

$$\frac{d^2y}{dx^2} = \pm \frac{3a^2}{x^{\frac{1}{2}}(2a-x)^{\frac{5}{2}}}.$$

Also the equations to the asymptotes as found in Ex. 1, Art. 233, are

$$y = \pm \sqrt{-}\,(x+a).$$

Hence the table becomes

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$\pm\sqrt{-}, 0, \pm$	$\pm\sqrt{-}, 0, \pm$	$\pm\sqrt{-}, \infty, \pm$
2	$2a$	$\pm, \infty, \pm\sqrt{-}$	$\pm, \infty, \pm\sqrt{-}$	$\pm, \infty, \pm\sqrt{-}$
3	$3a$	$\pm\sqrt{-}$	$\pm\sqrt{-}, 0, \pm\sqrt{-}$	$\pm\sqrt{-}$
4	a	$\pm a$	± 2	$\pm$
5	$+\infty$	$\pm\sqrt{-}\,\infty$	$\pm\sqrt{-}$	$\pm\sqrt{-}$
6	$-\infty$	$\pm\sqrt{-}\,\infty$	$\pm\sqrt{-}$	$\pm\sqrt{-}$

Hence, and by means of the asymptote, the curve is that delineated in fig. 34; OC = CA = a; OD = $3a$, OD being the abscissa corresponding to the maximum and minimum ordinates of the curve out of the plane of the paper.

Ex. 7. Discuss the Witch of Agnesi, the equation to which is

$$y^2 = 4a^2 \frac{2a-x}{x}.$$

$$y = \pm 2a\left(\frac{2a-x}{x}\right)^{\frac{1}{2}}.$$

Thus it appears that the curve is symmetrical with respect to the axis of x;

$$\frac{dy}{dx} = \mp \frac{2a^2}{x^{\frac{3}{2}}(2a-x)^{\frac{1}{2}}}; \quad \text{also } \frac{dy}{dx} = -\frac{4a^3}{x^2y};$$

$$\frac{d^2y}{dx^2} = \pm \frac{2a^2(3a-2x)}{x(2ax-x^2)^{\frac{3}{2}}}.$$

Also the equations to the asymptotes as found in Ex. 2, Art. 233, are

$$y = \pm \sqrt{-}\,2a;$$

that is, the asymptotes are two straight lines out of the plane of the paper, parallel to the axis of x, and at distances $\pm 2a$ from it.

The table is as follows:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$\pm\sqrt{-}, \infty, \pm$	$\pm\sqrt{-}, \infty, \pm$	$\pm\sqrt{-}, \infty, \pm$
2	$2a$	$\pm, 0, \pm\sqrt{-}$	$\pm, \infty, \pm\sqrt{-}$	$\pm$
3	$\frac{3a}{2}$	$\pm$	$\pm$	$\pm, 0, \mp$
4	$+\infty$	$\pm\sqrt{-}\,2a$	$\pm\sqrt{-}\,0$	$\pm\sqrt{-}$
5	$-\infty$	$\pm\sqrt{-}\,2a$	$\pm\sqrt{-}\,0$	$\pm\sqrt{-}$

An examination of which table shews that the curve is that drawn in fig. 35, where OC $=$ CA $= a$; OB $=$ OB$'$ $= 2a$.

Ex. 8. Discuss the curve whose equation is

$$y^2(x^2-a^2) = x^4.$$

$$\therefore \quad y = \pm\frac{x^2}{(x^2-a^2)^{\frac{1}{2}}}.$$

Since the given equation is not changed when we write $-x$ and $-y$ for $+x$ and $+y$ respectively, it appears that the curve is situated symmetrically in the four quadrants. Differentiating, we have

$$\frac{dy}{dx} = \pm\frac{x(x^2-2a^2)}{(x^2-a^2)^{\frac{3}{2}}}; \quad \text{also} \quad \frac{dy}{dx} = \frac{2x^3-xy^2}{y(x^2-a^2)};$$

$$\frac{d^2y}{dx^2} = \pm\frac{a^2(x^2+2a^2)}{(x^2-a^2)^{\frac{5}{2}}}.$$

To find the equations to the asymptote,

$$y = \pm\frac{x^2}{x\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}}} = \pm x\left(1-\frac{a^2}{x^2}\right)^{-\frac{1}{2}},$$

$$= \pm x\left\{1+\frac{a^2}{2x^2}+\dots\right\};$$

therefore the equations to the asymptotes are

$$y = \pm x;$$

and, as the sign of the next term is positive, the curve lies above the asymptote in the first quadrant.

When $x = 0$, $y = 0$; therefore the curve passes through the origin, at which point $\frac{dy}{dx} = \frac{0}{0}$, as appears from its second value given above, and is therefore to be evaluated.

$$\frac{dy}{dx} = \frac{2x^3 - xy^2}{y(x^2 - a^2)} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{6x^2 - y^2 - 2xy\frac{dy}{dx}}{(x^2 - a^2)\frac{dy}{dx} + 2xy}, \quad - - - - -$$

$$= \frac{0}{-a^2\frac{dy}{dx}}, \text{ when } x = y = 0;$$

$$\therefore \quad \frac{dy}{dx} = \pm\left(\frac{-0}{a^2}\right)^{\frac{1}{2}}, \text{ at the origin;}$$

which implies that two branches of the curve touch the axis of x at the origin, both of which are out of the plane of the paper.

The table of critical values is as follows:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$\pm\sqrt{-}, 0, \pm\sqrt{-}$	$\pm\sqrt{-}, 0, \pm\sqrt{-}$	$\pm\sqrt{-}$
2	$+a$	$\pm\sqrt{-}, \infty, \pm$	$\pm\sqrt{-}, \infty, \pm$	$\pm$
3	$-a$	$\pm, \infty, \pm\sqrt{-}$	$\pm, \infty, \pm\sqrt{-}$	$\pm$
4	$a\sqrt{2}$	$\pm 2a$	$\mp, 0, \pm$	$\pm$
5	$-a\sqrt{2}$	$\pm 2a$	$\pm, 0, \mp$	$\pm$
6	$+\infty$	$\pm\infty$	± 1	$\pm, 0, \mp$
7	$-\infty$	$\pm\infty$	± 1	$\pm, 0, \mp$

From 1 it appears that the curve passes through the origin, and has two branches, both of which are out of the plane of the paper, and which touch the axis of x; whence, as 2 and 3 shew, the curve recedes from the axis of x, until when $x = \pm a = \text{OA} = \text{OA}'$, $y = \pm \infty$, and there are two asymptotes parallel to the axis of y. For values of x beyond these lines, the curve is in the plane of reference, and returns towards the axis of x, until the ordinates reach minimum and maximum values when $x = a\sqrt{2}$, as is shewn by 4 and 5, whence it recedes again towards the asymptotes whose equations are $y = \pm x$, and intersects them at infinity in a point of inflexion, as shewn by 6 and 7, the curve lying above the asymptote in the first quadrant, and being symmetrically situated in the others. Its course is traced in fig. 77, where $\text{OA} = a$, $\text{OB} = \sqrt{2}a$, $\text{BC} = 2a$, and where the dotted line represents the curve out of the plane of reference.

If the equation to be discussed had been

$$y^2(a^2 - x^2) = x^4,$$

the branches of the curve which are in the plane of reference would have been out of it, and *vice versâ*. The continuity of curve is remarkable in both cases.

Ex. 9. Discuss the curve whose equation is

$$y^2 = x^2\frac{a+x}{a-x}.$$

$$y = \pm x\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}};$$

whence it appears that the curve is symmetrical with respect to the axis of x;

$$\frac{dy}{dx} = \pm\frac{a^2 + ax - x^2}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}};$$

$$\frac{d^2y}{dx^2} = \pm\frac{a^2(2a+x)}{(a+x)^{\frac{3}{2}}(a-x)^{\frac{5}{2}}};$$

and the equations to the asymptotes are

$$y = \pm\sqrt{-}(x+a).$$

Also since $\frac{dy}{dx} = 0$, when $x = \frac{a}{2}(1 \pm \sqrt{5})$, a careful inspection of the above quantities shews that the form of the curve is that drawn in fig. 78, the dotted branches being those out of the plane of the paper; $\text{OA} = \text{OB} = \text{BC} = a$,

$$\text{OF} = \frac{a}{2}\{1+\sqrt{5}\}, \qquad \text{OE} = \frac{a}{2}\{1-\sqrt{5}\}.$$

Ex. 10. Examine the Folium of Descartes, the equation to which is

$$y^3 - 3axy + x^3 = 0.$$

As shewn above in Ex. 3, Art. 235, the equation to the asymptote is

$$y = -x - a.$$

Also at the origin there is a double point, as shewn in Ex. 3, Art. 249.

Also since
$$\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax},$$

$\frac{dy}{dx} = 0$, if $ay = x^2$; that is, if $x = a\,(2)^{\frac{1}{3}}$, and $y = a\,(4)^{\frac{1}{3}}$.

Also $\frac{dy}{dx} = \infty$, if $ax = y^2$; that is, if $x = a(4)^{\frac{1}{3}}$, and $y = a(2)^{\frac{1}{3}}$; and the curve does not extend beyond these limits; it is such as is delineated in fig. 63.

Ex. 11. Trace the curve $y = \sin x$.

$$\frac{dy}{dx} = \cos x; \qquad \frac{d^2y}{dx^2} = -\sin x.$$

In tracing curves of this kind involving circular functions, the arc, of which the trigonometrical function is given, is to be measured along the coordinate axis; in the present case along the axis of x, since $\sin x$ is involved in the equation, and the ordinates are to be constructed corresponding to the arcs or abscissæ thus measured; π, we must remember, is the symbol for the arithmetical number 3.14159; and we must give to x such values as will render y a quantity capable of construction. Thus, in the equations above, let

$$x = \frac{\pi}{2} = 1.57079\ldots;$$

$$y = \sin x = 1, \qquad \frac{dy}{dx} = 0, \qquad \frac{d^2y}{dx^2} = -1.$$

Hence we may form the following table:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$-, 0, +$	1	$+, 0, -$
2	$\frac{\pi}{2}$	1	$+, 0, -$	$-$
3	π	$+, 0, -$	-1	$-, 0, +$
4	$\frac{3\pi}{2}$	-1	$-, 0, +$	$+$
5	2π	$-, 0, +$	1	$+, 0, -$

After which the values recur, and the curve is that drawn in fig. 79.

Ex. 12. Trace the curve whose equation is $y = e^{\sec x}$, e being the base of the Napierian logarithms.

$$\frac{dy}{dx} = e^{\sec x} \sec x \tan x;$$

$$\frac{d^2y}{dx^2} = e^{\sec x} \sec x \left\{(\sec x)^3 + 2(\sec x)^2 - \sec x - 1\right\}.$$

In the table h is a small increment of x.

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	e	$-, 0, +$	$+$
2	$\frac{\pi}{2} - h$	$+\infty$	$+\infty$	$+$
3	$\frac{\pi}{2} + h$	0	0	$+$
4	π	$\frac{1}{e}$	$+, 0, -$	$-$
5	$\frac{3\pi}{2} - h$	0	0	$+$
6	$\frac{3\pi}{2} + h$	$+\infty$	∞	$+$
7	2π	e	$-, 0, +$	$+$

The curve is that drawn in fig. 80, where OA = AB = BC = CD = DE $= \frac{\pi}{2}$, OF $= e$, BL $= \frac{1}{e}$. There are also two points of inflexion between A and C.

CHAPTER XI.

PROPERTIES OF PLANE CURVES, AS DEFINED BY EQUATIONS REFERRED TO POLAR COORDINATES.

SECTION 1.—*The mode of interpretation, and the equations, of curves referred to polar coordinates.*

261.] IN the present Chapter we shall investigate, for polar curves, formulæ somewhat analogous to those discussed in the last Chapter for curves referred to rectangular coordinates; but previously it is necessary to extend the usual mode of interpreting polar equations, so as to accommodate them in a greater degree to the law of continuity.

Let $r = f(\theta)$ be the equation to the curve. Then, taking a fixed point S as the origin, which is called the *pole*, and a fixed line SX passing through it as the line of origination, which is called the *prime radius*, see fig. 81, it is manifest that the moveable radius, which is symbolized by r, may revolve about S in either of two directions; and thus, if the only datum is that r makes an angle θ with the prime radius, it is undetermined whether r is above or below SX: that is, whether r revolves *up* from SX from right to left, or *down* from left to right. Hence arises the necessity of some symbol of the direction in which r turns, so that angles formed in one direction may be differently symbolized to those formed in another. This indefiniteness will be avoided if we call angles positive when measured *up* from SX, as in fig. 81: that is, when the radius vector revolves round S in the direction indicated by the curved arrow; and negative when they are measured *down* from SX, and the radius vector revolves in the direction indicated by the curved arrow in fig. 82. In this case then, + and −, as affecting angles, indicate the two different directions in which r can revolve in the plane of the paper.

Again, suppose that for a given value of θ, r is affected with a negative sign, a question arises, in what direction is the negative r to be measured? No doubt, if r is affected with a positive sign, the length of it, determined by the equation to the curve, is to be measured from the pole along the revolving radius vector which is inclined at the given angle to the prime radius; as e. g. if a polar equation between r and θ is such that, when $\theta = \frac{\pi}{4}$, $r = a$, then a length $= a$ is to be measured from the pole along the revolving radius, which is inclined at 45° to the prime radius. From analogy therefore to what has been said in Art. 189, on the signs $+$ and $-$, $-r$ must be measured along the radius vector produced backwards; i. e. if, when $\theta = \frac{\pi}{4}$, $r = -a$, a line equal to a must be measured from the pole along the revolving radius produced backwards: that is, in a direction making an angle of 225° with the prime radius. That we may the better avoid confusion on this subject, conceive the revolving radius to be an arrow of variable length, such as we have drawn in figs. 81 and 82, the pole being a fixed point in it; then, if θ is the angle between the prime radius and the part of the arrow towards the barbed end, lines measured from S in the direction SP will be positive, and in the direction SQ negative. If therefore r is affected with a positive sign, it is to be measured towards the barbed end, but if with a negative sign, towards the feathered end of the arrow. In the figures different positions of the arrow are drawn to indicate different positive and negative directions of r.

In the following Chapter we shall omit those particular values of r which are affected with $\pm\sqrt{-}$, as no satisfactory interpretation of such symbols in such a relation exists, and we shall consider those only which are affected with $\pm$; being careful however to make r revolve in both the positive and negative directions, otherwise at certain points the curve will appear to be discontinuous.

And for the purpose of illustration in the sequel, we must here insert an account of the mode of description, and the equations of some polar curves, many of which, having been treated of at length by old geometricians, possess no small historical interest.

262.] The Spiral of Archimedes.

Def.—If the length of the radius vector of a spiral is proportional to the angle through which it has moved from its originating position, the locus of its extremity is the Spiral of Archimedes.

Let a = the length of the radius, when the angle described is equal to unity*; and let r be its length after describing the angle θ; therefore the equation is

$$r = a\theta; \tag{1}$$

see fig. 83.

The curve therefore starts from the pole; and the radius vector, which at the beginning is equal to zero, is equal to SA, that is a, when it has revolved through the unit angle ASX; and at the end of the first complete revolution is equal to $2\pi a$; and this is the distance between the points at which any radius vector is cut by two successive convolutions of the curve. The dotted curve is that described by the generating point, as the radius vector revolves in the negative direction.

263.] The reciprocal spiral.

The reciprocal or hyperbolic spiral is so called from the form of its equation, which is

$$r = \frac{a}{\theta}. \tag{2}$$

The form of the curve is given in fig. 84. The radius vector $= \infty$ when $\theta = 0$, and the curve is asymptotic to the straight line B′AB, as will be shewn in the sequel. Also, when $\theta = 1 =$ A′SX, $r = a =$ SA′; also $r = 0$, when $\theta = \infty$; therefore, after an infinite number of revolutions, the curve falls into the pole. The curve has also the dotted branch arising from the revolution of r in the negative direction.

264.] The lituus.

This spiral is so called from its form as delineated in fig. 85. Its equation is

$$r = \frac{a}{\theta^{\frac{1}{2}}}. \tag{3}$$

The prime radius is an asymptote to the curve; which has a point of inflexion when $r =$ SB $= a\sqrt{2}$, as will be shewn hereafter. Also, when $\theta = 1 =$ ASX, $r =$ SA $= a$; there is an apparent discontinuity at the pole and at the extremity of the infinite branch, which arises from our not interpreting r when affected with $\pm\sqrt{-}$, as such it will be if the radius vector is made to revolve in a negative direction.

---

* The unit angle is that whose subtending arc is equal to the radius, and expressed in degrees = 57.29578. See Ex. 5, Art. 24.

265.] The logarithmic spiral.

DEF.—The logarithmic spiral is that whose radius vector increases in a geometric, as its angle increases in an arithmetic ratio.

Hence the equation is $$r = a^{\theta}. \qquad (4)$$

Therefore when $\theta = 0$, $r = 1 = $ SA; when $\theta = 1$, $r = a$; when $\theta = \infty$, $r = \infty$; when $\theta = -\infty$, $r = 0$; and therefore the spiral runs into its pole after an infinite number of revolutions in the negative direction; the spiral is also called the equiangular spiral from a property which will subsequently be proved; viz. that it cuts all its radii vectores at a constant angle; that is, the angle STP is constant, at whatever point P the tangent is drawn. Its form is delineated in fig. 86.

266.] The involute of the circle; fig. 87.

DEF.—The involute of the circle is the curve formed by the extremity of an inextensible string, as it is wrapped round the circumference of a circle.

If r is the radius vector of a curve, and p is the perpendicular from the pole on the tangent, it is frequently convenient to express the equation to the curve in terms of r and p. Such an equation is, as will be subsequently seen, a *differential* one; but expressing as it does an essential property of a curve, it is sufficient to individualize it, and thus to be a mathematical definition. The equation of the involute of the circle can be easily obtained in this form

Let SA $= a$, the radius of the circle; SP $= r$; SY $= p$; and let A be the point at which P, the generating point of the involute, is in contact with the circle.

Then from the geometry it is plain that QPY is a right angle, and consequently QP is parallel to SY; whence we have

$$\begin{aligned} \text{SP}^2 &= \text{SY}^2 + \text{PY}^2, \\ &= \text{SY}^2 + \text{SA}^2; \\ \therefore\quad r^2 &= p^2 + a^2. \qquad (5) \end{aligned}$$

267.] To find the equation to the circle in terms of p and r, any point being the pole; fig. 88.

Let CA, the radius of the circle, $= a$; SC $= c$, S being the pole; SP $= r$; SY $= p$.

$$\therefore \quad \text{SC}^2 = \text{SP}^2 + \text{PC}^2 - 2.\text{SP}.\text{PC}.\cos \text{SPC},$$

$$= \text{SP}^2 + \text{PC}^2 - 2.\text{SP}.\text{PC}.\cos \text{YSP},$$

$$c^2 = r^2 + a^2 - 2ra\frac{p}{r};$$

$$\therefore \quad p = \frac{r^2+a^2-c^2}{2a}. \tag{6}$$

Hence, if the pole is on the circumference, say at B, $c = a$; and the equation is

$$p = \frac{r^2}{2a}. \tag{7}$$

268.] To find the equation to the epicycloid, in terms of r and p.

From Art. 204, equations (37),

$$x = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$

$$y = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta;$$

$$\therefore \quad x^2+y^2 = r^2 = (a+b)^2+b^2-2b(a+b)\cos\frac{a}{b}\theta,$$

$$r^2 = a^2+2b(a+b)\left\{1-\cos\frac{a}{b}\theta,\right\}. \tag{8}$$

Also differentiating and reducing,

$$ds^2 = 2(a+b)^2\left\{1-\cos\frac{a}{b}\theta\right\}d\theta^2 = \frac{a+b}{b}(r^2-a^2)d\theta^2;$$

$$y\,dx - x\,dy = (a+b)(a+2b)\left\{\cos\frac{a}{b}\theta - 1\right\}d\theta$$

$$= -\frac{a+2b}{2b}(r^2-a^2)\,d\theta.$$

Also by equation (44), Art. 219,

$$p\,ds = y\,dx - x\,dy;$$

$$\therefore \quad p^2 = \frac{(a+2b)^2}{4b(a+b)}(r^2-a^2). \tag{9}$$

SECTION 2.—*Tangents and normals to polar curves.*

269.] Let $r = f(\theta)$ be the general type of the explicit equation to polar curves, and let us assume the figure drawn in fig. 89 to be the normal form of such curves; which figure the student is recommended to examine carefully, for the values of the lines in connexion with it will be deduced from the geometry of it.

Let S be the pole, SX the prime radius, APQ the curve, PSX $= \theta$, SP $= r$. Let XSP be increased by a small angle QSP $= d\theta$, then SQ $= f(\theta + d\theta) = r + dr$. From centre S, with radius SP = R, describe the small arc PR, subtending $d\theta$.

$$\therefore \quad \text{PR} = r\,d\theta, \tag{10}$$

$$\text{RQ} = dr. \tag{11}$$

Let PQ, the element of the arc of the curve, be represented by ds;

$$\therefore \quad \text{PQ}^2 = \text{PR}^2 + \text{RQ}^2,$$

$$ds^2 = dr^2 + r^2 d\theta^2. \tag{12}$$

Through the two points, P, Q, on the curve, let the straight line QPT be drawn; then, as the two points are infinitesimally near to each other, the line is a tangent, in accordance with the definition of a tangent given in the last Chapter; through P draw the normal PG, and through S draw TSG perpendicular to the radius vector SP, and SY perpendicular to the tangent PT. The lengths PT and PG are respectively called the Polar Tangent and the Polar Normal; SG is called the Polar Subnormal; ST the Polar Subtangent; and SY, the perpendicular from the pole on the tangent, is symbolized by p. The value of these lines we proceed to determine.

Since $\tan \text{PQR} = \dfrac{\text{PR}}{\text{RQ}}$, we have, from (10) and (11),

$$\tan \text{PQR} = \frac{r\,d\theta}{dr}. \tag{13}$$

And since SPT = SQT + PSQ = PQR + $d\theta$; therefore, SPT and PQR being in general finite angles, and $d\theta$ being an infinitesimal angle, we must neglect $d\theta$ in the above equation, and thus SPT = PQR; and SPT is the angle contained between the curve and the radius vector;

$$\therefore \quad \tan \text{SPT} = \frac{r\,d\theta}{dr}; \tag{14}$$

$$\frac{\sin \text{SPT}}{r\,d\theta} = \frac{\cos \text{SPT}}{dr} = \frac{1}{ds}, \tag{15}$$

by reason of Preliminary Theorem I, and equation (12) above.

Hence also the following values result:

$$\left.\begin{aligned} \text{ST} &= \text{polar subtangent} = \text{SP} \tan \text{SPT} = \frac{r^2\,d\theta}{dr}, \\ \text{SG} &= \text{polar subnormal} = \text{SP} \tan \text{SPG} = \text{SP} \cot \text{SPT} = \frac{dr}{d\theta}; \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} \text{PT} &= \text{polar tangent} = \text{SP} \sec \text{SPT} = \frac{r\,ds}{dr}, \\ \text{PG} &= \text{polar normal} = \text{SP} \operatorname{cosec} \text{SPT} = \frac{ds}{d\theta}; \end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned} \text{SY} &= p = \text{SP} \sin \text{SPY} = \frac{r^2\,d\theta}{ds} = \frac{r^2\,d\theta}{(dr^2 + r^2\,d\theta^2)^{\frac{1}{2}}}, \\ \text{PY} &= \text{SP} \cos \text{SPY} = \frac{r\,dr}{ds} = (r^2 - p^2)^{\frac{1}{2}}. \end{aligned}\right\} \quad (18)$$

Similarly may the values of other lines be determined.

270.] The value of p may be put under another form which is often convenient. Let u be the reciprocal of the radius vector, so that $u = \frac{1}{r}$; then

$$du = -\frac{dr}{r^2};$$

and from above, $\frac{1}{p^2} = \frac{dr^2 + r^2\,d\theta^2}{r^4\,d\theta^2} = \frac{1}{r^2} + \frac{1}{r^4}\frac{dr^2}{d\theta^2}$;

therefore substituting, in terms of u,

$$\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2}. \qquad (19)$$

The value of p in (18) might also have been deduced as follows, from the expression for p in equation (44), Art. 219, viz.:

$$p = y\frac{dx}{ds} - x\frac{dy}{ds};$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}; \qquad \therefore \quad \left.\begin{aligned} dx &= dr\cos\theta - r\sin\theta\,d\theta \\ dy &= dr\sin\theta + r\cos\theta\,d\theta \end{aligned}\right\}; \qquad (20)$$

$$\therefore \quad y\,dx - x\,dy = -r^2\,d\theta, \qquad ds = \pm\,\{dr^2 + r^2\,d\theta^2\}^{\frac{1}{2}};$$

$$\therefore \quad p = \pm\,\frac{r^2\,d\theta}{ds}. \qquad (21)$$

271.] It is frequently necessary to express the geometrical quantities of Article 269 in terms of p and r.

By similar triangles PQR, SPY,

$$\frac{\text{QR}}{\text{PY}} = \frac{\text{RP}}{\text{YS}} = \frac{\text{PQ}}{\text{SP}};$$

$$\text{or,} \qquad \frac{dr}{(r^2-p^2)^{\frac{1}{2}}} = \frac{r\,d\theta}{p} = \frac{ds}{r}; \tag{22}$$

$$\therefore \quad ds = \frac{r\,dr}{(r^2-p^2)^{\frac{1}{2}}}, \tag{23}$$

$$r^2\,d\theta = \frac{rp\,dr}{(r^2-p^2)^{\frac{1}{2}}}. \tag{24}$$

272.] **Examples illustrative of the preceding theory.**

Ex. 1. The Spiral of Archimedes.

$$r = a\theta; \qquad \therefore \quad dr = a\,d\theta.$$

$$\therefore \quad \frac{dr}{a} = \frac{r\,d\theta}{r} = \frac{ds}{(a^2+r^2)^{\frac{1}{2}}};$$

$$\therefore \quad p = \frac{r^2}{(a^2+r^2)^{\frac{1}{2}}}; \quad \text{the polar normal} = (a^2+r^2)^{\frac{1}{2}};$$

$$\text{the polar subnormal} = \frac{dr}{d\theta} = a.$$

Ex. 2. The circle, the pole being at the end of a diameter.

$$r = 2a\cos\theta; \qquad \therefore \quad dr = -2a\sin\theta\,d\theta;$$

$$\frac{-dr}{2a\sin\theta} = \frac{r\,d\theta}{r} = \frac{ds}{2a};$$

$$\therefore \quad p = \frac{r^2}{2a} = r\cos\theta = x = \text{the rectangular abscissa};$$

$$\frac{r\,d\theta}{dr} = -\frac{r}{2a\sin\theta} = -\cot\theta.$$

Ex. 3. The logarithmic spiral; see fig. 86.

$$r = a^{\theta}; \qquad \therefore \quad dr = \log_e a . a^{\theta}\,d\theta = \log_e a . r\,d\theta;$$

$$\frac{dr}{\log a} = \frac{r\,d\theta}{1} = \frac{ds}{\{1+(\log_e a)^2\}^{\frac{1}{2}}};$$

$$\therefore \quad \frac{r\,d\theta}{dr} = \frac{1}{\log_e a} = \tan \text{SPT};$$

which is a constant; and therefore the curve cuts all its radii vectores at a constant angle; and accordingly it is called the Equiangular Spiral.

Also
$$p = \frac{r^2\,d\theta}{ds} = \frac{r}{\{(\log_e a)^2+1\}^{\frac{1}{2}}};$$

which may be written in the form,
$$p = mr; \tag{25}$$

and this is the equation to the equiangular spiral in terms of p and r, and wherein m is the sine of the constant angle contained between the radius vector and the curve.

Ex. 4. The lituus.
$$r = \frac{a}{\theta^{\frac{1}{2}}}; \qquad \therefore \quad dr = -\frac{a\,d\theta}{2\theta^{\frac{3}{2}}} = -\frac{r\,d\theta}{2\theta};$$
$$\therefore \quad \frac{dr}{1} = \frac{-r\,d\theta}{2\theta} = \frac{\pm\,ds}{(1+4\theta^2)^{\frac{1}{2}}};$$
$$\therefore \quad p = \frac{r^2\,d\theta}{ds} = \frac{2a^2r}{(r^4+4a^4)^{\frac{1}{2}}}. \tag{26}$$

Ex. 5. To find the relation between p and r in the conics, the focus being the pole.

The general equation in terms of r and θ is
$$r = \frac{2ae}{1+e\cos\theta};$$

wherein $2a$ is the distance from the focus to the directrix. Taking formula (19),
$$u = \frac{1+e\cos\theta}{2ae}; \qquad \therefore \quad \frac{du}{d\theta} = \frac{-\sin\theta}{2a};$$
$$\therefore \quad u^2 + \frac{du^2}{d\theta^2} = \frac{1}{4a^2e^2} + \frac{1}{4a^2} + \frac{\cos\theta}{2a^2e},$$
$$= \;-\;-\;-\;-\;+\frac{1}{ae}\left(u - \frac{1}{2ae}\right);$$
$$\therefore \quad \frac{1}{p^2} = \frac{u}{ae} - \frac{1-e^2}{4a^2e^2},$$
$$= \frac{1}{aer} - \frac{1-e^2}{4a^2e^2};$$

and the equation represents an ellipse, a parabola, or a hyperbola, according as e is less than, equal to, or greater than, unity.

Hence the equation to the parabola is
$$p^2 = ar.$$

SECTION 3.—*Asymptotes to polar curves.*

273.] Curves referred to polar coordinates admit of rectilinear and curvilinear asymptotes, in the same manner as those referred to rectangular coordinates. As curvilinear asymptotes however are of little use in determining the course of a curve, we shall say nothing of them in general, and shall describe only one remarkable species, viz. the asymptotic circle.

As a rectilinear asymptote is a tangent to a curve at an infinite distance, the formulæ of Art. 269 enable us to determine it.

If for any *finite* value of θ, say $\theta = a$, r is infinite, then either the radius vector itself, or a line parallel to it, is an asymptote to the curve; and the polar subtangent, which is equal to $r^2 \frac{d\theta}{dr}$, becomes in this case the perpendicular distance from the pole on the tangent; thus, if the value of $\frac{r^2 d\theta}{dr}$, corresponding to $\theta = a$ and $r = \infty$, is finite, the line can be constructed; if $\frac{r^2 d\theta}{dr} = 0$, the radius vector itself is the asymptote; and if it is equal to ∞, the asymptote, being at an infinite distance from the pole, cannot be constructed. An inspection of fig. 90 will render this plain; in which SP is the infinite radius vector, TL the asymptote, ST the value of $\frac{r^2 d\theta}{dr}$, when $\theta = a$, and $r = \infty$. If θ has many values for which r is infinite, there may be many rectilinear asymptotes. Hence, to determine them, we must find what finite values of θ render $r = \infty$. If the polar subtangents, corresponding to such infinite values of r and finite values of θ, are finite, the curve has rectilinear asymptotes which may be constructed in the way explained above.

It is to be borne in mind that when $\frac{r^2 d\theta}{dr}$ is positive, the asymptote lies *below* the radius vector, as in fig. 90; and if it is negative, the asymptote lies *above* it, as in fig. 91.

Or, in other words, according as $r^2 \frac{d\theta}{dr}$ is positive or negative, so is the perpendicular on the asymptote to be drawn *in consequentia* or *in antecedentia*.

274.] Examples illustrative of the preceding theory.

Ex. 1. To find the positions of the asymptotes of the hyperbola whose polar equation is

$$\frac{(\cos\theta)^2}{a^2} - \frac{(\sin\theta)^2}{b^2} = \frac{1}{r^2}.$$

$$\therefore \quad r = \infty, \text{ when } \tan\theta = \pm\frac{b}{a}.$$

Therefore the asymptotes are inclined to the prime radius at $\tan^{-1}\left(\pm\frac{b}{a}\right)$. Also

$$r^2\frac{d\theta}{dr} = \pm\frac{ab\,\{b^2(\cos\theta)^2 - a^2(\sin\theta)^2\}^{\frac{1}{2}}}{(a^2+b^2)\sin\theta\cos\theta};$$

which is equal to 0, at the critical angles; therefore both asymptotes pass through the pole.

Ex. 2. To determine the position of the asymptote to the Conchoid of Nicomedes; see Art. 196, equation (18).

$$r = a\sec\theta + b.$$

$r = \infty$, when $\theta = \frac{\pi}{2}$; the asymptote therefore is perpendicular to the prime radius.

Also since $$\frac{1}{r} = \frac{\cos\theta}{a+b\cos\theta},$$

$$-\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{a\sin\theta}{(a+b\cos\theta)^2};$$

$$\therefore \quad \frac{r^2 d\theta}{dr} = a, \quad \text{when } \theta = \frac{\pi}{2}.$$

The asymptote therefore cuts the prime radius at right angles, and at a distance a from the pole in the positive direction; see fig. 36.

Ex. 3. To determine the asymptotes to the lituus.

$$r = \frac{a}{\theta^{\frac{1}{2}}}; \qquad \therefore \quad r = \infty, \text{ when } \theta = 0;$$

$$\frac{1}{r} = \frac{(\theta)^{\frac{1}{2}}}{a}; \qquad \therefore \quad -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{2a(\theta)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{r^2 d\theta}{dr} = -2a\theta^{\frac{1}{2}} = 0, \text{ when } \theta = 0.$$

The prime radius therefore is an asymptote, as delineated in fig. 85.

Ex. 4. To determine the asymptotes of the reciprocal spiral.

$$r = \frac{a}{\theta}; \qquad \therefore \quad r = \infty, \text{ when } \theta = 0.$$

Also
$$\frac{1}{r} = \frac{\theta}{a}, \qquad \therefore \quad -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{a};$$

$$\therefore \quad \frac{r^2 d\theta}{dr} = -a.$$

The asymptote therefore is a line parallel to the prime radius, at a distance a from it, to be measured *in antecedentia*, since $\frac{r^2 d\theta}{dr}$ is negative; see fig. 84.

275.] Asymptotic circles.

Suppose the equation to a polar curve to be such that r approaches to a finite limit, say a, as θ is infinitely increased; then the curve approaches more and more nearly to a circle whose radius is a, which circle is said to be asymptotic to the curve; and if the curve approaches to it from the outside, the circle is called an interior asymptotic circle, and if from the inside, an exterior asymptotic circle.

Ex. 1.
$$r = a\left(\frac{1+\theta}{\theta}\right),$$

which may be written in the form,

$$r = a + \frac{a}{\theta}.$$

First, let θ be positive; then r is always greater than a; and when $\theta = 0$, r is ∞, and $r^2\frac{d\theta}{dr} = -a$, shewing that the straight line parallel to the prime radius at a distance a above it is an asymptote to the curve; and when $\theta = \infty$, $r = a$; whence we have an interior asymptotic circle such as is drawn in fig. 91.

Secondly, let θ be negative; then

$$r = a - \frac{a}{\theta};$$

and therefore when $\theta = 0$, $r = -\infty$, and r is negative as θ increases until $\theta = 1$, in which case $r = 0$, and thence r is always less than a until $\theta = \infty$, when $r = a$. Thus we have the curve dotted in the figure, and with an exterior asymptotic circle of radius SA $= a$; the continuity of the two branches of the curve is worth remarking.

Ex. 2. To determine the asymptotic circle to the curve,

$$\theta(2ar-r^2)^{\frac{1}{2}} = 1;$$

$$\therefore \quad \theta = \infty, \text{ when } r = 2a, \text{ and when } r = 0.$$

Therefore the circle whose radius is $2a$ is asymptotic to the curve; and as r must be always less than $2a$, otherwise θ would be affected with $\sqrt{-}$, the asymptotic circle is exterior to the curve.

SECTION 4.—*Direction of curvature, and points of inflexion.*

276.] On an inspection of the figures numbered 92 and 93 it is manifest that, if a curve referred to polar coordinates is concave towards the pole, as r increases, p increases also, and therefore $\frac{dr}{dp}$ is positive; and if the curve is convex towards the pole, as r increases, p decreases, and *vice versâ*, and therefore $\frac{dr}{dp}$ is negative. If therefore the equation to the curve is given in the form $r = f(\theta)$, in order to determine whether the curve is concave or convex towards the pole, we must transform the equation into its equivalent between r and p, by means of the relations given in (19) or (21) of Art. 270, and thence find $\frac{dr}{dp}$; and for all values for which $\frac{dr}{dp}$ is positive, the curve is concave towards the pole; and for all values for which $\frac{dr}{dp}$ is negative, the curve is convex towards the pole; and therefore if at any point $\frac{dr}{dp}$ changes sign by passing through 0 or ∞, at such a point the direction of curvature changes and there is a point of inflexion; hence, to determine such points, we must equate $\frac{dr}{dp}$ to 0 and to ∞, and examine whether $\frac{dr}{dp}$ changes sign; if it does, there is a point of inflexion.

Ex. 1. To determine the point of inflexion of the lituus.

$$r = \frac{a}{\theta^{\frac{1}{2}}}.$$

By equation (26), Art. 272, $p = \dfrac{2a^2 r}{(4a^4+r^4)^{\frac{1}{2}}}$;

$$\therefore \quad \frac{dp}{dr} = \frac{2a^2(4a^4-r^4)}{(4a^4+r^4)^{\frac{3}{2}}},$$

$= 0$, if $r = a\sqrt{2}$, and changes sign from $+$ to $-$; the curve therefore having been concave towards the pole for values of r less than $a\sqrt{2}$, changes its direction of curvature at that point, and becomes convex towards the pole; see fig. 85, SB $= a\sqrt{2}$.

Ex. 2. To prove that the equiangular spiral is always concave towards the pole.

$$r = a^{\theta};$$

and by (25), Art. 272, $p = mr$; therefore $\dfrac{dp}{dr} = m$, which is always positive, and therefore the curve is always concave towards the pole.

SECTION 5.—*On tracing polar curves by means of their equations.*

277.] Having discussed all the peculiarities which curves referred to polar coordinates *generally* admit of, we are now in a condition to analyse the equations, and to give general rules for tracing the curves of which they are the mathematical expressions and definitions.

1) If the equation is of the form $r = f(\theta) \pm \phi(\theta)$, so that $r = f(\theta)$ is diametral to the curve to be traced, we had better trace separately the two curves $r = f(\theta)$ and $r = \phi(\theta)$, and then by addition and subtraction of the radii vectores trace the required curve. Thus if it is required to draw the curve whose equation is $r = a(2 \pm \sin\theta)$, the circle whose radius is $2a$ is diametral to the required curve, and its radii are to be increased and diminished by $a\sin\theta$ corresponding to the several values of θ; see fig. 94.

2) Investigate the several values of θ which make $r = 0$, and $= \infty$; and in the latter case, if the value of θ is finite, determine whether the polar subtangent is finite or not, as this is the criterion whether the rectilinear asymptote can be constructed or not. Give such particular values to θ as the equa-

tion suggests; as e.g. if the equation involves a function of 3θ, put $\theta = 15°, 30°, 45°$, &c.; or if the equation involves a function of $\frac{\theta}{4}$, put $\theta = 60°, 90°, 120°, 180°$, and so on. In general give to θ values such that r may be constructed; and, by giving to θ the values 0 and $n\pi$, we find the values of r when the curve cuts the prime radius, or the prime radius produced backwards; and make r revolve in both directions.

3) It is convenient to find $\frac{dr}{d\theta}$, as it is the ratio of the corresponding increments of r and θ; and therefore, if it is positive, as θ increases, r increases; and, if it is negative, r decreases as θ increases, and *vice versâ*. And if $\frac{dr}{d\theta} = 0$, we have no increase of r corresponding to an increase of θ; that is, the curve is at right angles to the radius vector; which is also manifest from equation (14), Art. 269, because at such a point $\tan \textsc{spt} = \infty$. And if $\frac{dr}{d\theta} = 0$, and changes its sign, we have a maximum or minimum value of r, the point corresponding to which is called an *apse*; of which there are instances in the ellipse, if the focus is the pole, at the extremities of the major axis: and of the circle, if the centre is the pole, every point is an apse.

4) Nothing more need be said on the subject of rectilinear asymptotes and asymptotic circles; or

5) On the direction of curvature and points of inflexion.

278.] Hence then to trace a curve referred to polar coordinates,

I. Investigate, arrange, and tabulate with their proper signs, all the particular values of θ which render $r = 0$, and $= \infty$; and equal to a value that may be constructed without difficulty.

II. Find $\frac{dr}{d\theta}$; examine its sign, and the values of θ at which it is equal to 0, and to ∞, and whether it changes its sign; if it does, at such points there are maximum and minimum radii vectores.

III. Determine whether any finite values of θ render $r = \infty$; if so, find the value of $r^2 \frac{d\theta}{dr}$ corresponding to this value of θ, and construct the asymptote. Examine whether there is an asymptotic circle.

IV. Transform the equation into its equivalent between r and p; find $\frac{dr}{dp}$, and examine its sign, for the purpose of determining whether the curve is convex or concave towards the pole; also examine whether $\frac{dr}{dp}$ changes its sign by passing through 0 or ∞, for a point at which such a change takes place will be a point of inflexion.

V. Trace the curve in a similar manner, by making r to revolve in a negative direction.

279.] Examples illustrative of the preceding theory.

Ex. 1. Trace the curve $r = a \sin 2\theta$.

$$\frac{dr}{d\theta} = 2a \cos 2\theta.$$

The curve has no rectilinear asymptotes, because no value of θ makes $r = \infty$; hence we may tabulate as follows:

	θ	r	$\frac{dr}{d\theta}$
1	0	$-, 0, +$	$+$
2	$\frac{\pi}{4}$	a	$+, 0, -$
3	$\frac{\pi}{2}$	$+, 0, -$	$-$
4	$\frac{3\pi}{4}$	$-a$	$-, 0, +$
5	π	$-, 0, +$	$+$

which values are sufficient to enable us to draw the curve.

We have thus examined the course of the curve through two right angles; and as $\sin 2\theta$ has passed through all its values, it is unnecessary to tabulate further, as the tracing point will describe equal curves in the other two quadrants.

Also the revolution of the radius vector in a negative direction produces the same curve. The curve is delineated in fig. 95.

It begins, as shewn by 1, from the pole s, and as θ increases

r increases, until $\theta = \frac{\pi}{4}$, where the radius vector attains to a maximum value a, as shewn by 2; afterwards the radius vector decreases, becomes equal to zero, when $\theta = \frac{\pi}{2}$, and passes into the fourth quadrant, because for all values of θ between $\frac{\pi}{2}$ and π r is negative, and, when $\theta = \frac{3\pi}{4}$, attains to a minimum value $-a$, and describing a loop exactly equal to that in the first quadrant, falls into the pole when $\theta = \pi$; afterwards in the third quadrant r is positive, so that the tracing point describes another equal loop in it, reaching a maximum when $\theta = \frac{5\pi}{4}$, and the curve falls into the pole when $\theta = \frac{3\pi}{2}$; after which the radius vector again becomes negative; and therefore, passing through the fourth quadrant, describes the loop in the second quadrant, which is exactly equal to those which have been already traced out in the other three quadrants.

Ex. 2. Trace the curve $r = a \sin 3\theta$.

$$\frac{dr}{d\theta} = 3a \cos 3\theta.$$

	θ	r	$\frac{dr}{d\theta}$
1	$\frac{\pi}{6}$	a	$+, 0, -$
2	$\frac{\pi}{3}$	$+, 0, -$	$-$
3	$\frac{\pi}{2}$	$-a$	$-, 0, +$
4	$\frac{2\pi}{3}$	$-, 0, +$	$+$
5	$\frac{5\pi}{6}$	$-a$	$+, 0, -$
6	π	$+, 0, -$	$-$

Whence the curve is manifestly that drawn in fig. 96. If the radius vector revolves in the negative direction, the same three loops will be traced out.

From this and the former example it appears that in all curves whose equations are of the form

$$r = a \sin n\theta,$$

the curve consists of n loops if n is an odd number, and of $2n$ loops if n is an even number.

Ex. 3. Trace the curve whose equation is

$$r = a \sin \frac{\theta}{2}.$$

$$\frac{dr}{d\theta} = \frac{a}{2} \cos \frac{\theta}{2};$$

therefore r is never greater than a; and $r = 0$, when $\theta = 0$, $= 2\pi$, $= \ldots = 2n\pi$.

Also $\frac{dr}{d\theta} = 0$, when $\theta = \pi$, $= 3\pi$, $= \ldots = (2n+1)\pi$.

Accordingly we have the following table:

	θ	r	$\frac{dr}{d\theta}$
1	0	−, 0, +	+
2	π	a	+, 0, −
3	2π	+, 0, −	−
4	3π	$-a$	−, 0, +
5	4π	−, 0, +	+

Hence the radius vector is zero, when $\theta = 0$, and attains a maximum value a, when $\theta = \pi$; whence it decreases, becoming 0, when $\theta = 2\pi$, until it reaches a minimum $-a$, when $\theta = 3\pi$; after which it increases, passing through zero, when $\theta = 4\pi$, and becomes a, when $\theta = 5\pi$; wherefore the curve is that drawn in fig. 97.

Ex. 4. Trace the curve whose equation is

$$r^2 = a^2 \{(\tan\theta)^2 - 1\}.$$

$$\therefore \quad r = \pm a \{(\tan\theta)^2 - 1\}^{\frac{1}{2}};$$

therefore r cannot be constructed whenever $(\tan\theta)^2$ is less than 1. Also as r is affected with $\pm$, the pole is the centre of the curve.

And as $r = \infty$, when $\theta = \frac{\pi}{2}$ and $= \frac{3\pi}{2}$, we must find $r^2\frac{d\theta}{dr}$ in order to determine the asymptote.

$$\frac{1}{r} = \frac{\pm 1}{a\{(\tan\theta)^2-1\}^{\frac{1}{2}}},$$

$$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{\mp\tan\theta\,(\sec\theta)^2}{a\{(\tan\theta)^2-1\}^{\frac{3}{2}}};$$

$$\therefore\quad r^2\frac{d\theta}{dr} = \pm\frac{a\{(\tan\theta)^2-1\}^{\frac{3}{2}}}{\tan\theta\,(\sec\theta)^2} = \pm a, \text{ when } \theta = \frac{\pi}{2}, \text{ and } = \frac{3\pi}{2}.$$

Hence there are two asymptotes perpendicular to the prime radius, at distances $\pm a$, from the pole.

	θ	r	$\frac{dr}{d\theta}$
1	$\frac{\pi}{4}$	$\pm\sqrt{-}$, 0, $\pm$	$\pm$
2	$\frac{\pi}{2}$	$\pm$, ∞, $\pm$	$\pm$
3	$\frac{3\pi}{4}$	$\pm$, 0, $\pm\sqrt{-}$	$\pm$

The curve therefore is that delineated in fig. 98.

Ex. 5. Trace the curve whose equation is

$$r = a\frac{\theta^2}{\theta^2-1}. \qquad \therefore\quad \frac{dr}{d\theta} = \frac{-2a\theta}{(\theta^2-1)^2}.$$

Also $r = \infty$, when $\theta = \pm 1$; therefore we must find $r^2\frac{d\theta}{dr}$ in order to determine the asymptote.

$$\frac{1}{r} = \frac{\theta^2-1}{a\theta^2}, \qquad -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{2}{a\theta^3};$$

$\therefore\quad r^2\frac{d\theta}{dr} = -\frac{a\theta^3}{2} = \mp\frac{a}{2}$, when $\theta = \pm 1$; the asymptotes therefore are inclined at ± 1 to the prime radius, and the perpendicular distances from the pole on them are $\mp\frac{a}{2}$.

Also, when $\theta = \infty$, $r = a$, and therefore the circle whose radius is a is asymptotic, and is an interior asymptotic circle, bceause r is greater than a.

As the radius vector revolves in the positive direction, $r = 0$, when $\theta = 0$, and is negative, and negatively increases, until $\theta = 1$, when $r = -\infty$; and changes to $+\infty$, approaching to the rectilinear asymptote, receding on one side of it, and returning on the other; after which, as θ increases, r decreases, until it attains its least value a, when $\theta = \infty$.

Again, as r revolves in a negative direction, it must be measured backwards from $\theta = 0$ to $\theta = -1$, at which latter angle $r = -\infty$, and then changes its sign to $+\infty$; that is, the branches of the curve have approached the rectilinear asymptote, and cut it at infinity; and as θ increases, r continually decreases and approaches to the asymptotic circle, of which the radius is a. See fig. 99, in which the dotted branches indicate the parts due to the negative revolution of r.

Ex. 6. Trace the curve whose equation is

$$r = a\frac{\theta + \sin\theta}{\theta - \sin\theta}. \qquad \frac{dr}{d\theta} = \frac{2a(\theta\cos\theta - \sin\theta)}{(\theta - \sin\theta)^2};$$

if θ is positive, r is positive, since the arc is greater than its sine. And since for all values of θ in the first and second quadrants $\sin\theta$ is positive, and for values in the third and fourth quadrants $\sin\theta$ is negative, therefore in the first and second quadrants r is greater than a, and in the third and fourth r is less than a.

And when $\theta = 0$, $\sin\theta = \theta$; and therefore when $\theta = 0$, $r = \infty$; hence, to determine the corresponding polar subtangent, we have

$$r^2\frac{d\theta}{dr} = \frac{a}{2}\,\frac{(\theta + \sin\theta)^2}{\theta\cos\theta - \sin\theta} = \frac{0}{0}, \text{ when } \theta = 0,$$

$$= \frac{a}{2}\,\frac{2(\theta + \sin\theta)(1 + \cos\theta)}{-\theta\sin\theta} = \frac{0}{0}, \text{ when } \theta = 0,$$

$$= a\frac{(1 + \cos\theta)^2 - \sin\theta\,(\theta + \sin\theta)}{-\sin\theta - \theta\cos\theta} = \infty, \text{ when } \theta = 0;$$

therefore the rectilinear asymptote cannot be drawn.

When $\theta = \infty$, $r = a$; therefore there is an asymptotic circle, whose radius is a. Hence we tabulate as follows:

	θ	r
1	0	∞
2	$\frac{\pi}{2}$	$a\frac{\pi+2}{\pi-2}$
3	π	a
4	$\frac{3\pi}{2}$	$a\frac{3\pi-2}{3\pi+2}$
5	2π	a
6	$\frac{5\pi}{2}$	$a\frac{5\pi+2}{5\pi-2}$
7	3π	a
8	∞	a

It appears then that the curve starts from infinity, as delineated in fig. 100, and periodically, when $\theta = \pi$, $= 2\pi$, $= \ldots$, passes through the points A and B, which are the extremities of the diameter of the circle whose centre is the pole and whose radius is a; to which circle the curve continually approaches, being outside in the first and second quadrants, and inside in the third and fourth. There is then this peculiarity, that the curve on the outside is gradually becoming nearer and nearer to the circle, and the curve on the inside is receding from the diameter as θ increases and approaching to coincidence with the circle.

CHAPTER XII.

ON THE CURVATURE OF PLANE CURVES.

SECTION 1.—*Curves referred to rectangular coordinates.*

280.] IMAGINE a tangent to be drawn at a point in a plane curve, which is such that the curve lies entirely on one side of the tangent; then the curve is said to be *convex* towards that side of space on which the tangent lies, and *concave* towards the other side; such is our definition of concavity and convexity; and on such a conception were investigated in Chapter X the analytical criteria for determining the direction of curvature. Let us moreover suppose that at the point of the curve under consideration there is no discontinuity, or indeterminateness of derived-functions; then, as the curve deviates from the tangent line, such a deviation may be greater or less, or, in other words, the curve may be more or less bent; herein then we have a new affection, viz. the amount of bending or of curvature, as it is called: the nature of which we propose to examine in the present Chapter.

And to consider it from another point of view; an infinitesimal element of the curve commencing from a given point being straight, it is in its length coincident with the tangent line at that point; and the next element being inclined at an angle to the former one deviates from the tangent. Now let the two consecutive elements be of equal lengths, and from the extremity of the second let a perpendicular be drawn to the tangent: as this perpendicular is longer or shorter, so will the deviation be greater or less, and the curve will be more or less bent.

These terms however are but relative; and accordingly it is necessary to fix on some standard with which to compare such amount of bending, and to investigate some means by which the comparison may be made.

The circle naturally suggests itself for a standard; whatever its curvature or bending is, it is the same at all points of the same circle: and in different circles, as the radius changes, so does the curvature. As the radius increases, the deviation from a straight line becomes less and less; and in the limit vanishes when the radius becomes infinite; see Art. 186; and as the radius decreases, the curvature increases, and in the limit when the radius becomes zero, the curvature becomes infinite; for the circle becomes a point, and the curve at once returns into itself; the radius of the circle therefore, and its curvature or deviation from a straight line, are so related that one varies inversely as the other. If then r is the radius of a circle, the amount of deviation of that circle from a straight line is a function of the reciprocal of r; let us give some definite name to this deviation, and in order that we may have a measure of it, let us define it. *Curvature* is the name which we shall adopt; and our mathematical definition of it is the simplest function of the reciprocal of r, viz.,

$$\text{The curvature of a circle} = \frac{1}{r};$$

so that the curvature $= 1$, when $r = 1$; and therefore the curvature of a circle whose radius is unity is the unit-curvature.

Now when a finite arc of a circle is given, we can in many ways determine the radius of the circle; but when the arc is infinitesimal, the following is best adapted to our present conceptions.

The relation between an arc, the radius, and the angle subtended at the centre is, see Art. 24, (15),

$$\text{the arc} = \text{the radius} \times \text{the angle}.$$

Let (x, y), $(x+dx, y+dy)$ be any two points on a circle infinitesimally near to each other; and let ds be the arc between the two points; let normals be drawn to the circle at the points (x, y), $(x+dx, y+dy)$, and let $d\psi$ be the angle at the centre contained between these normals, and let r be the radius: then

$$ds = r\,d\psi; \tag{1}$$

$$r = \frac{ds}{d\psi}; \tag{2}$$

and the curvature of the circle $= \dfrac{1}{r} = \dfrac{d\psi}{ds}$.

281.] Let us now extend these principles to any plane curve.

Suppose the two points (x, y), $(x+dx, y+dy)$ to be on a plane curve; and at neither of them let there be a point of discontinuity or of inflexion; then, although the ratio given in (2) may be no longer constant at all points of a curve, but may vary as we pass from one point to another; yet when the distance ds is infinitesimal, $\frac{d\psi}{ds}$ will assume some determinate value, which we may call the *curvature of the curve* at the point; perhaps it may be said that it is a measure of the *mean* curvature of the arc, but the difference between that and the *actual* curvature at the point (x, y) is infinitesimal, and therefore must be neglected, so that the two become identical.

Imagine then two normals to be drawn at two consecutive points of the curve; these will generally meet at a finite distance; let $d\psi$ be the small angle included between them, and let ρ be the distance from the curve of the point at which they intersect; then, by reason of (2), and introducing $\pm$ so as to include all the possible simultaneous changes of s and ψ, we have

$$\rho = \pm \frac{ds}{d\psi}; \tag{3}$$

From the analogy of the circle, ρ is called the *radius of curvature,* and the point at which the two consecutive normals intersect is called the *centre of curvature*; and therefore we have the following definition:

The distance from the curve at which two consecutive normals of a plane curve intersect is the radius of curvature of the curve at that point.

282.] To determine the analytical values of the radius of curvature.

Let the equation to the curve be $y = f(x)$; see fig. 101; and let P, Q be the two points on it, infinitesimally near to one another, at which the normals are PΠ, QΠ; Π their point of intersection, which is therefore the centre of curvature; through G draw a line KG parallel to QΠ; then $PQ = ds$, $P\Pi Q = d\psi = PGK = d.\tan^{-1}\left(\frac{dx}{dy}\right)$; $P\Pi = Q\Pi = \rho =$ the radius of curvature;

$$\therefore \quad ds = \pm \rho\, d.\tan^{-1}\left(\frac{dx}{dy}\right), \tag{4}$$

$$= \pm \rho \frac{d^2x\,dy - d^2y\,dx}{dy^2} \cdot \frac{dy^2}{dx^2 + dy^2};$$

$$\therefore \quad ds^3 = \pm \rho\,(d^2x\,dy - d^2y\,dx), \tag{5}$$

$$\rho = \frac{\pm\, ds^3}{d^2x\,dy - d^2y\,dx}; \tag{6}$$

which expression is the most general value of ρ, as in it neither x, y, nor s is equicrescent.

If x is equicrescent, $d^2x = 0$; and we have

$$\rho = \frac{\mp\, ds^3}{d^2y\,dx} = \mp \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}; \tag{7}$$

And, if y is equicrescent, $d^2y = 0$, and

$$\rho = \frac{\pm\, ds^3}{d^2x\,dy} = \frac{\mp \left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}. \tag{8}$$

It is to be observed that ρ is an absolute length; some principle therefore must be adopted by which the ambiguity of sign in the preceding expressions may be removed. Let us suppose the type-curve to be that represented in figs. 47, 101, and 102; and let us assume the radius of curvature to be positive, when it is drawn downwards and towards the axis of x; and to be negative when it is drawn upwards from the curve and away from the axis of x, as in fig. 106. Now let us take the value of ρ given in (7): in the former case $\frac{d^2y}{dx^2}$ is negative, because the curve is concave downwards; and therefore, as ρ is to be positive,

$$\rho = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}; \tag{9}$$

and if in the result ρ is affected with a negative sign, it is evident that ρ is drawn upwards from the curve, and that the curve is convex downwards, as in fig. 106. Similarly in (8) the right-hand member is to have a positive sign, and we have

$$\rho = \frac{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}; \tag{10}$$

which may also be derived from (9) by a change of the equicrescent variable.

283. Examples illustrative of the preceding.

Ex. 1. Determine the radius of curvature of an ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}; \qquad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{a^4 y^2 + b^4 x^2}{a^4 y^2};$$

$$\frac{d^2 y}{dx^2} = -\frac{b^4}{a^2 y^3};$$

$$\therefore \quad \text{by (9)}, \quad \rho = -\frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}} = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4}.$$

The radius of curvature at extremity of axis major $= \dfrac{b^2}{a}$,

- - - - - - - - - - - - - - - - axis minor $= \dfrac{a^2}{b}$.

Ex. 2. In curves of the second degree whose equations are of the form $y^2 = 4mx + nx^2$, the radius of curvature varies as the cube of the normal.

$$y \frac{dy}{dx} = 2m + nx,$$

$$\therefore \quad y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = n, \qquad \text{and} \quad \frac{d^2 y}{dx^2} = \frac{-4m^2}{y^3}.$$

Now by (42), Art. 219, the normal $= y \left\{1 + \dfrac{dy^2}{dx^2}\right\}^{\frac{1}{2}}$; substituting which, we have

$$\frac{\text{the radius of curvature}}{(\text{the normal})^3} = -\frac{1}{y^3 \frac{d^2 y}{dx^2}} = \frac{1}{4m^2}.$$

Q. E. D.

Ex. 3. To find the radius of curvature of the cycloid, the starting point being the origin.

$$x = a \operatorname{versin}^{-1} \frac{y}{a} - \{2ay - y^2\}^{\frac{1}{2}}.$$

$$\frac{dx}{dy} = \frac{y}{\{2ay - y^2\}^{\frac{1}{2}}}; \qquad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{2a}{y};$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}; \qquad \therefore \quad \rho = 2(2ay)^{\frac{1}{2}}.$$

Therefore, and from Ex. 6, Art. 221, it appears that the radius of curvature is equal to twice the normal.

Ex. 4. In illustration of formula (6), let it be required to find the length of the radius of curvature of the ellipse whose equations are, see Art. 193,

$$x = a\cos\theta, \qquad y = b\sin\theta.$$

$$dx = -a\sin\theta\, d\theta, \qquad dy = b\cos\theta\, d\theta,$$

$$d^2x = -a\cos\theta\, d\theta^2, \qquad d^2y = -b\sin\theta\, d\theta^2;$$

$$\therefore \quad ds^2 = \{a^2(\sin\theta)^2 + b^2(\cos\theta)^2\}\, d\theta^2,$$

$$= \left\{\frac{a^2}{b^2}y^2 + \frac{b^2}{a^2}x^2\right\} d\theta^2;$$

$$d^2x\, dy - d^2y\, dx = -ab\, d\theta^3;$$

$$\therefore \quad \rho = \pm\frac{\{a^4y^2 + b^4x^2\}^{\frac{3}{2}}}{a^4b^4}.$$

284.] On comparing figures 47 and 101, it appears that

$$\tau + \psi = \frac{\pi}{2}; \qquad \therefore \quad d\tau + d\psi = 0;$$

$$\therefore \quad \rho = \pm\frac{ds}{d\psi} = \mp\frac{ds}{d\tau};$$

$d\tau$, which is equal to TLT′, fig. 101, is the angle between two tangents drawn at consecutive points on the curve; it is therefore the angle at which two successive elements are inclined to each other, and is called *the angle of contingence.*

285.] We proceed to determine other values of ρ which are required in the sequel.

$$\because \quad ds^2 = dx^2 + dy^2;$$

$$\therefore \quad ds\,d^2s = dx\,d^2x + dy\,d^2y. \qquad (11)$$

Also by (6),
$$\pm\frac{ds^3}{\rho} = dy\,d^2x - dx\,d^2y. \qquad (12)$$

Therefore squaring (11) and (12), and adding, we have

$$ds^2(d^2s)^2 + \frac{ds^6}{\rho^2} = (dx^2 + dy^2)\{(d^2x)^2 + (d^2y)^2\},$$

$$= ds^2\{(d^2x)^2 + (d^2y)^2\}; \qquad (13)$$

$$\therefore \quad \frac{ds^4}{\rho^2} = (d^2x)^2 + (d^2y)^2 - (d^2s)^2; \tag{14}$$

$$\frac{1}{\rho^2} = \frac{1}{ds^4}\{(d^2x)^2 + (d^2y)^2 - (d^2s)^2\}. \tag{15}$$

Whence we have the following values for ρ:

(α) Let s be equicrescent; then $d^2s = 0$; and

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2. \tag{16}$$

Also by virtue of (11), since $d^2y = -\frac{dx}{dy}d^2x$,

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds\,dy}\right)^2 = \left(\frac{d^2x}{ds^2}\right)^2\frac{ds^2}{dy^2}; \tag{17}$$

and $$\frac{1}{\rho^2} = \left(\frac{d^2y}{ds\,dx}\right)^2 = \left(\frac{d^2y}{ds^2}\right)^2\frac{ds^2}{dx^2}. \tag{18}$$

(β) Let x be equicrescent; then $d^2x = 0$, and therefore by reason of (11),

$$d^2s = \frac{dy}{ds}d^2y;$$

$$\therefore \quad \frac{1}{\rho^2} = \frac{dx^6}{ds^6}\left(\frac{d^2y}{dx^2}\right)^2,$$

introducing dx^4 into the denominator to shew that x is equicrescent; see Article 54.

(γ) Let y be equicrescent; then $d^2y = 0$, and by the same process as above,

$$\frac{1}{\rho^2} = \frac{dy^6}{ds^6}\left(\frac{d^2x}{dy^2}\right)^2.$$

The last two values of ρ are the same as (9) and (10).

Also from (14), multiplying through by ds^2, and replacing $ds\,d^2s$ from (11), we have

$$\begin{aligned}\frac{ds^6}{\rho^2} &= (d^2x)^2\,ds^2 + (d^2y)^2\,ds^2 + (d^2s)^2\,ds^2 - 2\,(d^2s)^2\,ds^2,\\ &= (d^2x)^2\,ds^2 + (d^2y)^2\,ds^2 + (d^2s)^2\,(dx^2 + dy^2)\\ &\qquad\qquad - 2\,ds\,d^2s\,(dx\,d^2x + dy\,d^2y),\\ &= (d^2x\,ds - d^2s\,dx)^2 + (d^2y\,ds - d^2s\,dy)^2,\\ &= ds^4\left\{\left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2\right\};\end{aligned}$$

$$\therefore \quad \frac{ds^2}{\rho^2} = \left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2; \tag{19}$$

which is identical with (16) when s is equicrescent.

Hence also, and from Art. 284, we have the following value for the angle of contingence:

$$d\tau = \pm \left\{\left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2\right\}^{\frac{1}{2}}, \tag{20}$$

$$= \pm \{(d.\cos\tau)^2 + (d.\cos\psi)^2\}^{\frac{1}{2}},$$

where $\cos\psi$ and $\cos\tau$ are the cosines of the angles between the line of the radius of curvature and the coordinate axes; see Article 218.

Also from (11) and (12), eliminating d^2x, we have

$$ds\,dy\,d^2s \pm \frac{ds^3}{\rho}\,dx = d^2y\,(dy^2 + dx^2);$$

$$\therefore \quad \frac{ds^3}{\rho}\,dx = ds^2\,d^2y - d^2s\,dy\,ds,$$

$$\frac{dx}{ds} = \frac{\rho}{ds}\,d\left(\frac{dy}{ds}\right) = \cos\tau. \tag{21}$$

Similarly,
$$\frac{dy}{ds} = \frac{\rho}{ds}\,d\left(\frac{dx}{ds}\right) = \cos\psi. \tag{22}$$

Hence also, if s is equicrescent,

$$\cos\tau = \rho\,\frac{d^2y}{ds^2}, \tag{23}$$

$$\cos\psi = \rho\,\frac{d^2x}{ds^2}. \tag{24}$$

286.] If the equation to the curve is given in the implicit form,

$$F(x, y) = c, \tag{25}$$

we may substitute as follows in the general value of ρ given in equation (6).

Differentiating (25), we have

$$\left(\frac{dF}{dx}\right)dx + \left(\frac{dF}{dy}\right)dy = 0, \tag{26}$$

$$\left(\frac{d^2F}{dx^2}\right)dx^2 + 2\left(\frac{d^2F}{dx\,dy}\right)dx\,dy + \left(\frac{d^2F}{dy^2}\right)dy^2 + \left(\frac{dF}{dx}\right)d^2x + \left(\frac{dF}{dy}\right)d^2y = 0; \tag{27}$$

$$\therefore \quad \left(\frac{d\mathrm{F}}{dx}\right) d^2x + \left(\frac{d\mathrm{F}}{dy}\right) d^2y$$

$$= -\left\{\left(\frac{d^2\mathrm{F}}{dx^2}\right) dx^2 + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) dx\,dy + \left(\frac{d^2\mathrm{F}}{dy^2}\right) dy^2\right\}. \qquad (28)$$

Now from (26),

$$\frac{dy}{\left(\frac{d\mathrm{F}}{dx}\right)} = \frac{-dx}{\left(\frac{d\mathrm{F}}{dy}\right)} = \frac{d^2x\,dy - d^2y\,dx}{\left(\frac{d\mathrm{F}}{dx}\right) d^2x + \left(\frac{d\mathrm{F}}{dy}\right) d^2y}, \qquad (29)$$

the last following by reason of Preliminary Theorem I, if we operate upon the fractions successively by the factors d^2x and d^2y, and add numerators and denominators; each of which fractions is again equal to, by reason of the same Preliminary Theorem,

$$\frac{\pm\, ds}{\left\{\left(\frac{d\mathrm{F}}{dx}\right)^2 + \left(\frac{d\mathrm{F}}{dy}\right)^2\right\}^{\frac{1}{2}}}. \qquad (30)$$

Whence, equating (29) and (30), and replacing the numerator and denominator of (29) by their equivalents from (28) and (5), we have

$$\frac{\pm\, ds}{\left\{\left(\frac{d\mathrm{F}}{dx}\right)^2 + \left(\frac{d\mathrm{F}}{dy}\right)^2\right\}^{\frac{1}{2}}} = \frac{1}{\rho}\,\frac{ds^3}{\left(\frac{d^2\mathrm{F}}{dx^2}\right) dx^2 + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) dx\,dy + \left(\frac{d^2\mathrm{F}}{dy^2}\right) dy^2};$$

and, replacing dx, dy, ds in terms of their proportionals given in (29) and (30), we have finally

$$\frac{1}{\rho} = \pm\,\frac{\left(\frac{d\mathrm{F}}{dy}\right)^2\left(\frac{d^2\mathrm{F}}{dx^2}\right) - 2\left(\frac{d\mathrm{F}}{dx}\right)\left(\frac{d\mathrm{F}}{dy}\right)\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) + \left(\frac{d\mathrm{F}}{dx}\right)^2\left(\frac{d^2\mathrm{F}}{dy^2}\right)}{\left\{\left(\frac{d\mathrm{F}}{dx}\right)^2 + \left(\frac{d\mathrm{F}}{dy}\right)^2\right\}^{\frac{3}{2}}}. \qquad (31)$$

For an example of this formula, let us take the equation to the hyperbola,

$$\mathrm{F}(x, y) = xy = k^2;$$

$$\left(\frac{d\mathrm{F}}{dx}\right) = y, \qquad \left(\frac{d\mathrm{F}}{dy}\right) = x;$$

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right) = 0, \qquad \left(\frac{d^2\mathrm{F}}{dx\,dy}\right) = 1, \qquad \left(\frac{d^2\mathrm{F}}{dy^2}\right) = 0;$$

$$\therefore \quad \frac{1}{\rho} = \pm\,\frac{2xy}{(x^2+y^2)^{\frac{3}{2}}}.$$

287.] The numerator of (31) is the Hessian of the curve.

At a point of inflexion the Hessian $= 0$; and as $\left(\frac{d\text{F}}{dx}\right)$ and $\left(\frac{d\text{F}}{dy}\right)$ do not both simultaneously vanish, so at a point of inflexion $\rho = -\infty$; that is, the curve degenerates into a straight line; and the radius of curvature, which corresponds to a point at which two consecutive elements are in one and the same straight line, is evidently infinite. The Hessian also $= 0$ at double points; but then it vanishes identically, because $\left(\frac{d\text{F}}{dx}\right) = \left(\frac{d\text{F}}{dy}\right) = 0$; and therefore in this latter case ρ is indeterminate, and (31) must be evaluated. This is also otherwise apparent. At a double point the two branches may not have the same curvature, and as the same analytical expression gives the radius of curvature of both, it must necessarily take at that point an indeterminate form. If we multiply (31) by the denominator of the right-hand member, the expression becomes of $3n-3$ dimensions in terms of x and y: if then ρ is constant, the points of equal curvature on a curve of the nth degree are on a curve of the $3(n-1)$th degree; and as the number of the points of intersection of these two curves is $3n(n-1)$, so on a curve of the nth degree there may be $3n(n-1)$ points at which the curvature is the same.

SECTION 2.—*On evolutes of plane curves referred to rectangular coordinates.*

288.] We have thus far determined the length of the radius of curvature, and the cosines of the angles at which its line is inclined to the coordinate axes; we shall now investigate the coordinates to the centre of curvature; and since it changes position as the point, at which the radius of curvature is drawn, moves continuously along the curve, it thereby describes a continuous curve, which we shall determine. This curve is, for a reason which will shortly appear, called the *evolute* of the original curve.

Let the equation to the curve be $y = f(x)$, and let (x, y) be the point on it at which the radius of curvature is drawn; (ξ, η) the centre of curvature; so that, in fig. 102, we have

$$\text{OM} = x, \qquad \text{ON} = \xi, \qquad \text{PTN} = \tau,$$

$$\text{MP} = y, \qquad \text{N}\Pi = \eta, \qquad \text{P}\Pi\text{R} = \psi, \qquad \text{P}\Pi = \rho;$$

then since $\sin\psi = \frac{dx}{ds}$, and $\cos\psi = \frac{dy}{ds}$, we have from the geometry of the figure,

$$\xi = x + \rho\frac{dy}{ds}, \qquad \eta = y - \rho\frac{dx}{ds}; \tag{32}$$

which formulæ determine the position of the centre of curvature corresponding to any point of the curve; and from eliminating x and y between these equations and the equation to the curve, viz. $y = f(x)$, there will result an equation involving ξ and η, which will represent the locus of the centre of curvature.

The equations (32) assume various forms, according to the value given to ρ; i. e. whether we express ρ by one or other of its values (6), (7), (8), (16). Thus applying the value of the radius of curvature given in (9), when x is equicrescent, we have

$$\xi = x - \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}\cdot\frac{dy}{dx}, \qquad \eta = y + \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}. \tag{33}$$

If the equation to the curve is given in the implicit form, the equations (32) must be modified according to the equation (31) and those by means of which (31) has been determined; but as the expressions are long, not often employed, and easily found, it is not worth while to insert them at present.

289.] Examples on evolutes.

Ex. 1. To determine the evolute of the parabola.

$$y^2 = 4ax;$$

$$\therefore \quad \frac{dy}{dx} = \frac{2a}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}.$$

Substituting which in (33),

$$\xi = x + \frac{y^2+4a^2}{y^2}\,\frac{y^3}{4a^2}\,\frac{2a}{y},$$

$$= x + \frac{4ax+4a^2}{2a},$$

$$= 3x+2a; \qquad \therefore \quad x = \frac{\xi-2a}{3};$$

$$\eta = y - \frac{y^2+4a^2}{y^2}\,\frac{y^3}{4a^2},$$

$$= \frac{-y^3}{4a^2}; \qquad \therefore \quad y = (-4a^2\eta)^{\frac{1}{3}};$$

substituting which in the equation to the parabola, we have

$$\eta^2 = \frac{4}{27a}(\xi-2a)^3;$$

the equation to a semi-cubical parabola, whose cusp is on the axis of x at a distance $2a$ from the vertex of the parabola; see fig. 103.

Ex. 2. To find the equation to the evolute of a circle.

$$x^2+y^2 = a^2;$$

$$\frac{dy}{dx} = -\frac{x}{y}; \qquad \therefore \quad 1+\frac{dy^2}{dx^2} = \frac{x^2+y^2}{y^2} = \frac{a^2}{y^2};$$

$$\frac{d^2y}{dx^2} = -\frac{a^2}{y^3};$$

$$\therefore \quad \left.\begin{aligned} \xi &= x - \frac{a^2}{y^2}\,\frac{y^3}{a^2}\,\frac{x}{y} = 0, \\ \eta &= y - \frac{a^2}{y^2}\,\frac{y^3}{a^2} = 0; \end{aligned}\right\}$$

$\therefore$ the centre itself is the evolute.

Ex. 3. To find the equation to the evolute of the ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}; \qquad \therefore \quad 1+\frac{dy^2}{dx^2} = \frac{a^4y^2+b^4x^2}{a^4y^2};$$

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3};$$

$$\therefore \quad \left.\begin{aligned} \xi &= x - \frac{a^4y^2+b^4x^2}{a^4y^2}\,\frac{a^2y^3}{b^4}\,\frac{b^2x}{a^2y}, \\ &= x - \frac{x}{a^4b^2}(a^4b^2-a^2b^2x^2+b^4x^2), \\ &= \frac{a^2-b^2}{a^4}x^3, \end{aligned}\right\} \qquad \therefore \quad \frac{x^2}{a^2} = \left(\frac{a\xi}{a^2-b^2}\right)^{\frac{2}{3}};$$

$$\left.\begin{aligned} \eta &= y - \frac{a^4y^2+b^4x^2}{a^4y^2}\,\frac{a^2y^3}{b^4}, \\ &= y - \frac{y}{a^2b^4}(a^2b^4-a^2b^2y^2+a^4y^2), \\ &= -\frac{a^2-b^2}{b^4}y^3; \end{aligned}\right\} \quad \therefore \quad \frac{y^2}{b^2} = \left(\frac{b\eta}{a^2-b^2}\right)^{\frac{2}{3}};$$

whence, by addition,

$$(a\xi)^{\frac{2}{3}} + (b\eta)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}};$$

the curve represented by which is delineated in fig. 104.

In reference to the properties of the evolute which are mentioned in Art. 229, I may observe that for all points within ECE′C′ four normals may be drawn to the ellipse; from points on the curve three normals may be drawn; and for all space outside the curve only two can be drawn.

Ex. 4. To find the equation to the evolute of the cycloid.

Let the starting point be the origin; then

$$x = a \text{ versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}};$$

$$\frac{dx}{dy} = \frac{y}{(2ay-y^2)^{\frac{1}{2}}}; \qquad \therefore \quad 1+\frac{dy^2}{dx^2} = \frac{2a}{y};$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2};$$

$$\therefore \quad \eta = y - \frac{2a}{y}\,\frac{y^2}{a} = -y,$$

$$y = -\eta;$$

$$\begin{aligned} \xi &= x + \frac{2a}{y}\cdot\frac{y^2}{a}\cdot\frac{(2ay-y^2)^{\frac{1}{2}}}{y}, \\ &= x + 2(2ay-y^2)^{\frac{1}{2}}, \\ &= x + 2(-2a\eta-\eta^2)^{\frac{1}{2}}; \end{aligned}$$

$$\therefore \quad x = \xi - 2(-2a\eta-\eta^2)^{\frac{1}{2}};$$

substituting which values in the equation to the cycloid, we have

$$\xi - 2(-2a\eta-\eta^2)^{\frac{1}{2}} = a \text{ versin}^{-1}\frac{-\eta}{a} - (-2ay-\eta^2)^{\frac{1}{2}},$$

$$\xi = a \text{ versin}^{-1}\frac{-\eta}{a} + (-2a\eta-\eta^2)^{\frac{1}{2}};$$

which is the equation to a cycloid, the highest point of which is the origin, equal to the original cycloid: ξ being parallel to the base, η taken along the axis in a negative direction, as is manifest on a comparison of the last equation with (31) in Art. 201, and of figs. 50 and 105; which last represents the positions of the original cycloid and its evolute.

Since $\eta = -y$, NH = MP; and therefore PH = 2PG, or the radius of curvature is equal to twice the normal; see Ex. 3, Art. 283. Also the radius of curvature at B = BC = 2BA = $4a$; also it is manifest that the radius of curvature at O = 0.

Ex. 5. To determine the equation to the evolute of the equitangential curve.

The equation to the curve is, see equation (27), Art. 200,

$$x = a \log \left\{ \frac{a + (a^2 - y^2)^{\frac{1}{2}}}{y} \right\} - (a^2 - y^2)^{\frac{1}{2}};$$

$$\frac{dy}{dx} = -\frac{y}{(a^2 - y^2)^{\frac{1}{2}}}; \qquad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{a^2}{a^2 - y^2};$$

$$\frac{d^2y}{dx^2} = \frac{a^2 y}{(a^2 - y^2)^2};$$

$$\therefore \quad \eta = y + \frac{a^2}{a^2 - y^2} \cdot \frac{(a^2 - y^2)^2}{a^2 y},$$

$$= \frac{a^2}{y}; \qquad \therefore \quad y = \frac{a^2}{\eta};$$

$$\xi = x + \frac{a^2}{a^2 - y^2} \cdot \frac{(a^2 - y^2)^2}{a^2 y} \cdot \frac{y}{(a^2 - y^2)^{\frac{1}{2}}},$$

$$= x + (a^2 - y^2)^{\frac{1}{2}},$$

$$= x + \left(a^2 - \frac{a^4}{\eta^2}\right)^{\frac{1}{2}},$$

$$\therefore \quad x = \xi - \frac{a}{\eta} (\eta^2 - a^2)^{\frac{1}{2}};$$

whence, substituting in equation above,

$$\xi - \frac{a}{\eta} (\eta^2 - a^2)^{\frac{1}{2}} = a \log \left\{ \frac{\eta + (\eta^2 - a^2)^{\frac{1}{2}}}{a} \right\} - \frac{a}{\eta} (\eta^2 - a^2)^{\frac{1}{2}};$$

$$\therefore \quad \frac{\xi}{a} = \log \left\{ \frac{\eta + (\eta^2 - a^2)^{\frac{1}{2}}}{a} \right\};$$

$$\therefore \quad \frac{\eta + (\eta^2 - a^2)^{\frac{1}{2}}}{a} = e^{\frac{\xi}{a}}, \qquad \frac{\eta - (\eta^2 - a^2)^{\frac{1}{2}}}{a} = e^{-\frac{\xi}{a}};$$

$$\therefore \quad \eta = \frac{a}{2}\left\{e^{\frac{\xi}{a}} + e^{-\frac{\xi}{a}}\right\};$$

the equation to the catenary, as is plain on comparing it with equation (25), Art. 199.

The relative position of the two curves is delineated in fig. 106; OM $= x$, MP $= y$; ON $= \xi$, NH $= \eta$; and since $\xi = x + (a^2 - y^2)^{\frac{1}{2}}$, and NP $= a$, it follows that the ordinate through N, the foot of the tangent, passes through H, the centre of curvature. Also since PH is the radius of curvature, and PG is the normal of the equitangential curve, PN × PG $=$ NP$^2 = a^2$; also PH$^2 = \eta^2 - a^2 = \dfrac{a^4}{y^2} - a^2 = \dfrac{a^2}{y^2}(a^2 - y^2)$.

Ex. 6. To determine the equation to the evolute of the hypocycloid, whose equation is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}; \qquad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}};$$

$$\frac{d^2y}{dx^2} = \frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}.$$

$$\therefore \quad \xi = x + \frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}\,\frac{3x^{\frac{4}{3}}y^{\frac{1}{3}}}{a^{\frac{2}{3}}}\,\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = x + 3x^{\frac{1}{3}}y^{\frac{2}{3}};$$

similarly, $$\eta = y + 3x^{\frac{2}{3}}y^{\frac{1}{3}};$$

$$\therefore \quad \xi + \eta = x + 3x^{\frac{2}{3}}y^{\frac{1}{3}} + 3x^{\frac{1}{3}}y^{\frac{2}{3}} + y,$$

$$= (x^{\frac{1}{3}} + y^{\frac{1}{3}})^3;$$

$$(\xi + \eta)^{\frac{1}{3}} = x^{\frac{1}{3}} + y^{\frac{1}{3}}.$$

Similarly, $$(\xi - \eta)^{\frac{1}{3}} = x^{\frac{1}{3}} - y^{\frac{1}{3}};$$

$$\therefore \quad (\xi+\eta)^{\frac{1}{3}} + (\xi-\eta)^{\frac{1}{3}} = 2x^{\frac{1}{3}},$$

$$(\xi+\eta)^{\frac{1}{3}} - (\xi-\eta)^{\frac{1}{3}} = 2y^{\frac{1}{3}};$$

$$\therefore \quad \{((\xi+\eta)^{\frac{1}{3}} + (\xi-\eta)^{\frac{1}{3}}\}^2 + \{((\xi+\eta)^{\frac{1}{3}} - (\xi-\eta)^{\frac{1}{3}}\}^2 = 4\{x^{\frac{2}{3}} + y^{\frac{2}{3}}\},$$

$$= 4a^{\frac{2}{3}};$$

$$(\xi+\eta)^{\frac{2}{3}} + (\xi-\eta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

Let the coordinate axes be turned through 45°, so that

$$\xi = \frac{\xi' - \eta'}{\sqrt{2}}, \quad \text{and} \quad \eta = \frac{\xi' + \eta'}{\sqrt{2}};$$

$$\therefore \quad \xi+\eta = \xi'\sqrt{2}, \qquad \xi-\eta = -\eta'\sqrt{2};$$

$$\therefore \quad \xi'^{\frac{2}{3}}+\eta'^{\frac{2}{3}} = (2a)^{\frac{2}{3}}.$$

Accordingly the evolute is another hypocycloid, the radius of whose base-circle is twice that of the original circle, and whose cusps are on lines bisecting the lines joining the cusps of the original curve; see fig. 107. Similarly may the evolute of this new hypocycloid be found, which will be another hypocycloid, the radius of whose base-circle will be $4a$.

Theoretically, the equations to the evolutes of all curves may be found by means of equations (33), but the difficulty of elimination is in all cases, save in two or three besides the above, so great as to be beyond the present powers of analysis.

290.] We proceed now to discuss general properties of the evolute, of which the current coordinates are ξ and η.

From equations (32), we have

$$\left.\begin{aligned} \xi-x &= \rho\frac{dy}{ds}, \\ \eta-y &= -\rho\frac{dx}{ds} \end{aligned}\right\}; \tag{34}$$

whence, squaring and adding,

$$(\xi-x)^2+(\eta-y)^2 = \rho^2. \tag{35}$$

Multiplying the former by dx, and the latter by dy, and adding

$$(\xi-x)\,dx+(\eta-y)\,dy = 0. \tag{36}$$

Again, multiplying the former by d^2x, and the latter by d^2y, and adding,

$$\begin{aligned}(\xi-x)\,d^2x+(\eta-y)\,d^2y &= \frac{\rho}{ds}(d^2x\,dy-d^2y\,dx), \\ &= ds^2; \end{aligned} \tag{37}$$

since by (5), $\quad \rho(d^2x\,dy-d^2y\,dx) = ds^3.$

Now the relation between (35), (36), and (37) is very remarkable; for (36) is the differential of (35), and (37) is the differential of (36), the differentiations being calculated on the supposition that x and y vary, while ξ, η, and ρ do not change. And what *geometrical* fact is hereby implied? The following: (35) is the equation to a *circle* of which ρ is the radius, and the coordinates to whose centre are ξ and η, and of which x and y are the current coordinates. Hence the radius and coordinates to the centre of this circle remain the same, when for x and y we have successively $x+dx$, $y+dy$, and $x+2dx+d^2x$, $y+2dy+d^2y$; this circle therefore passes through *three* consecutive points on the curve, and therefore there are three points, and which is the same thing, two consecutive elements, common to the circle and the curve. The circle is for an obvious reason called *the circle of curvature*. This result is in accordance with the principle of Art. 281; for although nothing was said as to a circle having points common with the curve, yet since ξ and η refer to the point of meeting of two consecutive normals, and each normal implies a tangent passing through two points, there must be three consecutive points in the curve for which ξ, η, and ρ do not vary. It is also to be observed, that (36) is the equation to the normal, if ξ and η are its current coordinates; and thus the centre of curvature is on the normal.

291.] Again, since the new curve is the locus of the point of intersection of any two consecutive normals of the original one, if the new curve is continuous, each normal must pass through two points in the new curve which are infinitesimally near to one another. Hence, in the expressions (34), ξ, η, ρ, x, y may all vary simultaneously, and we have

$$d\xi = dx + d\rho\frac{dy}{ds} + \rho\frac{d^2y\,ds - d^2s\,dy}{ds^2};$$

and substituting for ρ and d^2s from (5) and (11), we have

$$\left.\begin{aligned} d\xi &= \frac{dy}{ds}\,d\rho\,; \\ \text{and similarly,}\qquad d\eta &= -\frac{dx}{ds}\,d\rho \end{aligned}\right\}. \tag{38}$$

Whence it appears that we may differentiate (34) on the supposition that ρ, ξ, and η vary independently of x and y; that is, the normal passing through (x, y) passses through (ξ, η) and $(\xi + d\xi, \eta + d\eta)$, though of course, as is plain from the figure 102, the length of ρ varies.

292.] Squaring and adding the two equations (38) we have

$$d\xi^2 + d\eta^2 = d\rho^2. \tag{39}$$

Let σ be the length of the arc of the new curve, and $d\sigma$ an element of it, then by (23), Art. 218,

$$d\sigma^2 = d\xi^2 + d\eta^2;$$

$$\therefore \quad d\sigma^2 = d\rho^2,$$

$$d\sigma = \pm\, d\rho; \qquad (40)$$

And taking the positive sign, that the analytical expression may be accommodated to the curve in fig. 102, where AΠ $= \sigma$, and the element of the arc at Π $= d\sigma$, and where therefore σ and ρ are increasing simultaneously, we have

$$d\sigma - d\rho = 0;$$

$$\therefore \quad \sigma - \rho = \text{a constant} = c:$$

and therefore the difference in length between the radius of curvature of the original curve, and the length of the arc of the new curve is the same, at whatever point of the old curve the radius of curvature is drawn. Imagine then, see fig. 103, a perfectly flexible and inextensible string to be fixed at a point (ξ, η) of the new curve, say at Π, and of length equal to the radius of curvature of the old curve which abuts there, say equal to PΠ; then, if the string is wrapped round the curve, say towards A, just so much will be taken off from the string by the wrapping that the remainder will be equal to the radius of the old curve, corresponding to the point in the new curve at which the wrapping ends; and therefore if an inextensible string is unwrapped from the new curve, as e. g. from AΠ, the length of which is exactly equal to the length of the new curve +AO, which is constant and is the radius of curvature of OP at O, the extremity of it will generate the old curve, viz. OP. It is for this reason that the new curve is called the *evolute**, as being that from which the string is unwrapped, and the original curve is called the *involute* with respect to it.

293.] It is manifest that the lengths of all evolutes can be determined; that is, the lengths can be compared with straight lines, whence they are said to be *rectifiable*; for, it appears from what has preceded, that the length of the evolute is equal to the difference of the radii of curvature of the involute corresponding to its two extremities.

Of this we subjoin some examples:

---

* By French writers the evolute is named *développée*, and the Involute *développante*.

Ex. 1. To find the length of the evolute of the parabola, in terms of the coordinates of its extremities.

$$\text{OM} = x, \quad \text{MP} = y; \quad \text{ON} = \xi, \quad \text{N}\Pi = \eta; \quad \text{see fig. 103.}$$

Let the equation to the parabola be $y^2 = 4ax$.

Then by (9), $\rho = \dfrac{2(a+x)^{\frac{3}{2}}}{a^{\frac{1}{2}}}$; therefore by what has preceded,

$$\begin{aligned} \text{The length of A}\Pi &= \text{rad. of curv. at P} - \text{rad. of curv. at O}; \\ &= \frac{2(a+x)^{\frac{3}{2}}}{a^{\frac{1}{2}}} - 2a, \\ &= \frac{2}{a^{\frac{1}{2}}}\left(\frac{a+\xi}{3}\right)^{\frac{3}{2}} - 2a; \end{aligned}$$

since by Ex. 1, Art. 289, $x = \dfrac{\xi - 2a}{3}$.

Ex. 2. To determine the length of the fourth part of the evolute of the ellipse; see fig. 104.

$$\begin{aligned} \text{The length of EC} &= \text{rad. of curv. at B}' - \text{rad. of curv. at A}, \\ &= \frac{a^2}{b} - \frac{b^2}{a}, \text{ by Ex. 1, Art. 283}, \\ &= \frac{a^3 - b^3}{ab}; \end{aligned}$$

$$\therefore \text{ the length of the whole evolute} = 4\,\frac{a^3 - b^3}{ab}.$$

Ex. 3. To determine the length of the arc OΠ of the cycloid; see fig. 105.

$$\begin{aligned} \text{The length of O}\Pi &= \text{P}\Pi, \\ &= 2(2ay)^{\frac{1}{2}}, \text{ by Ex. 3, Art. 283}, \\ &= 2(-2a\eta)^{\frac{1}{2}}, \text{ by Ex. 4, Art. 289}; \end{aligned}$$

$\therefore$ if $\eta = -2a$, the length of OC $= 4a =$ rad. of curv. at B;

$\therefore$ the length of the whole cycloid $= 8a$.

Ex. 4. To establish a relation between σ, η, and ξ in the catenary.

In fig. 106, let $\text{A}\Pi = \sigma$;

$$\begin{aligned} \therefore \quad \text{A}\Pi^2 &= \Pi\text{P}^2, \\ &= \Pi\text{N}^2 - \text{NP}^2; \end{aligned}$$

$$\begin{aligned}
\therefore\quad \sigma^2 &= \eta^2 - a^2,\\
&= \frac{a^2}{4}\left\{e^{\frac{\xi}{a}} + e^{-\frac{\xi}{a}}\right\}^2 - a^2,\\
&= \frac{a^2}{4}\left\{e^{\frac{\xi}{a}} - e^{-\frac{\xi}{a}}\right\}^2;\\
\therefore\quad \sigma &= \frac{a}{2}\left\{e^{\frac{\xi}{a}} - e^{-\frac{\xi}{a}}\right\}.
\end{aligned}$$

Every plane curve manifestly has an evolute, and has only one; but it has an infinite number of involutes, because in the unwrapping of the string which has been wound round the curve, every point of the stretched string describes a curve which is the involute corresponding to that point.

294.] Again, multiplying the former of (38) by dx, and the latter by dy, and adding, we have

or, dividing the 2d by the first.

$$dx\,d\xi + dy\,d\eta = 0; \tag{41}$$

$$\therefore\quad \frac{d\eta}{d\xi} = -\frac{dx}{dy}.$$

And since $\frac{d\eta}{d\xi}$ and $\frac{dy}{dx}$ are respectively the tangents of the angles made with the axis of x by the tangents to the evolute, and to the involute, it follows that the tangent to the evolute is perpendicular to the tangent to the involute, or that the normal to the involute is a tangent to the evolute. This result might have been anticipated from what is said in Art. 290.

The radius of curvature at any point of the evolute may be expressed in terms of the radius of curvature of the corresponding point of the involute by the following process.

Let ρ and ρ' be the radii of curvature at corresponding points of the involute and the evolute: the angle contained between two consecutive normals to the evolute is equal to that contained between two consecutive normals, or between two consecutive tangents to the involute; that is, is equal to $\pm\, d\tau$ or to $\pm\, d\psi$; see Art. 218: and if $d\sigma$ is the length-element of the evolute, $d\sigma = d\rho$; also $ds = \pm\, \rho\, d\tau$; therefore

$$\begin{aligned}
\rho' &= \pm\frac{d\sigma}{d\tau}\\
&= \pm\frac{d\rho}{ds}\rho. \qquad (42)
\end{aligned}$$

Hence $\rho' = 0$, when $d\rho = 0$; that is, if the radius of curvature of a curve is constant, or attains to a maximum or minimum value, that of the evolute at the corresponding point vanishes, and its curvature is infinite.

Other properties of the involute are discussed in Vol. II of our Treatise.

295.] Let us however investigate these properties of curvature from a geometrical point of view.

The equations (35), (36), (37), connecting x, y, ρ, ξ and η, shew that the centre of curvature is the centre of a circle passing through three consecutive points in the curve. These *three* points are necessary to render the circle definite. If it passes through only two points, its centre may be *any where* on the normal which is perpendicular to the tangent passing through the two points, and thus there may be an infinite number of circles satisfying the condition: but if the circle is to pass through *three* points, its centre must be on the normal perpendicular to the tangent passing through the second and third points, as well as on the normal corresponding to the first and second points; and as these two normals will intersect in *one* point, this point must be the centre of the circle, and the circle becomes definite; in other words, the two consecutive elements of the curve which are delineated in fig. 108, viz. PQ and QR, will form two sides of a triangle, and by joining PR the triangle will be completed, and the circle described about this triangle will be a definite circle, and pass through the points P, Q, R, which are three consecutive points on the curve. We may on this conception determine the radius of curvature as follows:

The angle of contingence, or $d\tau$, is the angle between the two consecutive elements PQ and QR, whence by Art. 284,

$$d\tau = \pm \frac{ds}{\rho}. \tag{43}$$

Complete the parallelogram, of which PQ, QR are two adjoining sides, and draw the diagonal SQ, and the other lines as in the fig. 108; then

$$\text{PQ} = ds, \qquad \text{QR} = ds + d^2 s;$$

$$\text{PK} = dx, \quad \text{QT} = dx + d^2 x = \text{PE}, \quad \because \ \text{SP is parallel to RQ};$$

$$\therefore \ \text{EK} = -d^2 x;$$

QK $= dy$, RT $= dy + d^2y =$ SE, $\because$ SP is parallel to RQ;

$$\therefore \quad \text{SY} = d^2y.$$

Therefore d^2x and d^2y are the projections of SQ on the coordinate axes.

Now the area of the parallelogram PQRS

$$\begin{aligned} &= \text{PQ} \times \text{QR} \times \sin \text{PQR}, \\ &= ds\,(ds + d^2s) \sin d\tau, \\ &= \pm \frac{ds^3}{\rho}, \qquad (44) \end{aligned}$$

omitting d^2s because it is added to ds, and replacing $\sin d\tau$ by $d\tau$ in terms of equation (43).

Again, the area of the parallelogram PQRS

$$\begin{aligned} &= \text{the area of PQUV}, \\ &= \text{PK} \times \text{UQ} = \text{PK} \times \text{RW} = \text{PK (RT} - \text{WT)}, \\ &= dx \left\{ dy + d^2y - (dx + d^2x) \frac{dy}{dx} \right\}, \\ &= dx\, d^2y - dy\, d^2x. \qquad (45) \end{aligned}$$

Hence equating (44) and (45), we have

$$\frac{1}{\rho} = \pm \frac{d^2x\, dy - d^2y\, dx}{ds^3}. \qquad (46)$$

Hence if three points of a curve are in one and the same straight line, PQRS $= 0$, and the radius of the circle of curvature is infinite. Also if of three elements the first and third are equally inclined to the second, the circle of curvature passes through four consecutive points.

296.] And now let s be equicrescent; so that in fig. 109 PQ $=$ QR $= ds$, OM $= x$, MP $= y$, MN $= dx$, NL $= dx + d^2x$, NQ $= y + dy$, LR $= y + 2dy + d^2y$; let us complete as before the parallelogram PQRS, which is in this case equilateral; and therefore the radius of curvature lies along the line ΠSQ, which is coincident in direction with the diagonal SQ. Join PR, which is, as is plain from the geometry, perpendicular to SQ; then, if ρ is the radius of curvature, by the property of the circle,

$$2\rho : \text{RQ} :: \text{RQ} : \text{VQ};$$

$$\therefore \quad \rho = \frac{\text{RQ}^2}{2.\text{VQ}} = \frac{ds^2}{\text{SQ}}.$$

Draw SY, YQ, as in the figure; then it may be proved, as in the last Article, that

$$\text{SY} = d^2y \quad \text{and} \quad \text{YQ} = -d^2x;$$

$$\therefore \quad \text{SQ}^2 = (d^2y)^2 + (d^2x)^2;$$

$$\therefore \quad \frac{1}{\rho^2} = \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2.$$

Hence also we have a geometrical proof of the values of $\cos\tau$ and $\cos\psi$ determined in Art. 285, equations (23) and (24).

As SQ is coincident in direction with the radius of curvature,

$$\left.\begin{aligned} \cos\psi &= \frac{\text{YQ}}{\text{SQ}} = \rho\frac{d^2x}{ds^2}, \\ \cos\tau &= \frac{\text{SY}}{\text{SQ}} = \rho\frac{d^2y}{ds^2}. \end{aligned}\right\} \tag{47}$$

297.] Ere we conclude this part of our subject, we must investigate some general properties of evolutes; and I will assume the equation to the base-curve or involute to be, as heretofore, an algebraical equation of the nth degree, of the form (45), Art. 207: although the geometrical properties which follow are generally true of all curves.

Let the equation to the curve be given in the implicit form

$$F(x, y) = c. \tag{48}$$

Now if we express ξ and η, the current coordinates of the evolute, in terms of the partial-derived functions of this equation we have,

$$(\xi - x)\left\{\left(\frac{d^2F}{dx^2}\right)\left(\frac{dF}{dy}\right)^2 - 2\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right)\left(\frac{d^2F}{dx\,dy}\right) + \left(\frac{d^2F}{dy^2}\right)\left(\frac{dF}{dx}\right)^2\right\} \\ + \left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}\left(\frac{dF}{dx}\right) = 0, \tag{49}$$

$$(\eta - y)\left\{\left(\frac{d^2F}{dx^2}\right)\left(\frac{dF}{dy}\right)^2 - 2\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right)\left(\frac{d^2F}{dx\,dy}\right) + \left(\frac{d^2F}{dy^2}\right)\left(\frac{dF}{dx}\right)^2\right\} \\ + \left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}\left(\frac{dF}{dy}\right) = 0; \tag{50}$$

each of which expressions is evidently of $3(n-1)$ dimensions in terms of x and y, and does not admit of further reduction. And as the equation to the evolute is found by the elimination of x and y between these equations and (48), which is of n dimensions, the evolute itself is of $3n(n-1)$ dimensions.

Also the class of a curve, see Art. 223, is the same as that of the number of tangents which can be drawn to it from any given point. Now the number of normals which can be drawn to the involute from a given point is n^2; and as every normal to the involute is a tangent to the evolute, so the evolute is of the n^2 class, so far as tangents can be drawn in the plane of reference; yet this does not affect the relation which has heretofore been shewn to exist between the order and class of a given curve. Thus the evolute to a conic is of the sixth degree, see Ex. 3, Art. 289, and of the fourth class: the evolute to a general curve of the third degree is of the 18th degree, and of the ninth class.

As a normal drawn at any point of an involute is a tangent to the evolute, so if there is any peculiarity at a point of the involute a corresponding peculiarity will exist in the tangent of the evolute: the two curves will therefore have reciprocal properties. Some of these I proceed to indicate.

To a point of inflexion on the involute, an infinite branch of the evolute corresponds; because at a point of inflexion $\frac{d^2y}{dx^2} = 0$, and the radius of curvature is infinite; and the normal to the involute is the asymptote to this infinite branch of the evolute; and as the direction, in which the radius of curvature is drawn, changes at a point of inflexion, so does the infinite branch of the evolute return on the side opposite to that in which it went off; that is, the evolute travels round the sphere of infinite radius; see Art. 186, and fig. 24; and has at infinity a point of inflexion. Hence it appears that the evolute has as many rectilinear asymptotes as the involute has points of inflexion; and thus as the involute is of the nth degree, it may have $3n(n-2)$ points of inflexion within a finite distance, and therefore the evolute may have that number of different rectilinear asymptotes within a finite distance. The evolute of a conic therefore has no rectilinear asymptotes. The evolute of a curve of the third degree may have nine rectilinear asymptotes.

To a point on the involute at which the curvature is a maximum or a minimum, a cusp corresponds on the evolute; the cusp being at a distance from the involute equal to the maximum or minimum radius of curvature.

If two branches of the involute pass through the same point, and have a common tangent and different curvatures, the normal at that point is a double normal, and it is consequently a double tangent to the evolute.

At a cusp, the radius of curvature is zero, so that at that point the evolute meets the involute and cuts it at right angles. Hence the curvature of a curve at a cusp is infinite.

SECTION 3.—*On curvature of plane curves referred to polar coordinates.*

298.] To determine the length of the radius of curvature.

Let $r = f(\theta)$ be the equation to the curve; see fig. 110; now

$$\rho = \pm \frac{ds}{d\tau};$$

but $\tau =$ the angle $PKX = PSX + SPK$,

$$= \theta + \tan^{-1} \frac{r\,d\theta}{dr};$$

$$\therefore \quad d\tau = d\theta + \frac{d\left(\frac{r\,d\theta}{dr}\right)}{1 + \frac{r^2 d\theta^2}{dr^2}},$$

$$= \frac{r^2 d\theta^3 + r\,dr\,d^2\theta + 2\,dr^2 d\theta - r\,d\theta\,d^2 r}{dr^2 + r^2 d\theta^2};$$

and since from equation (12), Art. 269,

$$ds = (dr^2 + r^2 d\theta^2)^{\frac{1}{2}};$$

$$\rho = \pm \frac{(dr^2 + r^2 d\theta^2)^{\frac{3}{2}}}{r^2 d\theta^3 + r\,dr\,d^2\theta + 2\,dr^2 d\theta - r\,d\theta\,d^2 r}; \tag{51}$$

whence, if θ is equicrescent, $d^2\theta = 0$, and we have

$$\rho = \pm \frac{(dr^2 + r^2 d\theta^2)^{\frac{3}{2}}}{r^2 d\theta^3 + 2\,dr^2 d\theta - r\,d\theta\,d^2 r}; \tag{52}$$

and if r is equicrescent, $d^2 r = 0$, and we have

$$\rho = \pm \frac{(dr^2 + r^2 d\theta^2)^{\frac{3}{2}}}{r^2 d\theta^3 + r\,dr\,d^2\theta + 2\,dr^2 d\theta}; \tag{53}$$

The general value (51) may also be deduced from that of equation (6), Art. 282, by substituting $x = r\cos\theta$, $y = r\sin\theta$, and calculating the successive differentials of x and y, as is done in Ex. 4, Art. 77.

The following are examples of the formula (52).

Ex. 1. To find the length of the radius of curvature of the cardioide, $r = a(1-\cos\theta)$.

$$\frac{dr}{d\theta} = a\sin\theta, \qquad \frac{d^2r}{d\theta^2} = a\cos\theta;$$

$$\therefore \quad \rho^2 = \frac{\{a^2(\sin\theta)^2 + a^2(1-\cos\theta)^2\}^3}{\{a^2(1-\cos\theta)^2 + 2a^2(\sin\theta)^2 - a^2\cos\theta\,(1-\cos\theta)\}^2},$$

$$= \frac{(2a^2 - 2a^2\cos\theta)^3}{\{3a^2(1-\cos\theta)\}^2},$$

$$= \frac{8}{9}ar.$$

Ex. 2. To find the length of the radius of curvature of the Spiral of Archimedes, $r = a\theta$.

$$\frac{dr}{d\theta} = a, \qquad \frac{d^2r}{d\theta^2} = 0; \qquad \therefore \quad \rho = \pm\frac{(a^2+r^2)^{\frac{3}{2}}}{2a^2+r^2}.$$

299.] The expression for the radius of curvature may also be put under an elegant and simple form, as follows:

Let the equation to the curve be changed into its equivalent in terms of r and p, by means of either (19) or (21) of Art. 270; so that,

$$r = f(p).$$

Y Since $\tau = \theta + \text{SPY}$,

$$= \theta + \sin^{-1}\frac{p}{r}; \qquad \therefore \quad d\tau = d\theta + \frac{r\,dp - p\,dr}{r(r^2-p^2)^{\frac{1}{2}}};$$

and since by equations (23) and (24), Art. 271,

$$d\theta = \frac{p\,dr}{r(r^2-p^2)^{\frac{1}{2}}}, \qquad \text{and} \qquad ds = \frac{r\,dr}{(r^2-p^2)^{\frac{1}{2}}};$$

$$\therefore \quad d\tau = \frac{dp}{(r^2-p^2)^{\frac{1}{2}}}; \tag{54}$$

$$\therefore \quad \rho = \pm\frac{ds}{d\tau} = \pm\frac{r\,dr}{dp}. \tag{55}$$

300.] Which expression may also thus be obtained; see fig. 111: SP $= r$, SY $= p$, PO $= \rho$, the radius of curvature, O being the point at which two consecutive normals intersect. Draw SZ from S perpendicular to OP; then PZ $=$ SY $= p$; therefore from the geometry,

$$\begin{aligned} \text{SO}^2 &= \text{SP}^2+\text{PO}^2-2\text{SP}.\text{PO}.\sin \text{SPY}, \\ &= r^2+\rho^2-2r\rho\frac{p}{r}, \\ &= r^2+\rho^2-2p\rho. \end{aligned} \tag{56}$$

Then, since O and ρ remain the same for the next consecutive point in the curve, and r and p vary, we may differentiate on these conditions, and we have, $0 = 2r\,dr-2\rho\,dp$;

$$\therefore \quad \rho = \frac{r\,dr}{dp}. \tag{57}$$

The geometrical meaning of which expression may thus be shewn by fig. 112. Let PY, PY′ be two tangents drawn at consecutive points on the curve, SY, SY′, the corresponding perpendiculars from the pole, then YPY′ is the angle of contingence; and therefore

$$ds = \rho \times \text{angle YPY}';$$

but the angle YPY′ is subtended by the small arc UY′ $= dp$, at the distance PY′, which is equal to $(r^2-p^2)^{\frac{1}{2}}$; therefore the angle YPY′ may be replaced by

$$\frac{dp}{(r^2-p^2)^{\frac{1}{2}}};$$

and replacing ds by its value given above, there results

$$\rho = \frac{r\,dr}{dp}.$$

301.] A comparison of the preceding results, with what has been said in Art. 290, shews that O is the centre of a circle which passes through three consecutive points on the curve. Let this circle be drawn as in fig. 111; then PV, the part of the radius vector SP which is intercepted by the circle, is called the *chord of the circle of curvature*. Its value is thus determined;

$$\begin{aligned} \text{PV} &= 2\text{PU} = 2\rho\cos \text{OPS}, \\ &= 2\rho\sin \text{SPY} = 2\rho\frac{p}{r}, \\ &= 2p\frac{dr}{dp}. \end{aligned} \tag{58}$$

302.] Examples illustrative of the preceding Articles.

Ex. 1. To find the lengths of the radius and of the chord of curvature of the logarithmic spiral, whose equation between r and p is $p = mr$. See equation (25), Art. 272;

$$\therefore \quad \frac{dp}{dr} = m; \qquad \text{and} \quad \rho = \frac{r}{m}; \qquad \text{the chord} = 2mr\frac{1}{m} = 2r;$$

$$\therefore \quad \text{PV} = 2\,\text{PS}.$$

Ex. 2. The involute of the circle.

$$p^2 = r^2 - a^2, \qquad p\frac{dp}{dr} = r;$$

$$\therefore \quad \rho = p; \qquad \text{the chord of curvature} = \frac{2p^2}{r}.$$

Ex. 3. The Lemniscata of Bernoulli.

The polar equation is $r^2 = a^2 \cos 2\theta$. And if $u = r^{-1}$, then $a^2 u^2 = \sec 2\theta$.

$$\therefore \quad \left.\begin{aligned} 2a^2 u \frac{du}{d\theta} &= 2\sec 2\theta \tan 2\theta, \\ &= 2a^2 u^2 \tan 2\theta; \end{aligned}\right\} \qquad \frac{du}{d\theta} = u \tan 2\theta;$$

$$\therefore \quad u^2 + \frac{du^2}{d\theta^2} = u^2\{1 + (\tan 2\theta)^2\},$$
$$= a^4 u^6;$$

$$\therefore \quad \frac{1}{p^2} = \frac{a^4}{r^6}; \qquad \text{and} \qquad \frac{dp}{dr} = \frac{3r^2}{a^2};$$

$$\therefore \quad \rho = \frac{a^2}{3r}; \qquad \text{the chord of curvature} = \frac{2a^2 p}{3r^2} = \frac{2}{3}r.$$

SECTION 4.—*On evolutes of curves referred to polar coordinates.*

303.] The following is the most convenient method of determining the equation to the evolute of a polar curve.

Let the original equation be transformed into its equivalent between r and p, so that,

$$r = f(p); \tag{59}$$

then, bearing in mind what has been said in Section 2 of the present Chapter on the properties of evolutes, which are general and independent of the particular system of determining position, the line PO in fig. 111, which is a normal to the involute, is a tangent to the evolute; therefore, referring the evolute to polar coordinates, r' and p', if we take S to be the pole of the evolute, we have $\text{SO} = r'$, $\text{SZ} = p'$; and these are the coordinates to the evolute; and we have the following equations:

$$\rho = r\frac{dr}{dp}; \tag{60}$$

and from the geometry, $p' = (r^2-p^2)^{\frac{1}{2}}$; (61)

and from equation (56), Art. 300,

$$r'^2 = r^2+\rho^2-2\rho p; \tag{62}$$

from which four equations we can, theoretically at least, eliminate r, p, and ρ, and obtain a final equation involving only r' and p', which will be the polar equation to the evolute.

The several properties of evolutes, and the relation they bear to their respective involutes, might be deduced from a discussion of them referred to polar coordinates; but as they have been fully explained in the former part of this Chapter, it is not necessary to repeat them.

Ex. 1. To find the equation to the evolute of the logarithmic spiral.

$$p = mr.$$

$$\therefore\quad \frac{dp}{dr} = m, \qquad \rho = \frac{r}{m};$$

$$\therefore\quad p' = (1-m^2)^{\frac{1}{2}}r, \qquad r'^2 = r^2+\frac{r^2}{m^2}-2r^2;$$

$$\therefore\quad p' = mr';$$

which represents another equal logarithmic spiral.

The same result also follows from Ex. 1, Art. 302; for since PV = 2 PS, the perpendicular to SP from S meets the normal in the centre of curvature, or according to the type-figure 89, G is the centre of curvature; therefore the angle SGP, which is equal to SPT, is constant; and therefore as PG is a tangent to the evolute, the locus of G is a logarithmic spiral, the angle between the radius vector and tangent of which is equal to the corresponding angle of the original curve; and therefore the two curves are equal.

Ex. 2. To determine the evolute of the involute of the circle.

$$r^2 = p^2+a^2;$$

$$\frac{dr}{dp} = \frac{p}{r}; \qquad \therefore\quad \rho = p; \qquad \therefore\quad p' = a.$$

$$\begin{aligned} r'^2 &= r^2+p^2-2p^2 \\ &= a^2; \end{aligned}$$

$$\therefore\quad r' = p' = a.$$

Hence the evolute is a circle whose radius is a, as is manifest from the figure 87.

CHAPTER XIII.

ON CONTACT OF CURVES AND ON ENVELOPES.

SECTION 1.—*On the theory of contact of plane curves.*

304.] IN Art. 214 a tangent is defined to be that line which passes through two consecutive points on a curve; and therefore it follows that there are two points *common* to the tangent and to the curve; and in Art. 290, it is shewn that three points are common to the curve and to the circle of curvature. This property of curves having consecutive points in common, or, as it is called, having *contact*, it is our object to generalize, and with reference to it we define as follows:

Curves which have two consecutive points in common are said to have contact of the *first* order: those which have three consecutive points in common are said to have contact of the *second* order; and similarly two curves have contact of the nth order, if they have $(n+1)$ consecutive points in common.

Thus ordinarily, a tangent line has contact of the first order with a curve; and the circle of curvature has contact of the second order.

Curves which possess these relative properties are also called *osculating curves*, and curves are said to *osculate* to each other.

Nothing is said as to curves having only *one* point in common, because such a condition implies no more than that they intersect, and does not enable us to determine the relative direction of the curves.

305.] Hence then it appears that if for two curves whose equations are,

$$y = f(x), \qquad \eta = \phi(\xi), \tag{1}$$

we have the series of common points indicated in the following table,

$\left.\begin{matrix}(x,y)\\(\xi,\eta)\end{matrix}\right\}$, the two curves intersect;

$\left.\begin{matrix}(x,y), & (x+dx,\ y+dy)\\(\xi,\eta), & (\xi+d\xi,\ \eta+d\eta)\end{matrix}\right\}$, the two curves have contact of the first order;

$\left.\begin{matrix}(x,y), & (x+dx,\ y+dy), & (x+2\,dx+d^2x,\ y+2\,dy+d^2y)\\(\xi,\eta), & (\xi+d\xi,\ \eta+d\eta), & (\xi+2\,d\xi+d^2\xi,\ \eta+2\,d\eta+d^2\eta)\end{matrix}\right\}$,

the two curves have contact of the second order;

and similarly as to contact of the nth order, for which it is necessary that the successive differentials of the variables up to the nth should be equal in both curves.

These conditions become greatly simplified if we consider x and ξ to be equicrescent, and each to increase by the same augment, in which case the several differentials of x and ξ, after the first, vanish; and we have, if

$$\left.\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}\right\}, \text{ intersection of the curves;}$$

$$\left.\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}\right\} \text{ and } \frac{d\eta}{d\xi} = \frac{dy}{dx}, \text{ contact of the first order;}$$

$$\left.\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}\right\}, \frac{d\eta}{d\xi} = \frac{dy}{dx}, \text{ and } \frac{d^2\eta}{d\xi^2} = \frac{d^2y}{dx^2}, \text{ contact of the second order;}$$

and if, besides, all the several successive differential coefficients, up to the nth inclusively, are equal in both curves, there is contact of the nth order.

Hence, if two curves have contact of the first order, they have a point in common, and the same tangent at the point; and therefore the tangent has contact of the first order. And if two curves have contact of the second order, they have not only a common tangent at the common point, but the curvature is the same, and is turned in the same direction; that is, they have the same circle of curvature.

306.] Hereby then the criterion of the order of contact assumes a new form; it depends on the number of the successively-derived functions of the equations to the curves which are equal; hence we are led to the following mode of viewing the subject, from which many important properties may be deduced.

$$\text{Let} \qquad y = f(x), \qquad \eta = \phi(\xi), \tag{2}$$

be the equations to the two curves, which have a common point; that is, let $x = \xi$, $y = \eta$; and let y' and η' be the ordinates corresponding to the abscissa $x+h$; then

$$y' = f(x) + \frac{h}{1}f'(x) + \frac{h^2}{1.2}f''(x) + \dots + \frac{h^n}{1.2\dots n}f^n(x+\theta h), \tag{3}$$

$$\eta' = \phi(x) + \frac{h}{1}\phi'(x) + \frac{h^2}{1.2}\phi''(x) + \dots + \frac{h^n}{1.2\dots n}\phi^n(x+\theta h)*. \tag{4}$$

---

* θ is used as the general symbol of a proper fraction, and does not necessarily represent the same quantity in (3) and (4), or in the subsequent equations: this is manifest from the argument of Chapter IV.

Therefore if the contact is of the first order, that is, if $n = 2$,

$$f(x) = \phi(x), \quad f'(x) = \phi'(x); \text{ and}$$

$$y' - \eta' = \frac{h^2}{1.2}\{f''(x+\theta h) - \phi''(x+\theta h)\}; \tag{5}$$

that is, if h is infinitesimal, the difference between the ordinates corresponding to $x+h$ is an infinitesimal of the second order.

And no other line can pass between these two curves, unless it has with each of them a contact of at least the first order; for suppose it to be possible that the ordinate y_1 corresponding to the abscissa $x+h$ of the curve $y = \text{F}(x)$ should be such that $\text{F}'(x)$ is not equal to $f'(x)$, then

$$y' - y_1 = \frac{h}{1}\{f'(x+\theta h) - \text{F}'(x+\theta h)\}: \tag{6}$$

which difference is obviously greater than that given in (5), if h is infinitesimal; and therefore the curve $y = \text{F}(x)$ does not come between the curves $y = f(x)$ and $y = \phi(x)$.

If the contact is of the second order, then, besides the former conditions, we have

$$f''(x) = \phi''(x);$$

and, if $n = 3$, then subtracting (4) from (3), we have

$$y' - \eta' = \frac{h^3}{1.2.3}\{f'''(x+\theta h) - \phi'''(x+\theta h)\}; \tag{7}$$

that is, the difference between the ordinates corresponding to the abscissa $x+h$ is an infinitesimal of the third order.

And if there is another curve, $y = \text{F}(x)$, such that $\text{F}''(x)$ is not equal to $f''(x)$, although $\text{F}'(x) = f'(x)$; then, if y_1 is the ordinate of this third curve corresponding to the abscissa $x+h$,

$$y' - y_1 = \frac{h^2}{1.2}\{f''(x+\theta h) - \text{F}''(x+\theta h)\}; \tag{8}$$

which difference, being an infinitesimal of the second order, is obviously greater than that given by equation (7); and therefore this third curve does not come between the first two curves. Similarly, if the contact between two curves is of the nth order, the difference of the ordinates corresponding to $x+h$, when h is infinitesimal, is an infinitesimal of the $(n+1)$th order. Hence we have the following theorems:

" Two curves which have contact of the nth order are infinitely nearer to one another than two curves which have contact of an order lower than the nth."

"A curve which has contact of the nth order cannot come between two curves which have contact of an order higher than the nth."

"Two curves which have contact of the nth order with a third curve, have contact of at least that order with each other."

307.] An inspection of the equations (5) and (7) above, and of other equations formed in a similar manner, and giving the difference between the ordinates y' and η', corresponding to the several orders of contact, leads to the following theorem:

"If the contact of two curves is of an *odd* order, they touch and do not intersect; and if the contact is of an *even* order, they touch and intersect."

For suppose the contact to be of the nth order,

$$y' - \eta' = \frac{h^{n+1}}{1.2.3\ldots(n+1)}\{f^{n+1}(x+\theta h) - \phi^{n+1}(x+\theta h)\}. \qquad (9)$$

Then, if n is *even*, $y' - \eta'$ changes its sign as h changes sign; and therefore $f(x-h) - \phi(x-h)$ and $f(x+h) - \phi(x+h)$ have different signs, and therefore the curves intersect at the point of contact. But if the contact is of an *odd* order, n is odd and $n+1$ is even, and $y' - \eta'$ does not change sign with h; that is, the curve which was nearer to the axis of x before contact is nearer to it afterwards, and the curves do not intersect.

308.] Suppose two curves, of one of which the equation is given, and contains certain fixed constants so that its form and position are completely determined; and the equation to the other involves arbitrary constants, in the determination of which we may make the curve fulfil certain conditions; we will shew that these latter may be satisfied by making the curve have with the former curve a contact, the order of which depends on the number of undetermined constants.

$$\text{Let} \qquad \mathrm{F}(x, y) = 0, \qquad (10)$$

be the equation to the former curve; and

$$f(\xi, \eta, c_1, c_2, \ldots\ldots c_n) = 0, \qquad (11)$$

that to the latter, in which $c_1, c_2, \ldots\ldots c_n$ are n arbitrary constants, and to be determined. By the theory of algebraical elimination it is plain that there must be n independent equations to determine the n unknown quantities; let n equations be formed by making the curve pass through n given points, that is, by substituting successively the coordinates to the given points for the current coordinates to the curve in its equation; and let us suppose these n points to be on the curve (10), and, which is allowable, to be infinitesimally consecutive points; then making x and ξ equicrescent, and variables with the same augments, in the two curves, by the latter part of Art. 305, all the successively-derived functions up to the $(n-1)$th of the equations of the two curves must be equal; up to the $(n-1)$th, I say, for thereby will n consecutive points be common, and sufficient conditions will have been introduced for the determination of all the constants.

And if the latter curve has with the former a contact of an order lower than the $(n-1)$th, the conditions will not be sufficient to determine all the n constants; and therefore the form and position of the latter curve may vary; and the number of curves satisfying the conditions is infinite..

309.] Suppose the equation to the latter curve to contain two arbitrary constants, and to be of the form,

$$f(\xi, \eta, c_1, c_2) = 0; \tag{12}$$

then, differentiating,

$$\left(\frac{df}{d\xi}\right) d\xi + \left(\frac{df}{d\eta}\right) d\eta = 0; \tag{13}$$

by means of which equations, when x and y are substituted for ξ and η, in combination with $\mathrm{F}(x, y) = 0$, we may determine c_1 and c_2, and find a curve which will have contact of the first order with the latter curve.

Ex. 1. To determine the values of the constants a and b, so that the straight line whose equation is

$$\frac{\xi}{a} + \frac{\eta}{b} = 1, \tag{14}$$

may have contact of the first order with the curve

$$\mathrm{F}(x, y) = 0. \tag{15}$$

Since (14) is to pass through a point on (15), we have

$$\frac{\xi - x}{a} + \frac{\eta - y}{b} = 0. \tag{16}$$

Also differentiating (14),

$$\frac{d\xi}{a}+\frac{d\eta}{b}=0; \tag{17}$$

and from (15), $$\left(\frac{dF}{dx}\right)dx+\left(\frac{dF}{dy}\right)dy=0; \tag{18}$$

and since $d\eta$ and $d\xi$ are to be equal to dy and dx, we have from the last equations

$$a\left(\frac{dF}{dx}\right)=b\left(\frac{dF}{dy}\right);$$

and multiplying (16) by the terms of this equality, there results

$$(\xi-x)\left(\frac{dF}{dx}\right)+(\eta-y)\left(\frac{dF}{dy}\right)=0,$$

which is the equation to the tangent to the curve at the point, and which has therefore contact of the first order with the curve.

Also from (17) $\dfrac{d^2\eta}{d\xi^2}=0$; and therefore if at any point on the curve $\dfrac{d^2y}{dx^2}=0$, the tangent will have contact of the second order; hence at a point of inflexion such is the case. Similarly if the several derived-functions of the equation to a curve up to the nth inclusive vanish at a certain point, the tangent at the point will have contact of the nth order with the curve.

Hence it appears that a straight line can generally have contact of only the first order with a curve, but may have contact of any order if the differential coefficients of the equation to the curve vanish at the point; that is, if many points of the curve are at that point in one and the same straight line.

310.] If the equation to the latter curve involves three constants, and is of the form

$$f(\xi,\eta,c_1,c_2,c_3)=0;$$

then, if ξ is equicrescent, we have,

$$\left(\frac{df}{d\xi}\right)d\xi+\left(\frac{df}{d\eta}\right)d\eta=0;$$

$$\left(\frac{d^2f}{d\eta^2}\right)\frac{d\eta^2}{d\xi^2}+2\left(\frac{d^2\xi}{d\eta\, d\xi}\right)\frac{d\eta}{d\xi}+\left(\frac{d^2f}{d\xi^2}\right)+\left(\frac{df}{d\eta}\right)\frac{d^2\eta}{d\xi^2}=0:$$

by means of which equations, when x and y have been substituted for ξ and η, in addition to $F(x,y)=0$, the constants may be eliminated; and the curve which has contact of the second order with the given curve may be determined.

Ex. 1. To determine the conditions of contact of a circle with a given curve.

For the sake of simplicity let the equation to the given curve be in the explicit form,

$$y = f(x); \tag{19}$$

and let the equation to the circle be,

$$(\xi-a)^2+(\eta-\beta)^2 = \rho^2; \tag{20}$$

in which three arbitrary constants are involved; hence ordinarily the circle may have contact of the second order with a given curve. Accordingly, differentiating twice, and taking neither variable to be equicrescent, we have

$$(\xi-a)\,d\xi+(\eta-\beta)\,d\eta = 0, \tag{21}$$

$$(\xi-a)\,d^2\xi+(\eta-\beta)\,d^2\eta+d\xi^2+d\eta^2 = 0; \tag{22}$$

whence, by elimination

$$\xi-a = \frac{-d\eta\,(d\xi^2+d\eta^2)}{d\eta\,d^2\xi-d\xi\,d^2\eta}; \tag{23}$$

$$\eta-\beta = \frac{d\xi\,(d\xi^2+d\eta^2)}{d\eta\,d^2\xi-d\xi\,d^2\eta}; \tag{24}$$

and therefore

$$\rho^2 = \frac{(d\xi^2+d\eta^2)^3}{(d\eta\,d^2\xi-d\xi\,d^2\eta)^2}; \tag{25}$$

and replacing η, ξ, and their differentials by y, x, and their differentials; and putting $dx^2+dy^2 = ds^2$, we have

$$a = x+\frac{ds^2\,dy}{dy\,d^2x-dx\,d^2y}; \tag{26}$$

$$\beta = y-\frac{ds^2\,dx}{dy\,d^2x-dx\,d^2y}; \tag{27}$$

$$\rho^2 = \frac{ds^6}{(dy\,d^2x-dx\,d^2y)^2}; \tag{28}$$

which are severally the coordinates to the centre, and the radius of the circle which has contact of the second order with the curve.

If x is taken to be equicrescent; and if the above expressions are modified accordingly, it will be seen on comparing them with (9) and (33) of the last Chapter, that they are the same as those determined for the coordinates of the centre of, and for the radius of curvature; and therefore the osculating circle, or that circle which has contact of the second order with a curve, is identical with the circle of curvature; and all the properties which were in the last Chapter proved to belong to the circle of curvature, are also equally true of the osculating circle.

Hence, and from Art. 307, it appears that a circle, which has contact of the second order with a curve at a given point, cuts the curve as well as touches it; that is, as at such a point the curvature continuously varies without a singular value, the osculating circle is wholly within the curve on the side towards which the curvature decreases, and is without it on that towards which the curvature increases.

311.] Let us consider whether under any, and if so, under what circumstances the osculating circle determined as above has contact of the third order with a given curve.

In (21) and (22), replacing ξ and η by x and y, we have

$$(x-a)\,dx+(y-\beta)\,dy = 0, \tag{29}$$

$$(x-a)\,d^2x+(y-\beta)\,d^2y+dy^2+dx^2 = 0; \tag{30}$$

therefore, differentiating again, since the contact is to be of the third order,

$$(x-a)\,d^3x+(y-\beta)\,d^3y+3\,(dx\,d^2x+dy\,d^2y) = 0; \tag{31}$$

whence, by cross-multiplication, from (29), (30) and (31), we have,

$$3\,(dx\,d^2x+dy\,d^2y)\,(dx\,d^2y-dy\,d^2x) + (d^3x\,dy-d^3y\,dx)\,(dx^2+dy^2) = 0; \tag{32}$$

and since $$ds^2 = dx^2+dy^2;$$

$$\therefore\quad ds\,d^2s = dx\,d^2x+dy\,d^2y;$$

therefore (32) becomes

$$3\,d^2s\,(dx\,d^2y-dy\,d^2x)+ds\,(d^3x\,dy-d^3y\,dx) = 0, \tag{33}$$

which condition must be satisfied, so that the circle may have contact of the third order.

Also since $$\rho = \frac{ds^3}{dx\,d^2y-dy\,d^2x},$$

$$\therefore\quad d\rho = \frac{3\,ds^2 d^2s\,(dx\,d^2y-dy\,d^2x)-ds^3\,(dx\,d^3y-dy\,d^3x)}{(dx\,d^2y-dy\,d^2x)^2}; \tag{34}$$

the numerator $= 0$, by virtue of equation (33); therefore $d\rho = 0$; and ρ is a maximum or a minimum, or a constant. Therefore if at any point of a curve the radius of curvature is a maximum or a minimum or constant, the osculating circle at that point has contact of the third order at least.

Thus in an ellipse the radius of curvature has singular values at the extremities of the principal axes; at these points therefore the circle of curvature has contact of the third order.

And if at any point on a curve the curvature is a maximum or a minimum, so that the contact is of an odd order, the osculating circle touches but does not intersect the curve: falling entirely within it when the curvature is a maximum, and entirely without it when the curvature is a minimum. In general too, in a closed curve which has no points of inflexion or cusps, such as an ellipse, the curvature has at least one maximum and one minimum value, and necessarily has the same number of minima as of maxima.

312.] Hence also it appears that, as the complete equation of the second degree of two variables is

$$\text{A}\eta^2 + \text{B}\xi\eta + \text{C}\xi^2 + \text{D}\eta + \text{E}\xi + \text{F} = 0,$$

which involves apparently six constants, but of which only *five* are arbitrary, because one fixes the standard of comparison, a central curve of the second degree, with all its terms complete, may have generally contact of the fourth order with a given curve; but as in the case of the parabola one relation is given amongst the constants, only four are arbitrary, and therefore a parabola can have contact of only the third order. Similarly with regard to the ellipse, if certain conditions are given, the number of disposable constants becomes diminished, and the order of contact is lowered; as, for instance, an ellipse with its major axis parallel to the axis of x has for its equation

$$\frac{(\xi-\alpha)^2}{a^2} + \frac{(\eta-\beta)^2}{b^2} = 1,$$

which has only four arbitrary constants, α, β, a, b; and therefore there can be contact of only the third order. Similarly, if in addition the ellipse is to be of a given eccentricity, a relation is given between a and b, and we have only three arbitrary constants, and the contact can be of only the second order.

Thus again suppose the problem to be, to determine the coordinates to the vertex and the latus rectum of a parabola, whose principal axis is parallel to the axis of x, and which has contact of the second order with the curve $y = f(x)$.

Let a and β be the coordinates to the vertex of the parabola, and $4m$ the latus rectum; then

$$(\eta-\beta)^2 = 4m(\xi-a); \tag{35}$$

$$(\eta-\beta)\frac{d\eta}{d\xi} = 2m, \qquad (\eta-\beta)\frac{d^2\eta}{d\xi^2}+\frac{d\eta^2}{d\xi^2} = 0;$$

whence, replacing ξ, η, and their differentials by x, y, and their differentials, we have

$$\beta = y+\frac{\frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}, \qquad a = x-\frac{\frac{dy}{dx}}{2\frac{d^2y}{dx^2}}; \tag{36}$$

$$4m = -\frac{2\left(\frac{dy}{dx}\right)^3}{\frac{d^2y}{dx^2}}; \tag{37}$$

By means of (36), in combination with the equation to the curve $y=f(x)$, we may find the locus of the vertex of the parabola, as the point of contact moves along the given curve.

It is required to determine the system of conics which shall have contact of the third order at the origin with the conic

$$x^2+2bxy+cy^2+2ey = 0.$$

Let the general equation to the required conics be

$$x^2+2\text{B}xy+\text{C}y^2+2\text{D}x+2\text{E}y+\text{F} = 0.$$

As these conics are to pass through the origin, $\text{F}=0$: and as they are to have the same tangent at the origin, $\frac{dy}{dx}$ is the same in both, when $x=y=0$; therefore $\text{D}=0$. And as $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ are at the origin to be the same in both, we have $\text{E}=e$, $\text{B}=b$; so that the equation to the required conic is

$$x^2+2bxy+\text{C}y^2+2ey = 0,$$

wherein C is undetermined; and therefore we have a system of conics fulfilling the conditions of the problem. If the required curve is a parabola, then $\text{C}=b^2$; and the equation to the parabola is

$$(x+by)^2+2ey = 0,$$

which is the original curve; and therefore no conic except itself can have contact of the third order with a parabola at a given point.

SECTION 2.—*The theory of envelopes.*

313.] In the discussion of the relative properties of evolutes and involutes, in Chapter XII, Arts. 294 and 303, it is proved that the normal to the involute is a tangent to the evolute; or, in other words, that as each normal to the involute passes through two consecutive points of the evolute, the latter curve may be imagined to be made up of an infinite number of infinitesimal straight lines, each of which is a part of a normal to the involute; thus we say that the evolute is formed by the intersection of consecutive normals. To take another case: let us conceive a system of straight lines infinite in number, and varying in position infinitesimally from each other, such that the perpendicular from a given point on each of them is the same; then the curve formed by the intersection of all is a circle. Or again, suppose that an infinite number of equal circles have their centres along a straight line, and infinitesimally near to each other; then they all intersect in, and by their intersections form, two straight lines parallel to the given line. Curves formed in this manner, by the ultimate intersection of straight lines or curves drawn according to some given law, are called *envelopes*, and are said to envelope the *family* of straight lines or curves. We proceed to discuss the general theory of them.

It is plain from algebraical as well as geometrical reasoning, that if an equation to a curve is given, involving one or more constants, as well as the current coordinates, the position and dimensions of the curve will be changed by a change in the constants, and yet generally the kind of curve will remain the same; that is, a variation of a constant may involve a specific though not a generic change of curve. A constant that enters into an equation, and varies in the way above explained, is called a *variable parameter*. Thus in the equation to a parabola, $y^2 = 4mx$, as m varies, the form of the parabola will vary, because its latus rectum varies, though its vertex and principal axis are unaltered. In the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

a and b may be variable parameters; in which case, changes of them will involve a change of an individual ellipse, though the family represented by the equation will remain that of ellipses still.

314.] Let the equation to the family of curves, of which it is our object to determine the envelope, involve only one parameter, and be

$$\text{F}(x, y, a) = 0, \tag{38}$$

in which a is the variable parameter; so that for every value of a we have some particular curve, but if we make a to vary infinitesimally and continuously, we have a series of curves, the position of each one differing infinitesimally from that of the next. Suppose a to receive a variation da, then the two curves whose equations are (38) and

$$\text{F}(x, y, a+da) = 0, \tag{39}$$

are in position infinitesimally near to another; but owing to the variation of a they will in general intersect in some point, which will be determined by x and y being the same in both (38) and (39), and which will be a point on the envelope. If therefore we eliminate a between (38) and (39), the resulting equation will involve only x and y, and will be the equation to the envelope.

Before however we proceed to apply the method, we may put (39) under a more convenient form. By (21), Art. 116,

$$\text{F}(x, y, a+da) = \text{F}(x, y, a) + da\, \text{F}'(x, y, a+\theta\, da) = 0;$$

and therefore by reason of equation (38)

$$\text{F}'(x, y, a+\theta\, da) = 0;$$

and therefore in the limit, when da is infinitesimal,

$$\text{F}'(x, y, a) = 0. \tag{40}$$

To determine therefore the envelope of the family of curves whose general equation is $\text{F}(x, y, a) = 0$, and of which the several individuals are formed by making a to vary, we must eliminate a between $\text{F} = 0$, and $\frac{d\text{F}}{da} = 0$.

The geometrical conception of such envelopes evidently requires that the particular curve and the envelope should have the same tangent at their common point. And this truth is also manifest from the following considerations:

Differentiate (38), making x, y and a to vary, then

$$\left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy + \left(\frac{d\text{F}}{da}\right) da = 0; \tag{41}$$

but by reason of (40), $\frac{dF}{d\alpha} = 0$; therefore, whether α varies or not, we have the same equation, viz.

$$\left(\frac{dF}{dx}\right) dx + \left(\frac{dF}{dy}\right) dy = 0,$$

whereby to determine $\frac{dy}{dx}$; and therefore the tangent is the same to the envelope and to each curve at their common point.

315.] Ex. 1. To determine the equation to the curve formed by the intersection of the straight lines whose equation is $y = \alpha x + \frac{m}{\alpha}$, where α varies.

Differentiating with respect to α, x and y being constant, we have

$$0 = x - \frac{m}{\alpha^2}; \qquad \therefore \quad \alpha = \pm \left(\frac{m}{x}\right)^{\frac{1}{2}};$$

$$\therefore \quad y = \pm \{\sqrt{(mx)} + \sqrt{(mx)}\};$$

$$y^2 = 4mx;$$

which is the equation to a parabola.

Ex. 2. To determine the envelope of the straight lines, of which the general equation is $y = \alpha x + (a^2 \alpha^2 + b^2)^{\frac{1}{2}}$, wherein α varies.

Differentiating with respect to α, we have

$$0 = x + \frac{a^2 \alpha}{(a^2 \alpha^2 + b^2)^{\frac{1}{2}}}; \qquad \therefore \quad \alpha = -\frac{b}{a} \frac{x}{(a^2 - x^2)^{\frac{1}{2}}};$$

substituting which in the general given equation, and reducing, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

On examination of these two examples it will be seen, that the determination of envelopes produced by straight lines is the inverse one to that of finding the equation to a tangent to a curve; for the two equations to the straight lines given in the two preceding examples are those known by the name of the "magical" equations to the tangents severally of the parabola and the ellipse. In this case then we have the equation to the tangent given, and the problem is, to determine the curve; in the other case the equation to the curve is given, and we have to determine that to the tangent. Hence the method of envelopes has been sometimes called "the inverse method of tangents."

The geometrical property involved in Ex. 1 is, "From a point in the axis of x, at a distance m from the origin, lines are drawn cutting the axis of y; at the points of intersection other lines are drawn perpendicular to these; to find the envelope of these latter lines." And that involved in Ex. 2 is, "To find the envelope of a series of straight lines drawn, so that the product of the two ordinates at distances $\pm a$ from the origin may be equal to b^2."

Ex. 3. To determine the envelope of all parabolas expressed

x^2 by the equation $y = ax - \frac{1+a^2}{2p}x^2$, wherein a varies.

Differentiating, with respect to a, we have

$$0 = x - \frac{a}{p}x^2; \qquad \therefore \quad a = \frac{p}{x};$$

$$\therefore \quad x^2 = 2p\left(\frac{p}{2} - y\right).$$

The envelope therefore is another parabola, having its focus at the origin.

Ex. 4. It is required to find the envelope of normals drawn to a given curve.

Let the equation to the curve be $y = f(x)$; then the equation to the normal is

$$(\eta - y)\frac{dy}{dx} + \xi - x = 0. \tag{42}$$

Differentiating, considering η and ξ, which are the current coordinates to the normal, to be constant, we have

$$-(\eta - y)\frac{d^2y}{dx^2} + 1 + \frac{dy^2}{dx^2} = 0. \tag{43}$$

From which, and from (42),

$$\eta = y + \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}, \qquad \xi = x - \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}\cdot\frac{dy}{dx}; \tag{44}$$

which are the same expressions as (33), Art. 288; and therefore the evolute is the envelope of the normals. The method pursued above is manifestly the same as that of Chapter XII, but expressed in different language.

316.] If the equation representing the family of curves involve many, say n, variable parameters, and these parameters are related by $(n-1)$ other and independent equations, which conditions are equivalent to there being only *one* variable parameter, instead of eliminating $(n-1)$ parameters, and then differentiating with respect to the remaining one, and proceeding as in the last Article, the following method is more elegant.

Let the equation to the family of curves be

$$\text{F}(x, y, a_1, a_2, a_3, \ldots\ldots a_n) = 0; \tag{45}$$

and let the $(n-1)$ equations of condition be

$$\left.\begin{aligned} f_1(a_1, a_2, \ldots\ldots a_n) &= 0, \\ f_2(a_1, a_2, \ldots\ldots a_n) &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \\ f_{(n-1)}(a_1, a_2, \ldots\ldots a_n) &= 0. \end{aligned}\right\} \tag{46}$$

Let $a_1, a_2, a_3, \ldots\ldots a_n$ vary by infinitesimal variations; then

$$\left(\frac{d\text{F}}{da_1}\right)da_1 + \left(\frac{d\text{F}}{da_2}\right)da_2 + \ \ldots\ldots + \left(\frac{d\text{F}}{da_n}\right)da_n = 0; \tag{47}$$

$$\left.\begin{aligned} \left(\frac{df_1}{da_1}\right)da_1 + \left(\frac{df_1}{da_2}\right)da_2 + \ldots\ldots + \left(\frac{df_1}{da_n}\right)da_n &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \\ \left(\frac{df_{(n-1)}}{da_1}\right)da_1 + \left(\frac{df_{(n-1)}}{da_2}\right)da_2 + \ldots + \left(\frac{df_{(n-1)}}{da_n}\right)da_n &= 0. \end{aligned}\right\} \tag{48}$$

Now, if x and y are the same in (45) and (47), they refer to the point in the envelope where the two particular curves of the family intersect; and therefore if the several variable parameters and their differentials can be eliminated, the resulting equation between x and y will represent the locus of the points of intersection, which will be the envelope required.

By equations (45) ...... (48) we have $2n$ different relations, from which $(2n-1)$ quantities, viz. $a_1, a_2, a_3, \ldots\ldots a_n$, $\frac{da_1}{da_n} \ldots\ldots \frac{da_{n-1}}{da_n}$, are to be eliminated; which is, of course, a possible problem. To solve it, multiply the $(n-1)$ equations in (48) by $(n-1)$ indeterminate multipliers $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_{n-1}$, and add all and (47) together; by which means there will be n conditions involved in one equation; and therefore we are at liberty to make n new conditions. Let these be that the coefficients of the differentials, viz. $da_1, da_2, \ldots\ldots da_n$, be equal to zero; whence we have

$$\left.\begin{aligned}
&\left(\frac{d\mathrm{F}}{da_1}\right)+\lambda_1\left(\frac{df_1}{da_1}\right)+\lambda_2\left(\frac{df_2}{da_1}\right)+\ldots+\lambda_{n-1}\left(\frac{df_{n-1}}{da_1}\right)=0,\\
&\left(\frac{d\mathrm{F}}{da_2}\right)+\lambda_1\left(\frac{df_1}{da_2}\right)+\lambda_2\left(\frac{df_2}{da_2}\right)+\ldots+\lambda_{n-1}\left(\frac{df_{n-1}}{da_2}\right)=0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\left(\frac{d\mathrm{F}}{da_n}\right)+\lambda_1\left(\frac{df_1}{da_n}\right)+\lambda_2\left(\frac{df_2}{da_n}\right)+\ldots+\lambda_{n-1}\left(\frac{df_{n-1}}{da_n}\right)=0;
\end{aligned}\right\} \quad (49)$$

and between the equations (45), (46), and (49), which are $2n$ in number, we may eliminate the $(2n-1)$ quantities, $a_1, a_2, \ldots\ldots a_n, \lambda_1, \lambda_2, \ldots\ldots \lambda_{n-1}$, and ultimately arrive at an equation between x and y only, which will be the required envelope to the family of curves.

In the particular case in which the general equation contains only two variable parameters, and one equation is given connecting them, the result assumes a form identical with that considered in Art. 168; and therefore, as is therein shewn, the ratio of the coefficients of the above differentials in the two differential equations is constant.

317.] Ex. 1. To find the envelope of a series of straight lines, such that the perpendiculars from the origin on them are equal to a given straight line c.

Let a be the angle which the perpendicular on any one of the lines makes with the axis of x; then the equation to the line is

$$x\cos a+y\sin a = c; \qquad (50)$$

whence, differentiating with respect to a,

$$y\cos a-x\sin a = 0; \qquad (51)$$

whence, squaring and adding (50) and (51), and bearing in mind the condition $(\cos a)^2+(\sin a)^2 = 1$, we have

$$x^2+y^2 = c^2;$$

the equation to a circle, which is manifestly correct.

Ex. 2. A straight line of given length slides down between two rectangular axes; to find the envelope of the line in all positions.

Let c be the length of the line; a and b the intercepts of the axes of x and y by the line: then the equation to the line is

$$\frac{x}{a}+\frac{y}{b} = 1; \tag{52}$$

wherein a and b are connected by the equation

$$a^2+b^2 = c^2. \tag{53}$$

Differentiating therefore (52) and (53) by making a and b to vary, we have

$$\frac{x}{a^2}\,da+\frac{y}{b^2}\,db = 0,$$

$$a\,da+b\,db = 0;$$

and therefore, by reason of the remark at the end of the last Article,

$$\frac{\dfrac{x}{a^2}}{a} = \frac{\dfrac{y}{b^2}}{b} = \frac{\dfrac{x}{a}}{a^2} = \frac{\dfrac{y}{b}}{b^2} = \frac{1}{c^2};$$

$$\therefore \quad a = x^{\frac{1}{3}}c^{\frac{2}{3}}, \qquad b = y^{\frac{1}{3}}c^{\frac{2}{3}};$$

$$\therefore \quad a^2+b^2 = (x^{\frac{2}{3}}+y^{\frac{2}{3}})\,c^{\frac{4}{3}};$$

$$\therefore \quad x^{\frac{2}{3}}+y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

Which curve is drawn in fig. 45, and of which therefore the length of the tangent intercepted between the two rectangular axes is constant.

Ex. 3. To find the envelope of a series of concentric and coaxal ellipses, of which the area is constant.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \qquad ab = c^2;$$

$$\therefore \quad \frac{x^2}{a^3}\,da+\frac{y^2}{b^3}\,db = 0, \qquad \frac{da}{a}+\frac{db}{b} = 0;$$

$$\therefore \quad \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{1}{2};$$

$$\therefore \quad a = x\sqrt{2}, \qquad b = y\sqrt{2};$$

$$\therefore \quad xy = \frac{c^2}{2};$$

the equation to a hyperbola referred to its asymptotes as axes.

Ex. 4. To find the envelope of a system of straight lines whose equation is

$$\frac{x}{a}+\frac{y}{b} = 1, \tag{54}$$

a and b being related by the equation

$$\frac{a}{l}+\frac{b}{m}=1. \tag{55}$$

$$\frac{x}{a^2}\,da+\frac{y}{b^2}\,db=0, \qquad \frac{da}{l}+\frac{db}{m}=0;$$

$$\therefore \quad \frac{\frac{x}{a^2}}{\frac{1}{l}}=\frac{\frac{y}{b^2}}{\frac{1}{m}}=\frac{\frac{x}{a}}{\frac{a}{l}}=\frac{\frac{y}{b}}{\frac{b}{m}}=1;$$

$$\therefore \quad a=(lx)^{\frac{1}{2}}, \qquad b=(my)^{\frac{1}{2}};$$

$$\therefore \quad \left(\frac{x}{l}\right)^{\frac{1}{2}}+\left(\frac{y}{m}\right)^{\frac{1}{2}}=1. \tag{56}$$

The equation to a parabola, referred to two tangents as coordinate axes, the intercepts of which by the curve are l and m.

The geometry of the problem is represented in fig. 112: OA $= a$, OB $= b$, OL $= l$, OM $= m$; therefore ML is the fixed line whose equation is (55), and of which a and b are current coordinates; and AB is the varying line whose equation is (54). The formation of the curve is manifest from the figure.

Ex. 5. The centres of a series of equal circles are on a given straight line; it is required to find the envelope.

Let the equation to the given straight line be

$$x\cos\alpha+y\sin\alpha=p;$$

and x, y being the coordinates to the centre of, and ξ, η the current coordinates to, the circles of radius c; their equation is

$$(\xi-x)^2+(\eta-y)^2=c^2;$$

whence, differentiating, we have

$$\cos\alpha\,dx+\sin\alpha\,dy=0,$$

$$(\xi-x)\,dx+(\eta-y)\,dy=0;$$

$$\therefore \quad \frac{\xi-x}{\cos\alpha}=\frac{\eta-y}{\sin\alpha}=\pm c=\frac{\xi\cos\alpha+\eta\sin\alpha-(x\cos\alpha+y\sin\alpha)}{(\cos\alpha)^2+(\sin\alpha)^2}$$

$$=\xi\cos\alpha+\eta\sin\alpha-p;$$

$$\therefore \quad \xi\cos\alpha+\eta\sin\alpha=p\pm c;$$

the equation to two straight lines parallel to the given line, and at distances $\pm$ c from it.

Ex. 6. From a given point on the circumference of a circle chords are drawn, and on these, as diameters, circles are described; it is required to find their envelope.

In fig. 124 let S be the given point in the circumference of the circle: from which let the chord SQ be drawn; and on SQ, as a diameter, let the circle SPQ be described; it is required to find the envelope of all circles described similarly to SPQ.

Let $\text{SP} = r$, $\text{PSA} = \theta$, $\text{SA} = 2a$, $\text{QSA} = \theta'$;

$$\therefore \quad \text{SQ} = 2a\cos\theta';$$

and since
$$\text{SP} = \text{SQ}\cos\text{PSQ},$$

$$r = 2a\cos\theta'\cos(\theta-\theta'); \tag{57}$$

differentiating which with respect to θ', since SP remains the same when θ' varies, we have

$$0 = 2a\{\cos\theta'\sin(\theta-\theta')-\sin\theta'\cos(\theta-\theta')\};$$

$$\therefore \quad \sin(\theta-2\theta') = 0, \qquad \text{and} \quad \theta' = \frac{\theta}{2};$$

whereby (57) becomes,

$$r = 2a\left(\cos\frac{\theta}{2}\right)^2,$$

$$= a(1+\cos\theta);$$

which is the equation to the cardioid.

Ex. 7. Again, suppose that on the radii vectores of the cardioid, as diameters, circles are described as in the last example; and again, on the radii vectores of the envelope, as diameters, circles are described, and so on continually; it is required to prove that the envelope ultimately is a circle whose radius $= 2a$.

Suppose in fig. 124 SQA to be a cardioid, and the circle SPQ to be described on SQ as a diameter; then, if $\text{SP} = r$, $\text{PSA} = \theta$, $\text{QSA} = \theta'$,

$$r = 2a\left(\cos\frac{\theta'}{2}\right)^2\cos(\theta-\theta').$$

Differentiating which with respect to θ', we have

$$0 = 2a\cos\frac{\theta'}{2}\left\{\sin(\theta-\theta')\cos\frac{\theta'}{2}-\cos(\theta-\theta')\sin\frac{\theta'}{2}\right\};$$

$$\therefore \quad 2a\cos\frac{\theta'}{2}\sin\left(\theta-\frac{3\theta'}{2}\right) = 0, \qquad \text{and} \;\therefore\; \theta' = \frac{2\theta}{3};$$

and the equation to the envelope is,

$$r = 2a\left(\cos\frac{\theta}{3}\right)^3.$$

And if a similar process is continued n times,

$$r = 2a\left(\cos\frac{\theta}{n}\right)^n;$$

and if $n = \infty$, $r = 2a$; that is, the ultimate envelope is a circle whose centre is s and radius is $2a$, and which is dotted in the figure.

SECTION 3.—*Reciprocal curves and the theory of reciprocation.*

318.] We are now able to complete, so far as the scope of our work requires, the inquiry into poles and polars which was opened in Articles 226 and 227; and we shall also be able to investigate some matters of interest connected with that inquiry. I shall take the requisite equations in a homogeneous and symmetrical form, so that the results may have all the elegance which they are capable of. Let us suppose the curve, which in these articles is called the base-curve, to be an algebraical curve of the nth degree, and of the form (49), Art. 208; and to be represented by the equation

$$\text{F}(x, y, z) = 0. \tag{58}$$

Let the pole be (ξ, η, ζ); then the equation of the first polar of this pole with respect to the curve (58) is

$$\xi\left(\frac{d\text{F}}{dx}\right) + \eta\left(\frac{d\text{F}}{dy}\right) + \zeta\left(\frac{d\text{F}}{dz}\right) = 0; \tag{59}$$

which we may, as in Art. 224, express by the equation $p_1 = 0$; and let the successive polars of the same pole be $p_2 = 0, \ldots$ $p_{n-1} = 0$, $p_n = 0$. Let us for the present confine our attention to the first polar, viz. (59); this is evidently a homogeneous equation of the $(n-1)$th order with respect to x, y, z; and is the equation to the curve which passes through the $n(n-1)$ points, real or imaginary, at which tangents drawn from the pole (ξ, η, ζ) meet the base-curve.

Let the pole move continuously along another curve, which I shall, as before, call the directrix; let the equation of this curve also be homogeneous and of the rth degree in terms of the coordinates ξ, η, ζ; and let its equation be

$$\Phi(\xi, \eta, \zeta) = 0; \tag{60}$$

then ξ, η, ζ, in (59) are subject to the condition (60.) And as the pole moves along the directrix, so the position of the first polar continuously varies; and as the variation, corresponding to an infinitesimal variation of the pole, is infinitesimal, so the several and consecutive polars intersect, and in their intersections envelope another curve: this result is of course true for each system of polars, but at present we are considering only the first polars, and are inquiring into the envelope of these. If the directrix is a straight line, it has already, in Art. 227, been shewn that the envelope is $(n-1)^2$ points.

319.] The degree of the first polar envelope may be determined as follows. Let us assume the degrees of the base-curve and of the directrix to be respectively n and r; then, as the position of the pole varies continuously, let us take the ξ-, η-, and ζ-differentials of (59) and (60): so that we have

$$\left(\frac{d\text{F}}{dx}\right) d\xi + \left(\frac{d\text{F}}{dy}\right) d\eta + \left(\frac{d\text{F}}{dz}\right) d\zeta = 0, \tag{61}$$

$$\left(\frac{d\Phi}{d\xi}\right) d\xi + \left(\frac{d\Phi}{d\eta}\right) d\eta + \left(\frac{d\Phi}{d\zeta}\right) d\zeta = 0; \tag{62}$$

and as no other relation is given between $d\xi$, $d\eta$, $d\zeta$, we have

$$\frac{\left(\frac{d\text{F}}{dx}\right)}{\left(\frac{d\Phi}{d\xi}\right)} = \frac{\left(\frac{d\text{F}}{dy}\right)}{\left(\frac{d\Phi}{d\eta}\right)} = \frac{\left(\frac{d\text{F}}{dz}\right)}{\left(\frac{d\Phi}{d\zeta}\right)}; \tag{63}$$

whence we have two equations in terms of $x, y, z, \xi, \eta, \zeta$; which are of $n-1$ dimensions in terms of x, y, z, and of $r-1$ dimensions in terms of ξ, η, ζ. If we eliminate ξ, η, ζ from these two equations and from equation (60), the resultant is of the order $r(r-1)(n-1)$ in terms of x, y and z; and therefore this is the degree of the first polar envelope. Thus if the directrix is a straight line, $r = 1$, and the degree of the polar envelope is zero: that is, as we have shewn in Art. 227, all the several first polars intersect in $(n-1)^2$ points. If $r = n = 2$, that is, if the base-curve and the directrix are conics, the first polar envelope is also a conic. Let us consider one or two examples of the formation of these first polar envelopes.

In the first place, I will take the base-curve to be the central conic whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0; \tag{64}$$

and I will assume the directrix to be the straight line whose equation is

$$\text{A}\xi+\text{B}\eta+\text{C}\zeta=0, \tag{65}$$

the pole being (ξ, η, ζ). Now the first polar of (64) is

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}=0, \tag{66}$$

of which line the envelope is to be found, the pole (ξ, η, ζ) being subject to the condition (65). Therefore from (65) and (66) we have

$$\text{A}\,d\xi+\text{B}\,d\eta+\text{C}\,d\zeta=0,$$

$$\frac{x}{a^2}d\xi+\frac{y}{b^2}d\eta+\frac{z}{c^2}d\zeta=0;$$

$$\therefore\quad x=\text{A}a^2,\qquad y=\text{B}b^2,\qquad z=\text{C}c^2; \tag{67}$$

which are the equations to a point; and therefore all the first polars of (64) with reference to a pole situated on the straight line whose equation is (65) pass through the point whose coordinates are given in (67).

Again, let us suppose the directrix to be a curve of the second degree, of which the equation is

$$\text{A}\xi^2+\text{B}\eta^2+\text{C}\zeta^2+2\text{E}\eta\zeta+2\text{G}\zeta\xi+2\text{H}\xi\eta=0; \tag{68}$$

and let us suppose the base-curve to be the central conic (64), as in the preceding example; then the equation to the polar of (64) is

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}=0; \tag{69}$$

of which the envelope is to be found, ξ, η, ζ being subject to (68); whence we have

$$(\text{A}\xi+\text{H}\eta+\text{G}\zeta)\,d\xi+(\text{H}\xi+\text{B}\eta+\text{E}\zeta)\,d\eta+(\text{G}\xi+\text{E}\eta+\text{C}\zeta)\,d\zeta=0,$$

$$\frac{x}{a^2}d\xi+\qquad\frac{y}{b^2}d\eta+\qquad\frac{z}{c^2}d\zeta=0;$$

whence we have

$$\frac{\dfrac{x}{a^2}}{\text{A}\xi+\text{H}\eta+\text{G}\zeta}=\frac{\dfrac{y}{b^2}}{\text{H}\xi+\text{B}\eta+\text{E}\zeta}=\frac{\dfrac{z}{c^2}}{\text{G}\xi+\text{E}\eta+\text{C}\zeta} \tag{70}$$

$$=\frac{1}{k}\ \text{(say)};$$

$$\therefore\quad \left.\begin{aligned} \mathrm{A}\xi+\mathrm{H}\eta+\mathrm{G}\zeta &= \frac{kx}{a^2} \\ \mathrm{H}\xi+\mathrm{B}\eta+\mathrm{E}\zeta &= \frac{ky}{b^2} \\ \mathrm{G}\xi+\mathrm{E}\eta+\mathrm{C}\zeta &= \frac{kz}{c^2} \end{aligned}\right\}; \tag{71}$$

$$\therefore\quad \xi\,\{\mathrm{ABC}+2\mathrm{EGH}-\mathrm{AE}^2-\mathrm{BG}^2-\mathrm{CH}^2\}$$
$$= k\left\{\frac{x}{a^2}(\mathrm{BC}-\mathrm{E}^2)+\frac{y}{b^2}(\mathrm{EG}-\mathrm{CH})+\frac{z}{c^2}(\mathrm{EH}-\mathrm{BG})\right\};$$

and similar values are true for η and ζ: substituting these in (69) we have

$$\frac{\mathrm{BC}-\mathrm{E}^2}{a^4}x^2+\frac{\mathrm{CA}-\mathrm{G}^2}{b^4}y^2+\frac{\mathrm{AB}-\mathrm{H}^2}{c^4}z^2$$
$$+2\frac{\mathrm{GH}-\mathrm{AE}}{b^2c^2}yz+2\frac{\mathrm{HE}-\mathrm{BG}}{c^2a^2}zx+2\frac{\mathrm{EG}-\mathrm{CH}}{a^2b^2}xy=0; \tag{72}$$

which is an equation of the second degree, and therefore represents a conic. Thus the envelope of all the first polars of a pole which is on a conic with respect to the central conic (64) is the conic (72); that is, as the pole moves along the conic (68) the envelope of the first polars with respect to (64) is another conic.

320.] The mutual relation of these two conics to each other by means of the central conic requires farther investigation. In the first place it is evident that to every point of the former conic a tangent of the latter corresponds: also let us take two points on the former conic infinitesimally near to each other; then the straight line passing through those two points is the tangent to the conic at the point: but the polars corresponding to all poles on this straight line pass, as we have shewn in the last Article, through the same point, that is, through the point of the second conic at which the two polars intersect which correspond to the poles on the former conic which are taken infinitesimally near to each other; and therefore this point is the pole of the tangent of the former conic. And a similar result is true of all points on this second conic; and therefore the former conic is the envelope of all the polars which correspond to the first polars of the points on the latter conic: or, in other words, the director-conic is the envelope of all the polars whose poles are on the conic envelope. The two conics therefore have reciprocal properties; each is the envelope of all the polars whose poles are on the other; and whatever peculiarities there are as to points on one conic, there are corresponding peculiarities as to the corresponding tangents of the other. Two conics thus related are called *reciprocal polar conics.*

In the preceding investigation I have taken the base-curve to be a central conic. The process will be simplified, and the results will be scarcely less general, if a circle is the base-curve; and this is usually assumed to be the case unless it is expressly stated to the contrary. In this case, $a = b = c$.

321.] Now properties similar to these are capable of extension to other curves; let us suppose the directrix to be a curve of the nth degree; and let us suppose the first polars of all the poles, which are points on it, to be drawn, with respect to the base-curve; all these polars will envelope another curve, which is the polar envelope; and all points on this are also manifestly the poles of the tangents of the original director. Thus these two curves are reciprocal to each other; and are called *reciprocal polars*: to a point on one a tangent on the other corresponds. And in these reciprocal relations very important properties have their origin: all properties of curves become double: that which is true of certain points on a curve, is also true, *mutatis mutandis*, of the corresponding tangents of the reciprocal curve. Some few of these reciprocal properties are indicated, although it is beyond the scope of my work to give a complete, or an approximately complete catalogue of them.

If the director curve is of the nth order, a straight line may cut it in n points; and as there is a different polar corresponding to each of these which touches the reciprocal curve, and as these polars will meet in a point which is the pole of the cutting line of the original curve, so to the reciprocal, from a given point, may n tangents be drawn: therefore the reciprocal polar is of the nth class; and the degree of a curve and the class of the reciprocal curve are the same. And since $n(n-1)$ tangents may be drawn from a given point to a curve of the nth degree, so is $n(n-1)$ the degree of the polar reciprocal of a curve of the nth degree.

If for one of two reciprocal polars a polygon is inscribed or circumscribed, then to each vertex of the inscribed polygon the side of another polygon circumscribing the reciprocal corresponds; and to each side of an inscribed polygon the vertex of a circumscribed polygon corresponds. Thus, a vertex of an inscribed polygon is to be changed into the side of another polygon circumscribed about the reciprocal polar; and a side of an inscribed polygon into the vertex of a polygon circumscribed about the reciprocal polar. The Theorems of Pascal and Brianchon are thus reciprocally involved.

It is plain also that the two reciprocal polars have the following corresponding and reciprocal properties. To a double point on one curve corresponds a double tangent on the other: to a multiple point of the *k*th order on one, a multiple tangent of the *k*th order on the other: to a cusp on one, a point of inflexion on the other. If many points on one curve are on a conic, the corresponding polars are tangents to another conic.

As a curve of the *n*th degree is determined which passes through $\frac{n(n+3)}{2}$ points, so is a curve of the *n*th class which touches $\frac{n(n+3)}{2}$ different lines. And as all curves of the *n*th degree which pass through $\frac{n(n+3)}{2}-1$ fixed points also pass through $\frac{(n-1)(n-2)}{2}$ other fixed points, so all curves of the *n*th class which touch $\frac{n(n+3)}{2}-1$ fixed straight lines also touch $\frac{(n-1)(n-2)}{2}$ other fixed lines.

Again, let $n =$ the degree of a given curve, $m =$ its class; $\delta =$ the number of its double points; $\kappa =$ the number of its cusps; $\tau =$ the number of its double tangents; $i =$ the number of its points of inflexion; then, from (134), Art. 252, and (142), Art. 253, we have

$$m = n(n-1)-2\delta-3\kappa, \tag{73}$$

$$i = 3n(n-2)-6\delta-8\kappa; \tag{74}$$

and in the reciprocal polar these become

$$n = m(m-1)-2\tau-3i, \tag{75}$$

$$\kappa = 3m(m-2)-6\tau-8i. \tag{76}$$

This last equation however is not independent, but is involved in the other three. Thus, three independent equations are given involving six unknown quantities; of these if three are given the other three can be found; so that we have altogether 60 different formulæ.

For other properties of reciprocal curves I must refer the reader to the original memoir by M. Poncelet in Crelle's Journal, Vol. IV, 1839; to several memoirs by Steiner and others in the same Journal; to Analytisch-Geometrische Entwickelungen of Plücker, Zweiter Band, p. 259; and to the Annales des Mathématiques of Terquem and Gerono, Vol. XII.

SECTION 4.—*On caustics.*

322.] A particular class of envelopes formed by straight lines exists in optics which is of too great importance to be passed over in silence in the present Chapter; we will therefore first give some general notions of the formation of such envelopes which are called Caustics, and then consider some particular examples and general properties of them.

In fig. 118 suppose S to be a source of light from which rays proceed and fall on a highly polished surface, which is perpendicular to the plane of the paper, and of which AP is a section made by the paper; and let SP be a type-ray of such a system incident on the surface at P. Now it is a physical property of a ray, that it is *reflected* or turned back in the direction PR; so that, if PO is the normal to the curve at P, the angle of incidence, as it is called, SPO is equal to the angle of reflexion RPO. The envelope of the lines of which PR is the type, is called the *caustic by reflexion* of the surface.

And again in fig. 122, suppose S to be the source of a system of rays, of which let SP be the type-ray; and suppose the rays to fall on a medium different to that in which S is, of which the bounding surface is perpendicular to the plane of the paper: and of which let AP be the section made by the paper; then by a physical law of optics, called the *law of refraction*, the ray SP does not proceed in the same straight line, but at P is bent or *refracted* into the direction Pr, which is so related to SP that, if nPN is the normal to the surface at P, $\sin \mathrm{SPN} = \mu \sin r\mathrm{P}n$, where μ is constant for a given medium, but varies for different media; that is, the sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction. The envelope of all the refracted rays is called the *caustic by refraction* of the given surface*.

323.] To determine the caustic by reflexion of a system of parallel rays falling on a plane curve.

Suppose the source of light to be at an infinite distance, such as the sun is, and therefore all the incident rays to be parallel; and first let us suppose them to be parallel to the axis of x. In fig. 114 let QP be the incident type-ray, and PR the reflected type-ray: PS being the reflecting curve, and PG its normal at the point P.

Let $y = f(x)$ be the equation to the reflecting curve, and η and ξ the current coordinates of the reflected ray; then the equation to PR is

$$\eta - y = \tan \text{PRG} \, (\xi - x);$$

of which straight line we have to find the envelope.

Since the angles of incidence and reflexion are equal, $\text{QPG} = \text{RPG}$, and therefore their complements are equal, viz. $\text{QPL} = \text{RPT}$; but $\text{QPL} = \text{PTG} = \tan^{-1} \dfrac{dy}{dx}$;

$$\begin{aligned} \therefore \quad \text{PRG} &= \text{RPT} + \text{RTP}, \\ &= 2.\text{RTP} = 2 \tan^{-1} \frac{dy}{dx}; \end{aligned}$$

$$\therefore \quad \tan \text{PRG} = \frac{2 \dfrac{dy}{dx}}{1 - \dfrac{dy^2}{dx^2}}.$$

and therefore the equation to the reflected ray is

$$\eta - y = \frac{2 \dfrac{dy}{dx}}{1 - \dfrac{dy^2}{dx^2}} (\xi - x); \tag{77}$$

$$\therefore \quad \xi - x = \frac{1}{2} \left\{ \frac{dx}{dy} - \frac{dy}{dx} \right\} (\eta - y). \tag{78}$$

Differentiating with respect to x,

$$1 = \frac{1}{2} \left\{ \frac{dx^2}{dy^2} \frac{d^2y}{dx^2} + \frac{d^2y}{dx^2} \right\} (\eta - y) + \frac{1}{2} \left\{ \frac{dx}{dy} - \frac{dy}{dx} \right\} \frac{dy}{dx}; \tag{79}$$

---

* In the figure the line AP is straight, but the matter of the text is expressed as though it were a curve.

$$\therefore \quad \eta - y = \frac{\frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}, \qquad \xi - x = \frac{1}{2}\,\frac{\frac{dy}{dx}\left(1-\frac{dy^2}{dx^2}\right)}{\frac{d^2y}{dx^2}}. \tag{80}$$

By means of which equations, and that to the reflecting curve, we may eliminate x and y, and thereby obtain a relation between ξ and η, which will be the equation to the caustic.

Ex. 1. Let the reflecting curve be the parabola.

$$y^2 = 4ax;$$

$$\frac{dy}{dx} = \frac{2a}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3};$$

$$\therefore \quad \eta - y = -\frac{4a^2}{y^2}\,\frac{y^3}{4a^2} = -y, \qquad \therefore \quad \eta = 0.$$

$$\xi - x = -\frac{1}{2}\,\frac{2a}{y}\,\frac{y^2-4a^2}{y^2}\,\frac{y^3}{4a^2},$$

$$= -\frac{4ax-4a^2}{4a},$$

$$= -x + a, \qquad \therefore \quad \xi = a.$$

All the reflected rays therefore pass through the focus, which is their envelope; the caustic therefore is a point.

Ex. 2. Let the reflecting curve be a circle.

$$x^2 + y^2 = a^2;$$

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{a^2}{y^3};$$

$$\therefore \quad \eta - y = -\frac{x^2}{y^2}\,\frac{y^3}{a^2} = -\frac{x^2y}{a^2},$$

$$\therefore \quad \eta = y\left(1-\frac{x^2}{a^2}\right) = \frac{y^3}{a^2},$$

$$\therefore \quad y = a^{\frac{2}{3}}\eta^{\frac{1}{3}}.$$

$$\xi - x = \frac{1}{2}\,\frac{x}{y}\left(1-\frac{x^2}{y^2}\right)\frac{y^3}{a^2};$$

$$\therefore \quad \xi = x\left\{\frac{a^2+2y^2}{2a^2}\right\} = x\,\frac{a^2+2a^{\frac{4}{3}}\eta^{\frac{2}{3}}}{2a^2},$$

$$\therefore \quad x = \frac{2a^{\frac{2}{3}}\xi}{a^{\frac{2}{3}}+2\eta^{\frac{2}{3}}}.$$

$$\therefore \quad \eta^{\frac{2}{3}} a^{\frac{4}{3}} + \frac{4a^{\frac{4}{3}}\xi^2}{(a^{\frac{2}{3}}+2\eta^{\frac{2}{3}})^2} = a^2,$$

$$\xi = \pm \frac{1}{2}(a^{\frac{2}{3}}+2\eta^{\frac{2}{3}})\{a^{\frac{2}{3}}-\eta^{\frac{2}{3}}\}^{\frac{1}{2}};$$

the equation to an epicycloid, the radius of whose fixed circle is $\frac{a}{2}$, and of whose generating circle is $\frac{a}{4}$, as will appear on eliminating θ between the two equations (37), Art. 204, having first replaced b by $\frac{a}{4}$ and a by $\frac{a}{2}$. See fig. 119.

The above examples are sufficient for illustration, but the difficulties of elimination are in most cases insurmountable; the semi-cubical parabola is another curve admitting of solution.

324.] Again, suppose the incident rays to be parallel to the axis of y, see fig. 115, of which let MP be the incident type-ray, and PR the reflected type-ray; let $y = f(x)$ be the equation to the reflecting curve; and let ξ, η be the current coordinates of the reflected ray; then the equation to PR is

$$\eta - y = \frac{1}{2}\left\{\frac{dy}{dx} - \frac{dx}{dy}\right\}(\xi - x); \tag{81}$$

of which line we have to determine the envelope.

Differentiating, and proceeding as in the last Article, it will be found that

$$\xi - x = -\frac{\frac{dy}{dx}}{\frac{d^2y}{dx^2}}, \qquad \eta - y = \frac{1}{2}\frac{1-\frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}. \tag{82}$$

By means of which, and the equation to the curve, we may eliminate x and y, and determine a relation between ξ and η, which will be the equation to the caustic.

Ex. 1. $$y^2 = 4ax.$$

$$\frac{dy}{dx} = \frac{2a}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3};$$

$$\xi - x = \frac{2a}{y}\frac{y^3}{4a^2} = \frac{y^2}{2a} = 2x;$$

$$\therefore \quad x = \frac{\xi}{3}.$$

$$\eta - y = -\frac{1}{2}\,\frac{y^2-4a^2}{y^2}\,\frac{y^3}{4a^2} = \frac{a-x}{2a}\,y;$$

$$\therefore \quad y = \frac{6a\eta}{9a-\xi}.$$

$$\therefore \quad \eta^2 = \frac{\xi}{27a}(9a-\xi)^2;$$

which represents a curve symmetrical with respect to the axis of x, passing through the origin where it touches the axis of y; with a double point on the axis of x at a distance $9a$ from the origin, and a loop between the origin and that point; and approaching to the semi-cubical parabola as an asymptotic curve.

Ex. 2. Find the equation to the caustic of the cycloid, the incident rays being perpendicular to the base.

Take the starting point o for the origin, as in fig. 117; then the equation to the curve is

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}};$$

$$\therefore \quad \frac{dy}{dx} = \frac{(2ay-y^2)^{\frac{1}{2}}}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{a}{y^2};$$

$$\therefore \quad \eta - y = -\frac{1}{2}\,\frac{2y^2-2ay}{y^2}\,\frac{y^2}{a},$$

$$= \frac{ay-y^2}{a};$$

$$\therefore \quad y = a - (a^2 - a\eta)^{\frac{1}{2}}.$$

$$\xi - x = \frac{y}{a}(2ay-y^2)^{\frac{1}{2}},$$

$$\therefore \quad x = \xi - (a\eta)^{\frac{1}{2}} + (a\eta-\eta^2)^{\frac{1}{2}};$$

$$\therefore \quad \xi - (a\eta)^{\frac{1}{2}} + (a\eta-\eta^2)^{\frac{1}{2}} = a \operatorname{versin}^{-1}\left\{1 - \left(\frac{a-\eta}{a}\right)^{\frac{1}{2}}\right\} - (a\eta)^{\frac{1}{2}},$$

$$\xi + (a\eta-\eta^2)^{\frac{1}{2}} = \frac{a}{2}\operatorname{versin}^{-1}\frac{2\eta}{a};$$

$$\therefore \quad \xi = \frac{a}{2}\operatorname{versin}^{-1}\frac{2\eta}{a} - (a\eta-\eta^2)^{\frac{1}{2}}; \qquad (83)$$

which equation is that to a cycloid, of which the starting point is the origin, and the radius of whose generating circle is one-half of that of the generating circle of the original cycloid.

Ex. 3. The logarithmic curve. $y = e^x$.

$$\frac{dy}{dx} = e^x, \qquad \frac{d^2y}{dx^2} = e^x;$$

$$\therefore \quad \xi - x = -1; \qquad \therefore \quad x = \xi + 1;$$

$$\eta - y = \frac{1}{2}\{e^{-x} - e^x\},$$

$$\therefore \quad \eta = \frac{1}{2}\{e^x + e^{-x}\},$$

$$= \frac{1}{2}\{e^{\xi+1} + e^{-(\xi+1)}\}.$$

The equation to the catenary; to the lowest point of which the abscissa $= -1$, and the ordinate $= 1$.

325.] The following general properties of caustics by reflexion, formed by a system of parallel rays, deserve consideration.

(1) The distance from the incident point in the reflecting curve to the point of intersection of two consecutive reflected rays, is equal to one-fourth of the chord of the circle of curvature at the point of incidence which is parallel to the incident ray.

Let QP, fig. 116, be an incident ray and PR be the reflected ray, P′ being the point where the next consecutive ray cuts it, and which is therefore a point in the caustic; let the circle drawn in the figure, and of which Π is the centre and PΠ is the radius, be the circle of curvature at the point P; then PF and PE are the chords of the circle through P which are parallel to the axes of x and y respectively, and let L and K be the bisecting points of PF and of PE.

Now, according to the notation of Art. 288, PL is equal to $\xi_1 - x$, and PK to $y - \eta_1$ of that Article;

$$\therefore \quad \text{PL} = -\frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}\frac{dy}{dx}, \qquad \text{PK} = -\frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}. \tag{84}$$

But from the formulæ (80), Art. 323,

$$(\text{PP}')^2 = (y-\eta)^2 + (x-\xi)^2,$$

$$= \left\{\frac{1}{2}\frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}\frac{dy}{dx}\right\}^2 = \left(\frac{\text{PL}}{2}\right)^2;$$

$$\therefore \quad \text{PP}' = \frac{1}{2}\,\text{PL}.$$

Similarly, if the incident rays are parallel to the axis of y, and Q′ is the point of the caustic on the reflected ray, it may be shewn by means of equations (82) that

$$\text{PQ}' = \frac{1}{2}\,\text{PK}.$$

The expressions throughout would have assumed a more symmetrical though less simple form if we had not considered x to be equicrescent.

(2) If the radiating point is such that a normal to the reflecting curve can be drawn through it, the caustic corresponding to the point where the normal meets the curve ultimately becomes a semi-cubical parabola.

For if the part of the reflecting curve which receives the rays parallel or nearly parallel to the normal through the source of light be taken very small compared to the distance of the origin of light from the curve, the system may be supposed to be one of parallel rays; and also whatever is the reflecting curve we may consider it to be identical with its circle of curvature at the point, so that the problem ultimately becomes, for the small distance, that solved in Ex. 2, Art. 323, wherein we may consider η to be small compared with a.

$$\begin{aligned}\therefore \quad 2\xi &= (a^{\frac{2}{3}} + 2\eta^{\frac{2}{3}})\,\{a^{\frac{2}{3}} - \eta^{\frac{2}{3}}\}^{\frac{1}{2}},\\ &= \text{- - - - -}\ a^{\frac{1}{3}}\left\{1 - \frac{\eta^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right\}^{\frac{1}{2}},\\ &= \text{- - - - -}\ a^{\frac{1}{3}}\left\{1 - \frac{\eta^{\frac{2}{3}}}{2a^{\frac{2}{3}}} - \ldots\right\},\\ &= a + \frac{3}{2}\,a^{\frac{1}{3}}\eta^{\frac{2}{3}} + \ldots\ldots;\end{aligned}$$

and neglecting terms involving powers of η higher than those retained, we have

$$2\xi - a = \frac{3}{2}\,a^{\frac{1}{3}}\eta^{\frac{2}{3}};$$

which is the equation to a semi-cubical parabola; the vertex of which is at a distance $\frac{a}{2}$ from the origin. An examination of figs. 117—120 at the point C, renders plain the geometrical form of the problem.

326.] The general form of the equation to a caustic of a circle by reflexion may be most conveniently determined as follows:

In fig. 118 let S be the source of light, and SP the incident type-ray, and PR the reflected type-ray, O being the centre of the circle. Let $OA = a$, $OS = b$, $SPO = RPO = \phi$, $POA = \theta$; then taking O as the origin, and OS as the axis of x, the equation to the reflected ray PR is

$$x \sin(\theta+\phi) + y\cos(\theta+\phi) + a\sin\phi = 0;$$

or $$x(\sin\theta\cot\phi+\cos\theta) + y(\cos\theta\cot\phi-\sin\theta) + a = 0; \tag{85}$$

and from the geometry of the figure

$$b\sin(\theta-\phi) = a\sin\phi,$$

$$\therefore \quad \cot\phi = \frac{a+b\cos\theta}{b\sin\theta}; \tag{86}$$

so that (85) becomes

$$x(a\sin\theta+b\sin 2\theta) + y(a\cos\theta+b\cos 2\theta) + ab\sin\theta = 0. \tag{87}$$

Differentiating with respect to θ, we have

$$x(a\cos\theta+2b\cos 2\theta) - y(a\sin\theta+2b\sin 2\theta) + ab\cos\theta = 0; \tag{88}$$

whence, eliminating from (87) and (88), we have

$$\left.\begin{aligned} x &= \frac{ab(2a+3b\cos\theta-b\cos 3\theta)}{2(a^2+3ab\cos\theta+2b^2)} \\ y &= \frac{ab^2(3\sin\theta-\sin 3\theta)}{2(a^2+3ab\cos\theta+2b^2)} \end{aligned}\right\}; \tag{89}$$

which are the equations to the caustic in terms of a subsidiary angle θ. In two cases they reduce themselves to the equations of an epicycloid.

(1) Let $b = \infty$; so that the source of light is at an infinite distance, and we have a system of parallel rays incident parallel to the axis of x. Then

$$\left.\begin{aligned} x &= \frac{a}{4}(3\cos\theta-\cos 3\theta) \\ y &= \frac{a}{4}(3\sin\theta-\sin 3\theta) \end{aligned}\right\}; \tag{90}$$

which are the equations to an epicycloid; see equations (37), Art. 204, the radii of the fixed and rolling circles being respectively $\frac{a}{2}$ and $\frac{a}{4}$. See fig. 119.

(2) Let $b = a$; in which case the source of light is at the extremity of the diameter of the circle, see fig. 120, and the equations (69) become,

$$\left.\begin{aligned} x &= \frac{a}{3}(2\cos\theta - \cos 2\theta) \\ y &= \frac{a}{3}(2\sin\theta - \sin 2\theta) \end{aligned}\right\}; \quad . \qquad (91)$$

which are the equations to a cardioid, see Art. 205, the radius of the fixed and generating circles being each $\frac{a}{3}$.

327.] Caustics by reflexion from curves expressed in terms of polar coordinates, and which have the origin of light at the pole, may be determined in the following manner; but as the general formulæ are complicated, we will illustrate the method by the particular case of the logarithmic spiral.

In fig. 121 let S be the pole of the spiral and the source of light, SP the incident, PR the reflected ray. Let R be the point in which two successive rays intersect, wherefore R is a point on the caustic; and it is also to be observed that PR is a tangent to the caustic. Let SP $= r$, SY $= p$; SR $= r'$, SZ $= p'$; PSX $= \theta$, RSX $= \theta'$; let the equation to the reflecting curve be $r = a^{\theta}$; and for convenience of writing, let $\log_e a = \text{A}$; therefore by Art. 272, Ex. 3,

$$\tan \text{SPN} = \frac{dr}{r\,d\theta} = \text{A}, \qquad (92)$$

and

$$r = (1+\text{A}^2)^{\frac{1}{2}} p. \qquad (93)$$

From the geometry, $\frac{p'}{r} = \sin \text{SPZ} = \sin 2\,\text{SPN}$,

$$= \sin(2\tan^{-1}\text{A}) = \frac{2\text{A}}{1+\text{A}^2};$$

$$\therefore \quad p' = \frac{2\text{A}r}{1+\text{A}^2}. \qquad (94)$$

Also since $\qquad \text{SRP} + \text{RPS} + \text{PSR} = 180^\circ$,

$$\therefore \quad \sin^{-1}\frac{p'}{r'} + 2\tan^{-1}\text{A} + \theta' - \theta = 180^\circ;$$

differentiating which, p' and θ varying, but r' and θ' being constant,

$$\frac{dp'}{(r'^2 - p'^2)^{\frac{1}{2}}} - d\theta = 0;$$

therefore from (92) and (94),

$$\frac{2\text{A}\,dr}{1+\text{A}^2}\,\frac{1}{(r'^2-p'^2)^{\frac{1}{2}}} = \frac{dr}{\text{A}r} = \frac{2\,dr}{(1+\text{A}^2)p'};$$

$$\therefore\quad r' = (1+\text{A}^2)^{\frac{1}{2}}p';$$

which is the equation to a logarithmic spiral, equal to the original one.

328.] We proceed now to consider some of the more general properties of caustics by refraction. Let (a, b) be the source of light; (x, y) the point on the surface at which the ray is incident; ξ, η the current coordinates of the refracted ray, and therefore of a point on the caustic; $\mu =$ the refractive index, that is, the ratio of the sine of the angle of incidence to that of the angle of refraction. Let $y = f(x)$ be the equation to the section by the paper of the bounding surface of the refractive medium, the surface being perpendicular to the paper; let r and r' be the distances of the point of incidence from the source of light, and from the point of the caustic; then $\frac{dx}{ds}, \frac{dy}{ds}$ are the cosines of the angles between the tangent to the curve and the coordinate axes: $\frac{x-a}{r}, \frac{y-b}{r}$ those of the angles between the incident ray and the coordinate axes; $\frac{\xi-x}{r'}, \frac{\eta-y}{r'}$ those of the angles between the refracted ray and the coordinate axes;

$$\therefore\quad \frac{(x-a)\,dx+(y-b)\,dy}{r\,ds} = \text{the sine of the angle of incidence,}$$

$$\frac{(\xi-x)\,dx+(\eta-y)\,dy}{r'\,ds} = \text{the sine of the angle of refraction;}$$

therefore by law of refraction,

$$\frac{(x-a)\,dx+(y-b)\,dy}{r\,ds} = \mu.\frac{(\xi-x)\,dx+(\eta-y)\,dy}{r'\,ds}; \tag{95}$$

which is the equation to the refracted ray, and of which ξ and η are the current coordinates; the envelope therefore may be found by eliminating x and y between the equation to the refracting curve, the equation (95), and its differential formed by making x and y to vary.

Also since
$$\left.\begin{aligned} r^2 &= (x-a)^2+(y-b)^2 \\ r'^2 &= (\xi-x)^2+(\eta-y)^2 \end{aligned}\right\}; \tag{96}$$

(95) becomes
$$dr+\mu\,dr' = 0; \tag{97}$$

therefore $r+\mu r'$ is a constant, or a maximum, or a minimum; but it cannot be a maximum, for such a value would be inconsistent with the geometrical possibility of the problem: therefore it is in general a minimum, and may sometimes be constant; the former case is that of an ordinary caustic; in the latter the refracted rays converge to a single focus.

329.] Hence also we may prove that all caustics are rectifiable.

Let ξ, η be the current coordinates, and $d\sigma$ the length of the element of the curve of the caustic, so that $d\sigma^2 = d\eta^2 + d\xi^2$; therefore $\frac{d\xi}{d\sigma}, \frac{d\eta}{d\sigma}$ are the cosines of the angles made by its tangent with the coordinate axes;

$$\therefore \quad \frac{d\eta}{d\sigma} = \frac{\eta-y}{r'}, \qquad \frac{d\xi}{d\sigma} = \frac{\xi-x}{r'};$$

therefore from (95), $dr = \mu\,\dfrac{d\xi\,dx+d\eta\,dy}{d\sigma}$; (98)

and differentiating the latter of (96),

$$r'dr' = (\xi-x)(d\xi-dx)+(\eta-y)(d\eta-dy);$$

$$\begin{aligned}\therefore \quad dr' &= \frac{d\xi}{d\sigma}(d\xi-dx)+\frac{d\eta}{d\sigma}(d\eta-dy),\\ &= d\sigma - \frac{d\xi\,dx+d\eta\,dy}{d\sigma},\\ &= d\sigma - \frac{dr}{\mu};\end{aligned}$$

$$\therefore \quad \mu d\sigma = dr+\mu dr',$$

$$\therefore \quad \mu\sigma = r+\mu r'+c. \tag{99}$$

An expression exactly analogous to that of Art. 292, and to which therefore a similar mode of explanation is applicable; and therefore the length of the caustic curve is equal to that of two straight lines increased by a constant which is to be determined by the data of the particular problem; but in all cases, if $\sigma_1\, r_1\, r_1'$, $\sigma_2\, r_2\, r_2'$, represent two sets of corresponding values,

$$\mu(\sigma_2-\sigma_1) = r_2-r_1+\mu(r_2'-r_1').$$

The law of refraction becomes that of reflexion, if $\mu = -1$; and therefore the properties of caustics by refraction proved above are likewise true of caustics by reflexion; attention must however be paid to an ambiguity of sign, of which no notice has been taken in the preceding investigation.

330.] To determine the caustic by refraction of rays refracted at a plane surface; see fig. 122.

Let S be the source of light; SP the incident ray; RPr the refracted ray; $\text{AS} = a$, $\text{AP} = y$; therefore the equation to the refracted ray is

$$\eta - y = -\tan \text{PTA}.\xi,$$

T being the point where PR intersects ASC.

And since $\sin \text{SPN} = \mu \sin r\text{P}n$, $\therefore \sin \text{PSA} = \mu \sin \text{PTA}$,

$$\frac{y}{(a^2+y^2)^{\frac{1}{2}}} = \mu \sin \text{PTA};$$

$$\therefore \tan \text{PTA} = \frac{y}{\{\mu^2 a^2 + (\mu^2-1) y^2\}^{\frac{1}{2}}}$$

therefore the equation to PR is

$$\eta - y = \frac{-y}{\{\mu^2 a^2 + (\mu^2-1) y^2\}^{\frac{1}{2}}}\xi. \tag{100}$$

Differentiating which with respect to y, and reducing

$$\mu^2 a^2 + (\mu^2-1) y^2 = \xi^{\frac{2}{3}} \mu^{\frac{4}{3}} a^{\frac{4}{3}},$$

$$y = \pm \frac{\mu^{\frac{2}{3}} a^{\frac{2}{3}}}{(\mu^2-1)^{\frac{1}{2}}} \{\xi^{\frac{2}{3}} - \mu^{\frac{2}{3}} a^{\frac{2}{3}}\}^{\frac{1}{2}}; \tag{101}$$

whence by elimination, and from (100),

$$\xi^{\frac{2}{3}} - (\mu^2-1)^{\frac{1}{3}} \eta^{\frac{2}{3}} = \mu^{\frac{2}{3}} a^{\frac{2}{3}}. \tag{102}$$

which is the equation to the evolute of a hyperbola or of an ellipse according as μ is greater or less than unity.

CHAPTER XIV.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO PROPERTIES OF CURVED SURFACES.

331.] An explanation of the mode of generation and of the equations of such curved surfaces and curves in space as are needed for illustration in this and the following Chapters, requires more room than we can afford to give; but it is the less necessary to introduce it, as the ordinary text-books contain sufficient information. It is however desirable to explain the equations to the straight line and the plane, in the forms which we shall employ, as a familiar knowledge of them is requisite to a due understanding of our processes.

(1) To find the equations to a straight line in space.

Let ξ, η, ζ be the current coordinates to the straight line; x, y, z the coordinates to a point through which the line passes; λ, μ, ν the direction-angles of the line; that is, the angles between a parallel line through the origin and the coordinate axes. And let r be the distance between $(x, y\ z)$ and (ξ, η, ζ); then the equations to the line are

$$\frac{\xi-x}{\cos\lambda} = \frac{\eta-y}{\cos\mu} = \frac{\zeta-z}{\cos\nu} = r, \tag{1}$$

the last of the equalities following by reason of Preliminary Theorem I.

If therefore the equations to a straight line are given under the form

$$\frac{\xi-x}{\text{L}} = \frac{\eta-y}{\text{M}} = \frac{\zeta-z}{\text{N}}, \tag{2}$$

each of these equalities is by reason of the same Preliminary Theorem equal to

$$\frac{r}{(\text{L}^2+\text{M}^2+\text{N}^2)^{\frac{1}{2}}}; \tag{3}$$

and therefore comparing (1) with (2) and (3),

$$\left.\begin{aligned}\cos\lambda &= \frac{\xi - x}{r} = \frac{\mathrm{L}}{(\mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2)^{\frac{1}{2}}}\\ \cos\mu &= \frac{\eta - y}{r} = \frac{\mathrm{M}}{(\mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2)^{\frac{1}{2}}}\\ \cos\nu &= \frac{\zeta - z}{r} = \frac{\mathrm{N}}{(\mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2)^{\frac{1}{2}}}\end{aligned}\right\}; \qquad (4)$$

whence $\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1$

and therefore L, M, N in (2) are proportional to the direction-cosines of the line, that is, to the cosines of the angles between the line and the coordinate axes.

(2) To find the equation to a plane.

A plane is a surface generated by a straight line revolving round another straight line which is at right angles to it.

Let the origin be at the point O in the straight line OQ, fig. 123, round which the perpendicular and generating line QP turns; and let λ, μ, ν be the direction-angles of OQ; let ξ, η, ζ be the current coordinates to any point P in the line QP which is in any position; and let $\mathrm{OP} = \rho$, $\mathrm{OQ} = \delta$; then the direction-cosines of OP are $\frac{\xi}{\rho}, \frac{\eta}{\rho}, \frac{\zeta}{\rho}$; and since OQP is a right angle, OQ = OP cos POQ, that is,

$$\delta = \rho\left\{\frac{\xi}{\rho}\cos\lambda + \frac{\eta}{\rho}\cos\mu + \frac{\zeta}{\rho}\cos\nu\right\};$$

$$\therefore \quad \xi\cos\lambda + \eta\cos\mu + \zeta\cos\nu = \delta; \qquad (5)$$

and as this relation is true for every point in QP, and in every position of QP, it is according to our definition, the equation to a plane; λ, μ, ν being the direction-angles of the normal to the plane, and δ the length of the perpendicular from the origin on the plane.

Equation (5) is evident by the theory of projections, the left-hand side of the equation being the sum of the projections of the broken line OMNPQ on the line OQ.

If therefore the equation to a plane is given in the form

$$\mathrm{A}\xi + \mathrm{B}\eta + \mathrm{C}\zeta = \mathrm{D}; \qquad (6)$$

on comparing this with (5), we have,

$$\frac{\mathrm{A}}{\cos\lambda} = \frac{\mathrm{B}}{\cos\mu} = \frac{\mathrm{C}}{\cos\nu} = \frac{\mathrm{D}}{\delta} = (\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2)^{\frac{1}{2}}: \qquad (7)$$

whence it appears that A, B, C, D are proportional respectively to the direction-cosines of the normal to the plane, and to the length of the perpendicular on the plane from the origin.

332] To find the equation to a tangent plane to a curved surface at a given point.

Let the equation to the surface be

$$\text{F}(x, y, z) = 0. \tag{8}$$

Our present object is to shew that if a straight line is drawn through a point on the surface (x, y, z), and through a second point $(x+dx,\ y+dy,\ z+dz)$ infinitesimally near to it and also on the surface, the locus of such tangent lines is in general a plane; and is what is called the tangent plane. Of course it is manifest that the number of points $(x+dx,\ y+dy,\ z+dz)$ contiguous to the first point is infinite, and so therefore is the number of tangent lines.

Let ξ, η, ζ be the current coordinates to one of the tangent lines, and x, y, z the coordinates to the point of contact on the surface; then the equations to the line are

$$\frac{\xi - x}{\text{L}} = \frac{\eta - y}{\text{M}} = \frac{\zeta - z}{\text{N}}; \tag{9}$$

as the line passes through the point $(x+dx,\ y+dy,\ z+dz)$, we have

$$\frac{dx}{\text{L}} = \frac{dy}{\text{M}} = \frac{dz}{\text{N}}; \tag{10}$$

and therefore by division,

$$\frac{\xi - x}{dx} = \frac{\eta - y}{dy} = \frac{\zeta - z}{dz} = \frac{r}{ds}, \tag{11}$$

r being the distance of (x, y, z) from any point (ξ, η, ζ) on the line, and

$$ds = \{dx^2 + dy^2 + dz^2\}^{\frac{1}{2}}; \tag{12}$$

that is, ds is the distance between the two points on the surface through which the line passes; (11) therefore are the equations to any straight line touching a surface at a given point.

But as the second point through which the line passes is on the surface, though it may have any position infinitesimally near to (x, y, z); so dx, dy, dz, must be consistent with the equation to the surface. If therefore at the point under consideration $\left(\frac{d\text{F}}{dx}\right)$, $\left(\frac{d\text{F}}{dy}\right)$, $\left(\frac{d\text{F}}{dz}\right)$ do not all vanish, then

$$\left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy + \left(\frac{d\text{F}}{dz}\right) dz = 0; \tag{13}$$

multiplying the several terms of which by the terms of the equalities (11), we have

$$(\xi-x)\left(\frac{dF}{dx}\right)+(\eta-y)\left(\frac{dF}{dy}\right)+(\zeta-z)\left(\frac{dF}{dz}\right)=0. \tag{14}$$

Now x, y, z, being the coordinates to the point of contact, are constant for a given point, and so are $\left(\frac{dF}{dx}\right)$, $\left(\frac{dF}{dy}\right)$, $\left(\frac{dF}{dz}\right)$ which are functions of x, y, z; and ξ, η, ζ being the current coordinates of the locus, it follows that (14) is of the same form as (6), and therefore represents a plane; and being the locus of the tangent lines to the surface represents the tangent plane.

Let us once for all make certain substitutions which for the purpose of abridging the notation will be convenient both in the sequel of the present volume, and in future parts of our Treatise. Let

$$\left(\frac{dF}{dx}\right)=U,\quad \left(\frac{dF}{dy}\right)=V,\quad \left(\frac{dF}{dz}\right)=W, \tag{15}$$

$$U^2+V^2+W^2=Q^2; \tag{16}$$

so that

$$U\,dx+V\,dy+W\,dz=0. \tag{17}$$

and thus (14) becomes

$$U(\xi-x)+V(\eta-y)+W(\zeta-z)=0. \tag{18}$$

333.] On comparing (14) with (6), and with equations (7), if α, β, γ are the direction-angles of the normal to the plane, and if p is the perpendicular on the plane from the origin, we have

$$\left.\begin{aligned}
\cos\alpha &= \frac{\left(\frac{dF}{dx}\right)}{\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}=\frac{U}{Q},\\
\cos\beta &= \frac{\left(\frac{dF}{dy}\right)}{\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}=\frac{V}{Q},\\
\cos\gamma &= \frac{\left(\frac{dF}{dz}\right)}{\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}=\frac{W}{Q};
\end{aligned}\right\} \tag{19}$$

$$p = \frac{x\left(\frac{dF}{dx}\right)+y\left(\frac{dF}{dy}\right)+z\left(\frac{dF}{dz}\right)}{\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}; \tag{20}$$

and $\left(\frac{dF}{dx}\right)$, $\left(\frac{dF}{dy}\right)$, $\left(\frac{dF}{dz}\right)$ are proportional to the direction-cosines of the normal to the tangent plane.

334.] If the equation to the surface is given in the explicit form $z = f(x,y)$; then $F(x,y,z) = f(x,y) - z = 0$;

$$\therefore\ \left(\frac{dF}{dx}\right) = \left(\frac{df}{dx}\right) = \left(\frac{dz}{dx}\right),\quad \left(\frac{dF}{dy}\right) = \left(\frac{df}{dy}\right) = \left(\frac{dz}{dy}\right),\quad \left(\frac{dF}{dz}\right) = -1;$$

in which case the equation to the tangent plane becomes

$$\zeta - z = (\xi - x)\left(\frac{dz}{dx}\right) + (\eta - y)\left(\frac{dz}{dy}\right), \tag{21}$$

and (19) and 20) must be modified accordingly.

335.] If the equation to the surface is a homogeneous function of n dimensions of the form,

$$F(x,y,z) = c;$$

then, since by the property of such functions proved in Art. 82,

$$x\left(\frac{dF}{dx}\right)+y\left(\frac{dF}{dy}\right)+z\left(\frac{dF}{dz}\right) = nc,$$

the equation to the tangent plane becomes

$$\xi\left(\frac{dF}{dx}\right)+\eta\left(\frac{dF}{dy}\right)+\zeta\left(\frac{dF}{dz}\right) = nc,$$

and (20) becomes

$$p = \frac{nc}{\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}} = \frac{nc}{Q}. \tag{22}$$

Also if the surface is expressed by an algebraical equation of the form,

$$F(x,y,z) = u_n + u_{n-1} + \dots\dots + u_1 + u_0 = 0, \tag{23}$$

where $u_n, u_{n-1}, \dots u_1, u_0$ are homogeneous functions of n, $n-1$, ... 1, 0 dimensions; then by a process exactly similar to that of Art. 222, except that in this case there are three variables, it may be shewn that the equation to the tangent plane is

$$\xi\left(\frac{d\text{F}}{dx}\right)+\eta\left(\frac{d\text{F}}{dy}\right)+\zeta\left(\frac{d\text{F}}{dz}\right)$$
$$= -\{u_{n-1}+2u_{n-2}+\ldots\ldots+(n-1)u_1+nu_0\}, \quad (24)$$

and is therefore an equation of only $(n-1)$ dimensions in terms of x, y, and z.

In the equation to the tangent plane, considering ξ, η, ζ to be constant, and the coordinates of a given point through which a series of tangent planes is drawn, x, y, z refer to the points of contact on the surface; hence we have the following theorems:

If through a given point planes are drawn, touching a given surface of the nth order, the points of contact lie on a surface of the $(n-1)$th order; and therefore

If through a given point planes are drawn touching a surface of the second order, all the points of contact lie in one plane.

In harmony with the nomenclature of Art. 226, the point (ξ, η, ζ) whence the tangent planes are drawn is called the pole: the surface whose equation is (24) is called the first polar surface, and the surface $\text{F}(x, y, z) = c$ is called the base-surface. Also in the same way that the first polar surface is derived from the base-surface and is of the $(n-1)$th order, so may other and successive polar surfaces with reference to the same pole be derived, and these will be of the $(n-2)$th, $(n-3)$th, ... order. Want of space however precludes me from entering on these subjects, although they are replete with interest.

336.] To find the equations to a normal of a curved surface.

A normal is a straight line drawn through any point of a curved surface, and at right angles to the tangent plane at that point.

Let ξ, η, ζ be the current coordinates of the normal, and x, y, z the coordinates to the point where it meets the surface; then, by Art. 333, the direction-cosines of the normal being proportional to $\left(\frac{d\text{F}}{dx}\right)$, $\left(\frac{d\text{F}}{dy}\right)$, $\left(\frac{d\text{F}}{dz}\right)$, its equations are

$$\frac{\xi-x}{\left(\frac{d\text{F}}{dx}\right)}=\frac{\eta-y}{\left(\frac{d\text{F}}{dy}\right)}=\frac{\zeta-z}{\left(\frac{d\text{F}}{dz}\right)}. \quad (25)$$

Also the form of the equations to the normal shews that it is the longest or the shortest line which can be drawn from a point on it to the surface.

If the equation to the surface is given in the explicit form, these equations, by means of Art. 334, become

$$\left.\begin{aligned}\xi-x &= -\left(\frac{dz}{dx}\right)(\zeta-z)\\ \eta-y &= -\left(\frac{dz}{dy}\right)(\zeta-z)\end{aligned}\right\}. \qquad (26)$$

In these equations, if ξ, η, ζ are constant, x, y, z refer to the points on a surface where normals drawn through a given point meet it, and the equations (25) or (26) are those to a curve in space which is the locus of such points of contact.

337.] From (25) it follows, that the equations to a line passing through the origin, and at right angles to the tangent plane, are

$$\frac{\xi}{\left(\frac{dF}{dx}\right)} = \frac{\eta}{\left(\frac{dF}{dy}\right)} = \frac{\zeta}{\left(\frac{dF}{dz}\right)}. \qquad (27)$$

By means of which equations, combined with those to the tangent plane and to the surface, we may determine the equation to the surface, which is the locus of the point of intersection of a tangent plane, with the perpendicular drawn to it from the origin.

338.] Examples illustrative of the preceding Articles.

Ex. 1. The ellipsoid whose equation is

$$F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \qquad (28)$$

$$\left(\frac{dF}{dx}\right) = \frac{2x}{a^2}, \qquad \left(\frac{dF}{dy}\right) = \frac{2y}{b^2}, \qquad \left(\frac{dF}{dz}\right) = \frac{2z}{c^2};$$

therefore by equation (14) the equation to the tangent plane is

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 1, \qquad (29)$$

which is plainly the equation, since the equation to the surface is a homogeneous function of two dimensions.

Also if (ξ, η, ζ) is the pole, the equation to the first polar is

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 1; \qquad (30)$$

which is the equation to a plane.

Hence also we have by (22),

$$\frac{1}{p} = \left\{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right\}^{\frac{1}{2}}. \tag{31}$$

And the equations to the normal are

$$\frac{a^2}{x}(\xi - x) = \frac{b^2}{y}(\eta - y) = \frac{c^2}{z}(\zeta - z). \tag{32}$$

The equations therefore to a line through the origin, and perpendicular to the tangent plane, are

$$\frac{a^2\xi}{x} = \frac{b^2\eta}{y} = \frac{c^2\zeta}{z}. \tag{33}$$

Whence may be found the equation to the surface, which is the locus of the point of intersection of these lines with the tangent planes.

For ξ, η, ζ being the same in (29) and (33), we have

$$\frac{a^2\xi}{x} = \frac{b^2\eta}{y} = \frac{c^2\zeta}{z} = \frac{a\xi}{\frac{x}{a}} = \frac{b\eta}{\frac{y}{b}} = \frac{c\zeta}{\frac{z}{c}} = \{a^2\xi^2 + b^2\eta^2 + c^2\zeta^2\}^{\frac{1}{2}}$$

$$= \frac{\xi^2}{\frac{\xi x}{a^2}} = \frac{\eta^2}{\frac{\eta y}{b^2}} = \frac{\zeta^2}{\frac{\zeta z}{c^2}} = \xi^2 + \eta^2 + \zeta^2;$$

$$\therefore \quad \{\xi^2 + \eta^2 + \zeta^2\}^2 = a^2\xi^2 + b^2\eta^2 + c^2\zeta^2; \tag{34}$$

which is the equation to the surface required.

Ex. 2. The elliptic paraboloid whose equation is

$$x - \frac{y^2}{4a} - \frac{z^2}{4b} = 0.$$

$$\left(\frac{dF}{dx}\right) = 1, \qquad \left(\frac{dF}{dy}\right) = -\frac{y}{2a}, \qquad \left(\frac{dF}{dz}\right) = -\frac{z}{2b};$$

therefore the equation to the tangent plane is

$$\xi + x - \frac{\eta y}{2a} - \frac{\zeta z}{2b} = 0; \tag{35}$$

and the equations to the line through the origin, and perpendicular to the tangent plane, are

$$\xi = -\frac{2a\eta}{y} = -\frac{2b\zeta}{z}. \tag{36}$$

The equation therefore to the locus of the point of intersection of (36) with (35) is

$$a\eta^2 + b\zeta^2 + \xi(\xi^2 + \eta^2 + \zeta^2) = 0. \tag{37}$$

Ex. 3. If the equation to the surface is $xyz = k^3$,

$$\left(\frac{dF}{dx}\right) = yz = \frac{k^3}{x}, \qquad \left(\frac{dF}{dy}\right) = \frac{k^3}{y}, \qquad \left(\frac{dF}{dz}\right) = \frac{k^3}{z}; \tag{38}$$

and therefore the equation to the tangent plane is

$$(\xi - x)\frac{k^3}{x} + (\eta - y)\frac{k^3}{y} + (\zeta - z)\frac{k^3}{z} = 0,$$

$$\frac{\xi}{x} + \frac{\eta}{y} + \frac{\zeta}{z} = 3. \tag{39}$$

And the intercepts of the coordinate axes by the tangent plane are, according to the notation of Art. 219,

$$\xi_0 = 3x, \qquad \eta_0 = 3y, \qquad \zeta_0 = 3z;$$

$$\begin{aligned} \therefore \quad \xi_0\eta_0\zeta_0 &= 27xyz, \\ &= 27k^3, \end{aligned}$$

that is, the volume of the pyramid contained between the tangent plane and the coordinate planes is constant.

The equations to the line through the origin, and perpendicular to the tangent plane, are

$$\xi x = \eta y = \zeta z = k(\xi\eta\zeta)^{\frac{1}{3}}; \tag{40}$$

therefore the equation to the locus of the point of intersection of (40) with (39) is

$$\xi^2 + \eta^2 + \zeta^2 = 3k(\xi\eta\zeta)^{\frac{1}{3}}. \tag{41}$$

339.] If at the point on the surface at which the tangent lines of equation (11) are drawn, $\left(\frac{dF}{dx}\right)$, $\left(\frac{dF}{dy}\right)$, and $\left(\frac{dF}{dz}\right)$ all vanish, equation (13) is satisfied independently of any relation between dx, dy and dz, and therefore does not give an equation whereby to eliminate them; in fact the direction-cosines of the normal at the point are indeterminate, and the tangent plane has no definite position. At such a point there will be a locus of tangent planes, to determine which we must seek for some other relation between dx, dy and dz, arising out of the equation to the surface. Such we have, if all the differential-coefficients of the second order do not vanish at the point in question, in the differential of (13), and which is also the third term of the expansion of F $(x+dx, y+dy, z+dz)$ in Art. 142, viz.

$$\left(\frac{d^2F}{dx^2}\right) dx^2 + \left(\frac{d^2F}{dy^2}\right) dy^2 + \left(\frac{d^2F}{dz^2}\right) dz^2 + 2\left(\frac{d^2F}{dy\,dz}\right) dy\,dz$$
$$+ 2\left(\frac{d^2F}{dz\,dx}\right) dz\,dx + 2\left(\frac{d^2F}{dx\,dy}\right) dx\,dy = 0;$$

and multiplying through by corresponding terms of equality (11), we have

$$\left(\frac{d^2F}{dx^2}\right)(\xi-x)^2 + \left(\frac{d^2F}{dy^2}\right)(\eta-y)^2 + \left(\frac{d^2F}{dz^2}\right)(\zeta-z)^2$$
$$+ 2\left(\frac{d^2F}{dy\,dz}\right)(\eta-y)(\zeta-z) + 2\left(\frac{d^2F}{dz\,dx}\right)(\zeta-z)(\xi-x)$$
$$+ 2\left(\frac{d^2F}{dx\,dy}\right)(\xi-x)(\eta-y) = 0; \quad (42)$$

an equation of a cone of the second degree, shewing therefore that the locus of the tangent lines is not a plane, but a cone of the second order.

Changing the origin to the point under consideration, the equation assumes the form

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2D\eta\zeta + 2E\zeta\xi + 2F\xi\eta = 0, \quad (43)$$

and the vertex of the cone is at the point of contact; and it may happen that the coefficients have such relations that the equation is decomposable into two factors of the first degree, in which case it will represent two planes.

Ex. 1. Determine the nature of the point at the origin of the surface whose equation is, $ay^2 + bz^2 + x(x^2+y^2+z^2) = 0$.

$$\left.\begin{aligned}\left(\frac{dF}{dx}\right) &= 3x^2+y^2+z^2 = 0,\\ \left(\frac{dF}{dy}\right) &= 2ay+2xy \quad = 0,\\ \left(\frac{dF}{dz}\right) &= 2bz+2xz \quad = 0,\end{aligned}\right\} \text{ at the origin;}$$

$$\left(\frac{d^2F}{dx^2}\right) = 6x = 0, \quad \left(\frac{d^2F}{dy^2}\right) = 2a+2x = 2a, \quad \left(\frac{d^2F}{dz^2}\right) = 2b+2x = 2b,$$

$$\left(\frac{d^2F}{dy\,dz}\right) = 0, \quad \left(\frac{d^2F}{dz\,dx}\right) = 2z = 0, \quad \left(\frac{d^2F}{dx\,dy}\right) = 2y = 0;$$

therefore equation (43) becomes

$$ay^2 + bz^2 = 0, \tag{44}$$

which is satisfied only by $y = 0$, $z = 0$; therefore (44) represents the axis of x, or the surface at the origin degenerates into a cuspal point formed round the axis of x.

Ex. 2. A surface is formed by the revolution of a parabola about an ordinate through its focus; it is required to find the nature of the points where it meets the axis of z.

The equation to the surface is

$$16m^2(x^2+y^2)-(z^2-4m^2)^2 = 0;$$

whence it appears that $x = y = 0$, when $z = \pm 2m$; and at such a point $\left(\frac{dF}{dx}\right) = \left(\frac{dF}{dy}\right) = \left(\frac{dF}{dz}\right) = 0$; and differentiating again, and substituting in equation (42), we shall find

$$\xi^2+\eta^2-(\zeta \pm 2m)^2 = 0;$$

the equation to two right-angled circular cones, whose axis is the axis of z and vertices at distances $\pm 2m$ from the origin.

If all the second differential-coefficients vanish at the point where the tangent plane is to be drawn, we must proceed to a third differentiation, or to the fourth term of the expansion in Art. 142; and thus we shall arrive at a cone of the third order.

CHAPTER XV.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO PROPERTIES OF CURVES IN SPACE.

340.] THE curves whose properties have been inquired into lie wholly in one plane; that is, all their elements and all their consecutive points have been entirely in the plane of xy; and we have considered them in reference to two fixed lines in that plane. It is manifest however that all curves are not subject to the restriction of having their elements in the same plane; there may be non-plane as well as plane curves, and as such they exist in space, and are conveniently referred to three coordinate axes meeting each other at right-angles and in one point; such are also called curves of double curvature, and for a reason which will be hereafter assigned. They may be determined in two ways: either by the intersection of two surfaces whose equations involving x, y, z are given, and therefore by the combination of these two equations; or, what amounts to the same thing, one of the variables, as e. g. z, may have been eliminated between these two equations, and an equation obtained involving only x and y, which will be the equation to the projection of the curve on the plane of xy; and so with the other variables; whereby three equations may be formed, each containing two variables, which will severally represent the projections of the curve on the coordinate planes, and any two of which equations will be sufficient to define the curve; and according as one or the other method is adopted the formulæ will assume different, though equivalent, shapes.

341.] To find the equations to a tangent line to a curve in space.

A tangent line is the straight line passing through two points on the curve which are infinitesimally near to each other.

Let ξ, η, ζ be the current coordinates to the tangent line, and first let the two points through which the line is to pass be at a finite distance Δs apart; and let them be (x, y, z), $(x + \Delta x,\ y + \Delta y,\ z + \Delta z)$; then the equations to the line are

$$\frac{\xi - x}{\Delta x} = \frac{\eta - y}{\Delta y} = \frac{\zeta - z}{\Delta z} = \frac{r}{\Delta s}, \tag{1}$$

where r is the distance between the two points (x, y, z) and (ξ, η, ζ). When these two points become infinitesimally near to one another, the line becomes a tangent, and its equations become

$$\frac{\xi - x}{dx} = \frac{\eta - y}{dy} = \frac{\zeta - z}{dz} = \frac{r}{ds}; \tag{2}$$

where $$ds = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}},$$

and is the differential of the arc, or the length-element of the curve.

On comparing these equations with those of (4) in Art. 331, if λ, μ, ν are the direction-angles of the tangent,

$$\cos\lambda = \frac{dx}{ds}, \quad \cos\mu = \frac{dy}{ds}, \quad \cos\nu = \frac{dz}{ds}. \tag{3}$$

If the equations to the curve are two equations, say of the forms

$$f(x, z) = 0, \qquad \phi(y, z) = 0,$$

$\frac{dx}{dz}$ and $\frac{dy}{dz}$ can be found by differentiation, and equations (2) and (3) can be determined for the particular curve.

If the curve is determined by means of the equations to two surfaces of the forms,

$$F_1(x, y, z) = 0, \qquad F_2(x, y, z) = 0; \tag{4}$$

then, since

$$\left.\begin{aligned} \left(\frac{dF_1}{dx}\right) dx + \left(\frac{dF_1}{dy}\right) dy + \left(\frac{dF_1}{dz}\right) dz &= 0, \\ \left(\frac{dF_2}{dx}\right) dx + \left(\frac{dF_2}{dy}\right) dy + \left(\frac{dF_2}{dz}\right) dz &= 0 \end{aligned}\right\}; \tag{5}$$

we have by elimination the following system of equations,

$$\frac{dx}{\left(\frac{dF_1}{dy}\right)\left(\frac{dF_2}{dz}\right) - \left(\frac{dF_1}{dz}\right)\left(\frac{dF_2}{dy}\right)} = \frac{dy}{\left(\frac{dF_1}{dz}\right)\left(\frac{dF_2}{dx}\right) - \left(\frac{dF_1}{dx}\right)\left(\frac{dF_2}{dz}\right)}$$

$$= \frac{dz}{\left(\frac{dF_1}{dx}\right)\left(\frac{dF_2}{dy}\right) - \left(\frac{dF_1}{dy}\right)\left(\frac{dF_2}{dx}\right)}; \tag{6}$$

whence, multiplying the several terms of equality (2) by the several terms of this equality, dx, dy, dz will divide out, and we shall have the equations to the tangent in terms of the partial differential-coefficients of the intersecting surfaces. Similarly may the direction cosines in (3) be determined.

342.] To find the equation to the normal plane to a curve in space.

The plane perpendicular to the tangent line, and passing through the point of contact, is called the normal plane. Let ξ, η, ζ be its current coordinates, and (x, y, z) be the point of contact through which it passes; then, since it is to be perpendicular to the line whose direction cosines are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, its equation is

$$(\xi-x)\,dx+(\eta-y)\,dy+(\zeta-z)\,dz = 0. \tag{7}$$

343.] To find the equation to the osculating plane to a curve in space.

In curves such as we have discussed in previous Chapters, all the points lie in one plane; and therefore the curves are called plane curves. This property however does not hold good for all curves in space; although every three consecutive points must be in one plane, yet the fourth may be out of it; or in other words, every two consecutive tangents are in the same plane, but the next consecutive tangent is in general in a different one; our object is to determine the equation to the plane which contains two consecutive tangents, and which is called the osculating plane, and is defined as follows:

The osculating plane is the plane containing three consecutive points on a curve.

Let the equation to the plane be

$$\text{A}\xi+\text{B}\eta+\text{C}\zeta = \text{D}, \tag{8}$$

and let it pass through the three points on the curve (x, y, z), $(x+dx, y+dy, z+dz)$, $(x+2dx+d^2x, y+2dy+d^2y, z+2dz+d^2z)$; whence we have

$$\text{A}x+\text{B}y+\text{C}z = \text{D}, \tag{9}$$

$$\text{A}\,dx+\text{B}\,dy+\text{C}\,dz = 0, \tag{10}$$

$$\text{A}\,d^2x+\text{B}\,d^2y+\text{C}\,d^2z = 0; \tag{11}$$

whence, subtracting (9) from (8),

$$\text{A}(\xi-x)+\text{B}(\eta-y)+\text{C}(\zeta-z) = 0; \tag{12}$$

and from (10) and (11) we have,

$$\frac{A}{dy\,d^2z - dz\,d^2y} = \frac{B}{dz\,d^2x - dx\,d^2z} = \frac{C}{dx\,d^2y - dy\,d^2x}; \qquad (13)$$

whence, dividing (12) by the several terms of equality (13), we have

$$(dy\,d^2z - dz\,d^2y)(\xi - x) + (dz\,d^2x - dx\,d^2z)(\eta - y)$$
$$+ (dx\,d^2y - dy\,d^2x)(\zeta - z) = 0; \qquad (14)$$

which is the equation to the osculating plane.

344.] The method by which we have deduced this equation is the same as if we had defined the osculating plane to be that in which two consecutive tangents lie, as will be apparent from what follows.

Let the equation to the plane passing through (x, y, z) be

$$A(\xi - x) + B(\eta - y) + C(\zeta - z) = 0;$$

and since it is to be that in which two consecutive tangents lie, whose direction cosines are respectively

$$\frac{dx}{ds}, \quad \frac{dy}{ds}, \quad \frac{dz}{ds}, \quad \frac{dx + d^2x}{ds + d^2s}, \quad \frac{dy + d^2y}{ds + d^2s}, \quad \frac{dz + d^2z}{ds + d^2s};$$

we have the conditions

$$A\,dx + B\,dy + C\,dz = 0,$$
$$A(dx + d^2x) + B(dy + d^2y) + C(dz + d^2z) = 0;$$

whence, by subtraction,

$$A\,d^2x + B\,d^2y + C\,d^2z = 0;$$

which two relations between A, B, C are the same as those above marked (10) and (11), whence equality (13) follows, and therefore the equation to the osculating plane is the same.

345] It is manifest from (7) that all straight lines passing through a point of contact, and perpendicular to the tangent line, lie in the normal plane; two of these normal lines have peculiar properties in relation to the osculating plane, viz. that which is perpendicular to it, and that which lies in it, and is therefore the line of intersection of it by the normal plane. The latter is called the *principal normal*, and the former has the distinctive name of *binormal*, being, as it is, perpendicular to two consecutive elements of the curve, while all other normals are perpendicular to only one.

To find the equations to the binormal.

Let l, m, n be its direction-angles; then, as it is perpendicular to the osculating plane,

$$\frac{\cos l}{dy\,d^2z - dz\,d^2y} = \frac{\cos m}{dz\,d^2x - dx\,d^2z} = \frac{\cos n}{dx\,d^2y - dy\,d^2x}$$

$$= \frac{1}{\{(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2\}^{\frac{1}{2}}}; \quad (15)$$

The denominator of which last expression may be modified as follows:

$$(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2$$
$$= (dx^2 + dy^2 + dz^2)\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2\}$$
$$- (dx\,d^2x + dy\,d^2y + dz\,d^2z)^2; \quad (16)$$

but since $$ds^2 = dx^2 + dy^2 + dz^2, \quad (17)$$

$$\therefore \quad ds\,d^2s = dx\,d^2x + dy\,d^2y + dz\,d^2z; \quad (18)$$

and therefore the right-hand member of (16) becomes

$$ds^2\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}; \quad (19)$$

and if s is equicrescent,

$$ds^2\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2\}; \quad (20)$$

whence the equations of the binormal are

$$\frac{\xi - x}{dy\,d^2z - dz\,d^2y} = \frac{\eta - y}{dz\,d^2x - dx\,d^2z} = \frac{\zeta - z}{dx\,d^2y - dy\,d^2x}. \quad (21)$$

346.] To find the equations to the principal normal.

Let its equations be

$$\frac{\xi - x}{\text{L}} = \frac{\eta - y}{\text{M}} = \frac{\zeta - z}{\text{N}}; \quad (22)$$

then, by reason of its being perpendicular to the tangent line, and of its lying in the osculating plane, we have

$$\text{L}\,dx + \text{M}\,dy + \text{N}\,dz = 0, \quad (23)$$

$$\text{L}(dyd^2z - dzd^2y) + \text{M}(dzd^2x - dxd^2z) + \text{N}(dxd^2y - dyd^2x) = 0; \quad (24)$$

whence we have

$$\frac{\text{L}}{dy\,(dx\,d^2y - dy\,d^2x) - dz\,(dz\,d^2x - dx\,d^2z)} = \ldots\ldots \quad (25)$$

or, $$\frac{\text{L}}{dx\,(dx\,d^2x + dyd^2y + dz\,d^2z) - d^2x\,(dx^2 + dy^2 + dz^2)} = \ldots\ldots;$$

and since $$ds^2 = dx^2 + dy^2 + dz^2, \quad (26)$$

$$ds\,d^2s = dx\,d^2x + dy\,d^2y + dz\,d^2z, \quad (27)$$

we have
$$\frac{\text{L}}{dx\,ds\,d^2s - ds^2\,d^2x} = \dots\dots$$

or,
$$\frac{\text{L}}{d\left(\frac{dx}{ds}\right)} = \frac{\text{M}}{d\left(\frac{dy}{ds}\right)} = \frac{\text{N}}{d\left(\frac{dz}{ds}\right)}; \qquad (28)$$

and therefore the equations to the principal normal are
$$\frac{\xi - x}{d\left(\frac{dx}{ds}\right)} = \frac{\eta - y}{d\left(\frac{dy}{ds}\right)} = \frac{\zeta - z}{d\left(\frac{dz}{ds}\right)}; \qquad (29)$$

and if s is equicrescent,
$$\frac{\xi - x}{d^2x} = \frac{\eta - y}{d^2y} = \frac{\zeta - z}{d^2z}. \qquad (30)$$

Therefore, if λ, μ, ν are the direction-angles of the principal normal, we have
$$\frac{\cos\lambda}{d\left(\frac{dx}{ds}\right)} = \frac{\cos\mu}{d\left(\frac{dy}{ds}\right)} = \frac{\cos\nu}{d\left(\frac{dz}{ds}\right)} = \frac{ds}{\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}^{\frac{1}{2}}}, \quad (31)$$
as will be found on reduction.

347.] Examples on the preceding.

Ex. 1. The curve formed by the intersection of an ellipsoid by a central plane.
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \text{A}x + \text{B}y + \text{C}z = 0;$$
$$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz = 0, \qquad \text{A}\,dx + \text{B}\,dy + \text{C}\,dz = 0;$$
$$\therefore \quad \frac{dx}{\text{C}\frac{y}{b^2} - \text{B}\frac{z}{c^2}} = \frac{dy}{\text{A}\frac{z}{c^2} - \text{C}\frac{x}{a^2}} = \frac{dz}{\text{B}\frac{x}{a^2} - \text{A}\frac{y}{b^2}};$$
therefore the equations to the tangent line are
$$\frac{\xi - x}{\text{C}\frac{y}{b^2} - \text{B}\frac{z}{c^2}} = \frac{\eta - y}{\text{A}\frac{z}{c^2} - \text{C}\frac{x}{a^2}} = \frac{\zeta - z}{\text{B}\frac{x}{a^2} - \text{A}\frac{y}{b^2}};$$
and the equation to the normal plane is
$$(\xi - x)\left(\text{C}\frac{y}{b^2} - \text{B}\frac{z}{c^2}\right) + (\eta - y)\left(\text{A}\frac{z}{c^2} - \text{C}\frac{x}{a^2}\right) + (\zeta - z)\left(\text{B}\frac{x}{a^2} - \text{A}\frac{y}{b^2}\right) = 0.$$

Ex. 2. The helix; see fig. 125.

Let $\text{OA} = \text{OB} = a$ be the radius of the base-cylinder of the helix, and $\phi = \text{AON}$ be the angle between the plane of xz and the radius of the cylinder drawn to the point (x, y, z), and whose projection on the plane of xy is ON; and let $\text{OM} = x$, $\text{MN} = y$, $\text{NP} = z$; and let k be the tangent of the angle at which the thread of the helix is inclined to the plane of xy; so that $\text{NP} = k \times$ the arc AN; whereby the equations to the curve are

$$x = a\cos\phi, \qquad y = a\sin\phi, \qquad z = ka\phi; \tag{32}$$

$$\left.\begin{aligned} \therefore\; dx &= -a\sin\phi\, d\phi, & dy &= a\cos\phi\, d\phi, & dz &= ka d\phi, \\ d^2x &= -a\cos\phi\, d\phi^2, & d^2y &= -a\sin\phi\, d\phi^2, & d^2z &= 0, \end{aligned}\right\} \tag{33}$$

the differentiations being performed on the supposition that ϕ is equicrescent; therefore the equations to the tangent are

$$\frac{\xi - x}{-a\sin\phi} = \frac{\eta - y}{a\cos\phi} = \frac{\zeta - z}{ka}. \tag{34}$$

The equation to the normal plane is

$$-a(\xi - x)\sin\phi + a(\eta - y)\cos\phi + ka(\zeta - z) = 0,$$

$$\eta x - \xi y + ka(\zeta - z) = 0; \tag{35}$$

when $\xi = \eta = 0$, $\zeta = z$; the normal plane therefore cuts the axis of z at a distance from the origin, equal to the z of the helix at which it is drawn.

Also
$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2, \\ &= (1 + k^2)\, a^2\, d\phi^2; \end{aligned} \tag{36}$$

therefore if λ, μ, ν are the direction-angles of the tangent

$$\left.\begin{aligned} \cos\lambda &= \frac{dx}{ds} = \frac{-\sin\phi}{\{1 + k^2\}^{\frac{1}{2}}}, \\ \cos\mu &= \frac{dy}{ds} = \frac{\cos\phi}{\{1 + k^2\}^{\frac{1}{2}}}, \\ \cos\nu &= \frac{dz}{ds} = \frac{k}{\{1 + k^2\}^{\frac{1}{2}}}. \end{aligned}\right\} \tag{37}$$

The tangent therefore is always inclined at the same angle to the axis of z.

Hence also the equation to the osculating plane is

$$ka^2 \sin\phi\, (\xi - x) - ka^2 \cos\phi\, (\eta - y) + a^2 (\zeta - z) = 0,$$

or
$$k(\xi y - \eta x) + a(\zeta - z) = 0. \tag{38}$$

Also from (37) and (36), taking s to be equicrescent,

$$\left.\begin{aligned}\frac{d^2x}{ds^2} &= -\frac{\cos\phi}{(1-k^2)^{\frac{1}{2}}}\frac{d\phi}{ds} = -\frac{\cos\phi}{a(1-k^2)},\\ \frac{d^2y}{ds^2} &= -\frac{\sin\phi}{(1-k^2)^{\frac{1}{2}}}\frac{d\phi}{ds} = -\frac{\sin\phi}{a(1-k^2)},\\ \frac{d^2z}{ds^2} &= 0;\end{aligned}\right\}\qquad(39)$$

therefore the direction-cosines of the principal normal are, by reason of (31), $\cos\phi$, $\sin\phi$, and 0. The principal normal is therefore perpendicular to the axis of z, and coincident with the radius of the base-cylinder drawn to the point (x, y, z).

348.] In connexion with the subject of the osculating plane, it is convenient to determine the analytical condition, that a curve in space may be wholly in one plane; or in other words, that every four consecutive points on the curve may be in one plane.

Let the equation to the plane be $\text{A}x+\text{B}y+\text{C}z = \text{D}$; then

$$\left.\begin{aligned}\text{A}\,dx+\text{B}\,dy+\text{C}\,dz &= 0,\\ \text{A}\,d^2x+\text{B}\,d^2y+\text{C}\,d^2z &= 0,\\ \text{A}\,d^3x+\text{B}\,d^3y+\text{C}\,d^3z &= 0;\end{aligned}\right\}\qquad(40)$$

whence by cross-multiplication,

$$dx\,(d^2y\,d^3z-d^2z\,d^3y)+dy\,(d^2z\,d^3x-d^2x\,d^3z)+dz\,(d^2x\,d^3y-d^2y\,d^3x)=0;\quad(41)$$

which condition becomes, if z is taken to be an equicrescent variable,

$$\frac{d^2x}{dz^2}\frac{d^3y}{dz^3}-\frac{d^2y}{dz^2}\frac{d^3x}{dz^3}=0;\qquad(42)$$

the geometrical meaning of which condition will be explained hereafter.

349.] Of lines which can be drawn on a surface, and which are therefore generally curves of double curvature, two classes require notice in this place; although they will be discussed at greater length in future parts of our Treatise; and when we have more means at our command.

The first are geodesic lines, or geodesics as they are often called; they are those lines on a surface at all points of which the principal normal is coincident with the normal to the surface. And therefore their differential equations are

$$\frac{d.\frac{dx}{ds}}{\left(\frac{d\text{F}}{dx}\right)}=\frac{d.\frac{dy}{ds}}{\left(\frac{d\text{F}}{dy}\right)}=\frac{d.\frac{dz}{ds}}{\left(\frac{d\text{F}}{dz}\right)}.\qquad(43)$$

Hereafter it will be seen that they are the shortest or the longest lines which can be drawn from one point on a surface to another. And as they are manifestly of great importance, from this point of view, in geodesy, so have they therefrom derived their name.

The second lines are lines of greatest slope, (*lignes de plus grande pente* of M. Monge); that is, if a surface is referred to three coordinate planes, one of which, say that of xy, is horizontal, the line of greatest slope starting from a given point on the surface is that curve each element of which makes with the plane of xy a greater angle than any other element on the surface abutting at the same point: and thus, since all the tangent lines at any point of a surface lie in the tangent plane at that point, that line which is perpendicular to the intersection of the tangent plane with the plane of xy makes the greatest angle with the plane of xy, and is therefore the line of greatest slope.

Let $\text{F}(x, y, z) = 0$ be the equation to the surface; then the equation to the tangent plane is

$$(\xi-x)\left(\frac{d\text{F}}{dx}\right)+(\eta-y)\left(\frac{d\text{F}}{dy}\right)+(\zeta-z)\left(\frac{d\text{F}}{dz}\right) = 0; \qquad (44)$$

the intersection of this with the plane of xy is the line

$$\xi\left(\frac{d\text{F}}{dx}\right)+\eta\left(\frac{d\text{F}}{dy}\right) = x\left(\frac{d\text{F}}{dx}\right)+y\left(\frac{d\text{F}}{dy}\right)+z\left(\frac{d\text{F}}{dz}\right); \qquad (45)$$

and if dx, dy, dz are the projections on the axes of an element common to both the surface and the line of greatest slope, then as the projection of this on the plane of xy is perpendicular to (45) we have

$$\left(\frac{d\text{F}}{dx}\right)dy-\left(\frac{d\text{F}}{dy}\right)dx = 0; \qquad (46)$$

and this differential equation combined with the equation to the surface will give the equations to the line of greatest slope.

Let the surface be a sphere of radius a; then

$$\text{F}(x, y, z) = x^2+y^2+z^2-a^2 = 0;$$

so that (46) becomes $\quad x\,dy-y\,dx = 0;$

$$\therefore \quad \frac{x}{y} = \frac{c}{b};$$

if the initial values of x and y are c and b; that is, the line of greatest slope is a meridianal arc.

CHAPTER XVI.

THE GENERAL AND PARTIAL-DIFFERENTIAL EQUATIONS OF SURFACES, GENERATED BY LINES MOVING ACCORDING TO GIVEN LAWS.

350.]* In the present Chapter I propose to consider a few simple properties of surfaces, which are generated by straight lines and circles moving according to given laws; which lines, as they produce the surface, are called *generators*. The general theory is as follows:

Suppose that we have two equations involving x, y, z and two constants c_1 and c_2, and that they are of the forms,

$$\text{F}_1(x, y, z) = c_1, \qquad \text{F}_2(x, y, z) = c_2; \tag{1}$$

each of which represents a surface; and they, when taken conjointly, represent the line of intersection of the two surfaces. But if c_1 and c_2 are variable parameters, and dependent on each other by means of another equation,

$$f(c_1, c_2) = 0, \tag{2}$$

then, as c_1 and c_2 vary, the line of intersection of the two surfaces (1) varies, and by a continuous variation generates a surface, the equation of such a surface being found by the substitution of (1) in (2), whereby we have

$$f(\text{F}_1, \text{F}_2) = 0. \tag{3}$$

The form however which such problems actually assume is generally somewhat different: a geometrical condition is given which is equivalent to the equation (2); thus, for instance, the generator may be a straight line which is to pass through a given curve, and move parallel to itself, or be parallel to a given plane and pass through two given lines, in which cases the curves through which the generator passes are called *directors*. The process of elimination is as follows:

---

* To those who desire further information on subjects connected with the discussions of the present Chapter I must recommend "Application d'Analyse à la Géométrie, par G. Monge; 5$^{\text{me}}$ edition, par M. Liouville, Paris, 1850."

Let (1) be the equations to the generator involving two *independent* variable parameters, c_1 and c_2; and let the equations to the *director* be,

$$\Phi_1(x, y, z) = 0, \qquad \Phi_2(x, y, z) = 0; \tag{4}$$

then, as the generator is to pass through the director, x, y, z are at that common point the same in (1) and (4); eliminating therefore x, y, z, which refer to that common point, between these four equations, there will result a relation between c_1 and c_2 of the same form as (2), in which they must be replaced by their values in (1), and the resulting equation between x, y, z is that to the surface.

Again, the condition to which the generator is subject frequently is that it should circumscribe a given surface: the generator therefore must touch the given surface at their common point. Let the equation to the surface which is to be circumscribed be

$$u = 0; \tag{5}$$

the direction-cosines of its normal at any point are proportional to

$$\left(\frac{du}{dx}\right), \qquad \left(\frac{du}{dy}\right), \qquad \left(\frac{du}{dz}\right); \tag{6}$$

and the direction-cosines of the tangent of the generator are, Art. 341, equation (6), proportional to

$$\left.\begin{array}{l} \left(\frac{dF_1}{dy}\right)\left(\frac{dF_2}{dz}\right) - \left(\frac{dF_1}{dz}\right)\left(\frac{dF_2}{dy}\right), \\ \left(\frac{dF_1}{dz}\right)\left(\frac{dF_2}{dx}\right) - \left(\frac{dF_1}{dx}\right)\left(\frac{dF_2}{dz}\right), \\ \left(\frac{dF_1}{dx}\right)\left(\frac{dF_2}{dy}\right) - \left(\frac{dF_1}{dy}\right)\left(\frac{dF_2}{dx}\right); \end{array}\right\} \tag{7}$$

which we will symbolize respectively by P, Q, R; and as these lines are to be perpendicular to each other at the point of contact, so that the generator may touch the director-surface, we have the condition

$$P\left(\frac{du}{dx}\right) + Q\left(\frac{du}{dy}\right) + R\left(\frac{du}{dz}\right) = 0, \tag{8}$$

at their common points, which is the equation to the curve of contact; and its intersection with (5) gives us a director through which the generator is to pass. Between therefore (1), (5) and (8), we may eliminate the coordinates which refer to their common points of contact, and get a relation between c_1 and c_2, for which we may substitute the general values of the coordinates given by (1). I propose now to consider those properties of the surfaces thus generated which the Differential Calculus enables us to elucidate.

SECTION 1.—*On surfaces generated by the motion of straight lines.*

351.] Surfaces generated by the motion of straight lines are generally termed *ruled* surfaces (*surfaces réglées*), and of them there are two distinct classes: according as two consecutive generating lines do or do not intersect each other; or in other words, according as two consecutive generators are in the same or in different planes. Surfaces of the former class are termed developable, and those of the latter skew surfaces (*surfaces gauches.*).

The equations to a straight line being

$$\frac{x-x'}{\text{L}} = \frac{y-y'}{\text{M}} = \frac{z-z'}{\text{N}}, \qquad (9)$$

six constants are apparently involved; of which however only four are indeterminate, because the equations can be put into the forms

$$\left.\begin{aligned} x &= az+a, \\ y &= \beta z+b; \end{aligned}\right\} \qquad (10)$$

and of these four variable parameters, two fix the direction of the line and two fix its position. To eliminate these and to find the equation to the surface, five conditions are required, two of which are of necessity the equations (10) of the generator, and the other three are indeterminate, and may be given by means of the equations of three directors. Hence no ruled surface can in general have more than three directors; and to determine the surface from the equations to the generator, such conditions, or others equivalent to them in number, must be given. In general the directors may be such that two consecutive generators do not intersect, in which case the surface generated is skew; when however two successive generators intersect, the analytical condition of this being the case satisfies one of the relations which are required amongst the parameters, and leaves only two to be satisfied by the equations of the fixed directors. Developable surfaces cannot therefore in general have more than two directors.

352.] On developable surfaces.

Since in developable surfaces every generating line and its consecutive line are in the same plane, this plane is the tangent plane to the surface at every point along the first line: for consider any point on the first generating line; the tangent plane at that point passes through the next consecutive point on the line, and therefore contains the line; and as the tangent plane also passes through an indefinite number of points infinitesimally near to the point at which it is drawn, it also passes through a point on the consecutive generating line; and this line is in the same plane with the first generating line; therefore the tangent plane which contains the first line also contains this latter line; the tangent plane therefore touches the surface along the whole length of the generating line. Hence also we have the following property of such surfaces:

Let any number of generating lines be represented by G_1, G_2, G_3, ……; then the surface is made up of the infinitesimal plane areas contained between G_1 and G_2, between G_2 and G_3, …… Now the plane area lying between G_1 and G_2 may be brought into the same plane with that lying between G_2 and G_3, by being turned through a small angle about G_2; and similarly, by turning this last area about G_3, may all the areas between G_1 and G_3 be brought into the same plane without any discontinuity. Let these operations be performed for all the elements, then all will be brought into the same plane; and if we suppose any thin flexible and inextensible film to be laid on such a surface, it will be unfolded into a plane without tearing, rumpling, or doubling. For this reason such surfaces have obtained the expressive title of Developable Surfaces.

If all the generating lines of such surfaces meet in one point the surface is called *conical*, and the point is called the vertex of the cone; and if that point is at an infinite distance, so that all the generators are parallel, the surface is called *cylindrical*. Of these surfaces also independent definitions are given in the following Articles.

353.] On conical surfaces.

A conical surface is generated by a straight line which passes through a given point, and through a given director curve.

Let a, b, c be the coordinates to the given point, which is the vertex of the cone; then the equations to the generating line are

$$\frac{x-a}{\text{L}} = \frac{y-b}{\text{M}} = \frac{z-c}{\text{N}}; \tag{11}$$

in which a, b, c are constant; and L, M N, which are constant for any one position of the generator, vary as the generator passes from one position to another.

Let the equations to the director be

$$\text{F}_1(x', y', z') = 0, \qquad \text{F}_2(x', y', z') = 0; \tag{12}$$

and therefore, as the generator has to pass through (x', y', z'), its equations become,

$$\frac{x-a}{x'-a} = \frac{y-b}{y'-b} = \frac{z-c}{z'-c}; \tag{13}$$

whence

$$\left.\begin{aligned} x'-a &= \frac{x-a}{z-c}(z'-c), \\ y'-b &= \frac{y-b}{z-c}(z'-c) \end{aligned}\right\}; \tag{14}$$

between which equations and (12), eliminating x', y', z', we have a function of the form,

$$\text{F}\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0; \tag{15}$$

which is the general functional equation of conical surfaces.

Equation (15) may also be written in the explicit form

$$\frac{x-a}{z-c} = f\left(\frac{y-b}{z-c}\right); \tag{15*}$$

and may also be put into the symmetrical form,

$$\text{F}\left(\frac{y-b}{z-c}, \frac{z-c}{x-a}, \frac{x-a}{y-b}\right) = 0. \tag{16}$$

If the origin is taken at the vertex of the cone, $a = b = c = 0$, and the last three equations severally become

$$\text{F}\left(\frac{x}{z}, \frac{y}{z}\right) = 0; \qquad \frac{x}{z} = f\left(\frac{y}{z}\right);$$

$$\text{F}\left(\frac{y}{z}, \frac{z}{x}, \frac{x}{y}\right) = 0;$$

which are homogeneous functions of 0 dimensions; such functions therefore represent conical surfaces.

Two particular forms which the director takes require notice. Firstly, if the director is a curve in the plane of xy, let us suppose its equation to be

$$\mathrm{F}(x_0, y_0) = 0;$$

but from (14), $x_0 = \dfrac{az-cx}{z-c}$, $y_0 = \dfrac{bz-cy}{z-c}$;

so that the equation to the conical surface is

$$\mathrm{F}\left(\frac{az-cx}{z-c}, \frac{bz-cy}{z-c}\right) = 0.$$

Again, suppose the cone to be circumscribed about a given surface; then each generating line touches the surface; and if the equation to the surface is $\mathrm{F}(x', y', z') = 0$, we have

$$(x'-a)\left(\frac{d\mathrm{F}}{dx'}\right) + (y'-b)\left(\frac{d\mathrm{F}}{dy'}\right) + (z'-c)\left(\frac{d\mathrm{F}}{dz'}\right) = 0. \qquad (17)$$

and this and the equation to the surface are the equations to the director curve of the conical surface.

Equation (17) is that of the first polar of the surface with reference to (a, b, c) the pole, and gives the line of contact of all the generating lines of the cone with the surface. So if an eye is placed at the point (a, b, c), the visible part of the surface is separated from the unseen part by this line of contact. In surfaces of the second order the line of contact is a plane curve.

354.] Equations (15) and (16) are the general equations to all conical surfaces. They contain an undetermined functional symbol because the director curve is undetermined; and the variables enter under the functional symbol in a particular combination. Now that function may be continuous or discontinuous. If it is continuous, partial derived functions may be found, and by means of them, as in Article 53, the functional symbol may be eliminated, and a differential expression will arise, which gives a general property of all conical surfaces. Thus, from (15) we have, see Art. 53, Ex. 2,

$$(x-a)\left(\frac{dz}{dx}\right) + (y-b)\left(\frac{dz}{dy}\right) = z-c; \qquad (18)$$

and from (16), see Art. 53,

$$(x-a)\left(\frac{d\mathrm{F}}{dx}\right) + (y-b)\left(\frac{d\mathrm{F}}{dy}\right) + (z-c)\left(\frac{d\mathrm{F}}{dz}\right) = 0, \qquad (19)$$

which is indeed identical with the preceding, as the theory of Art. 50 shews. Now the geometrical property which this equation expresses is this: all tangent planes of a conical surface pass through the vertex of the cone, and also contain a generating line of the cone. Sometimes this geometrical property is assumed to be (see Monge, Application de l'Analyse &c.) the defining property of conical surfaces; in which case the mathematical translation is the preceding differential equation (19), and from it by integration the integral equation of the surface is found. This question is considered in Vol. II of our treatise.

355.] Examples of conical surfaces.

Ex. 1. To find the equation to a cone whose director is a circle in the plane of xy.

Let the equations to the circle and to the generator be

$$x_0^2 + y_0^2 = k^2, \tag{20}$$

and

$$\frac{x-a}{\text{L}} = \frac{y-b}{\text{M}} = \frac{z-c}{\text{N}}; \tag{21}$$

therefore, when $z = 0$, $\quad x_0 = a - \frac{\text{L}}{\text{N}}c, \quad y_0 = b - \frac{\text{M}}{\text{N}}c;$

squaring and adding which, by means of (20), and replacing the variable parameters by their values from (21), we have

$$(cx-az)^2 + (cy-bz)^2 = k^2(z-c)^2;$$

which is the general equation to a cone of which the director is a circle in the plane of xy.

If the line joining the centre of the circular director and the vertex is at right angles to the circle, the cone is called right; in which case $a = b = 0$, and the equation is

$$x^2 + y^2 = \frac{k^2}{c^2}(z-c)^2;$$

where $\frac{k}{c}$ is the tangent of the semi-vertical-angle.

Ex. 2. To find the equation to a cone circumscribing a given ellipsoid.

Let the vertex of the cone be (x_0, y_0, z_0), and the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{22}$$

Let the equations to the generator be

$$\frac{\xi-x_0}{\mathrm{L}} = \frac{\eta-y_0}{\mathrm{M}} = \frac{\zeta-z_0}{\mathrm{N}};$$

and as the generators are to touch the ellipsoid, their equations take the form,

$$\frac{\xi-x}{x_0-x} = \frac{\eta-y}{y_0-y} = \frac{\zeta-z}{z_0-z}; \tag{23}$$

and as the points (ξ, η, ζ), (x_0, y_0, z_0) are in the tangent plane to the ellipsoid,

$$\frac{\xi x}{a^2}+\frac{\eta y}{b^2}+\frac{\zeta z}{c^2} = 1, \tag{24}$$

$$\frac{x_0 x}{a^2}+\frac{y_0 y}{b^2}+\frac{z_0 z}{c^2} = 1; \tag{25}$$

whence, operating on the equality (23), and reducing by means of (22), (24) and (25), we have

$$\frac{\xi-x}{x_0-x} = \frac{\eta-y}{y_0-y} = \frac{\zeta-z}{z_0-z} = \frac{\frac{\xi x_0}{a^2}+\frac{\eta y_0}{b^2}+\frac{\zeta z_0}{c^2}-1}{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}-1},$$

$$= \frac{\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}-1}{\frac{\xi x_0}{a^2}+\frac{\eta y_0}{b^2}+\frac{\zeta z_0}{c^2}-1}; \tag{26}$$

$$\therefore \left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}-1\right)\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}-1\right)$$

$$= \left\{\frac{\xi x_0}{a^2}+\frac{\eta y_0}{b^2}+\frac{\zeta c_0}{c^2}-1\right\}^2; \tag{27}$$

which is the equation to the circumscribing cone.

As (x, y, z) is the point on the ellipsoid common to it and to the cone, (25) is the equation to the plane of contact; and it, and equation (22), are those to the director-curve.

Ex. 3. As another example let us investigate the relations between the coefficients of the equation of the second degree when it represents a cone. The equations to all conical surfaces are subject to the condition (19). Let the general equation of the second degree be

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{A}_1 yz+2\mathrm{B}_1 zx+2\mathrm{C}_1 xy$$
$$+2\mathrm{A}_2 x+2\mathrm{B}_2 y+2\mathrm{C}_2 z+\mathrm{K} = 0;$$

$$\therefore \left(\frac{d\mathrm{F}}{dx}\right) = 2(\mathrm{A}x+\mathrm{C}_1 y+\mathrm{B}_1 z+\mathrm{A}_2), \quad \left(\frac{d\mathrm{F}}{dy}\right) = \ldots, \quad \left(\frac{d\mathrm{F}}{dz}\right) = \ldots;$$

So that (19) becomes, after reduction,

$$(aA+bC_1+cB_1+A_2)x + (aC_1+bB+cA_1+B_2)y$$
$$+ (aB_1+bA_1+cC+C_2)z + aA_2+bB_2+cC_2+K = 0;$$

and as this relation is true independently of x, y, z, the coefficients of the several terms must be equal to zero: whence we have

$$\begin{aligned} Aa+C_1b+B_1c+A_2 &= 0, \\ C_1a+Bb+A_1c+B_2 &= 0, \\ B_1a+A_1b+Cc+C_2 &= 0, \\ A_2a+B_2b+C_2c+K &= 0. \end{aligned}$$

Of these four conditions the first three shew that the surface has a centre, and that the vertex of the cone, viz. the point (a, b, c), is at the centre; and from them the coordinates of the centre may be determined. And the last shews that in the transformation of the equation to the centre as origin the constant term disappears. Thus the equation to the surface becomes after transformation a homogeneous equation of two dimensions in terms of x, y, z.

356.] To find the equation to cylindrical surfaces.

A cylindrical surface is generated by a straight line which moves parallel to itself, and always passes through a given director curve.

Let x, y, z be the current coordinates to the surface; x', y', z' the coordinates to a point on the director; L, M, N proportional to the direction cosines of the generating line; then the equations to the generating line are

$$\frac{x-x'}{L} = \frac{y-y'}{M} = \frac{z-z'}{N}. \tag{28}$$

Let the equations to the director be

$$F_1(x', y', z') = 0, \qquad F_2(x', y', z') = 0; \tag{29}$$

from (28) we have

$$x' = x - \frac{L}{N}(z-z'), \qquad y' = y - \frac{M}{N}(z-z'); \tag{30}$$

from which four equations eliminating x', y', z', we have a result of the form

$$F(Nx-Lz,\ Ny-Mz) = 0, \tag{31}$$

which is the general equation to cylindrical surfaces.

Equation (31) may also be put in the form

$$x - lz = f(y - mz), \tag{32}$$

which is the explicit form of the general equation to cylindrical surfaces.

Also (31) may be written in the following form,

$$\mathrm{F}(\mathrm{N}y - \mathrm{M}z,\ \mathrm{L}z - \mathrm{N}x,\ \mathrm{M}x - \mathrm{L}y) = 0. \tag{33}$$

Two particular cases require notice. Firstly, let the director be a plane curve in the plane xy, and let its equation be

$$x_0 = f(y_0); \tag{34}$$

then, from (28),

$$x_0 = x - \frac{\mathrm{L}}{\mathrm{N}} z = x - lz, \qquad y_0 = y - \frac{\mathrm{M}}{\mathrm{N}} z = y - mz; \tag{35}$$

and if these are substituted in (34), we have

$$x - lz = f(y - mz).$$

Again, let us suppose the cylinder to circumscribe a surface whose equation is $\mathrm{F}_1(x', y', z') = 0$; then, as the generator is to be perpendicular to the normal at the point of contact,

$$\mathrm{L}\left(\frac{d\mathrm{F}_1}{dx'}\right) + \mathrm{M}\left(\frac{d\mathrm{F}_1}{dy'}\right) + \mathrm{N}\left(\frac{d\mathrm{F}_1}{dz'}\right) = 0, \tag{36}$$

from which, the equation to the surface, and the equations to the generators, x', y', z' are to be eliminated; and the resulting equation in terms of x, y, z is that to the cylindrical surface.

357.] Now (32) and (33) are the general equations to all cylindrical surfaces. They contain an undetermined functional symbol, under which the coordinates enter as subject-variables in a particular combination, and which is itself determined when the law of the director curve is given: but whatever that law is, the general form of the function in reference to the subject-variables is the same. These forms also hold good, whether the function is continuous or discontinuous. If the function is continuous, two partial derived functions, say the z and x, and the z and y, may be formed from (32), and the undetermined function may be eliminated; whereby we shall have

$$l\left(\frac{dz}{dx}\right) + m\left(\frac{dz}{dy}\right) = 1. \tag{37}$$

Or we may operate on (33) as in Article 58, and we shall have

$$\mathrm{L}\left(\frac{d\mathrm{F}}{dx}\right) + \mathrm{M}\left(\frac{d\mathrm{F}}{dy}\right) + \mathrm{N}\left(\frac{d\mathrm{F}}{dz}\right) = 0, \tag{38}$$

which is in fact identical with (37); see Art. 50. Now these equations express properties which are true of all cylindrical surfaces whose directors are continuous curves; and what is the geometrical interpretation of them? Let us take (38); L, M, N are proportional to the direction-cosines of the generating line, and the partial derived functions are proportional to the direction-cosines of the normal at (x, y, z): and (38) shews that these lines are perpendicular to each other: or, in other words, the tangent plane at every point contains the generating line of the cylinder which passes through that point.

358.] Examples of cylindrical surfaces.

Ex. 1. To find the equation to the cylinder whose director is the ellipse,

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2} = 1. \tag{39}$$

If we substitute for x_0 and y_0 from (35), we have

$$\frac{(\mathrm{N}x-\mathrm{L}z)^2}{a^2\,\mathrm{N}^2}+\frac{(\mathrm{N}y-\mathrm{M}z)^2}{b^2\,\mathrm{N}^2} = 1; \tag{40}$$

which is the general equation to an oblique elliptical cylinder. If the generator is perpendicular to the plane of xy, the cylinder is called right; and as in that case $\mathrm{L}=0$, $\mathrm{M}=0$, the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

Hence the equation to an oblique circular cylinder is

$$(\mathrm{N}x-\mathrm{L}z)^2+(\mathrm{N}y-\mathrm{M}z)^2 = \mathrm{N}^2a^2. \tag{41}$$

Ex. 2. To find the equation to a cylinder circumscribing a given ellipsoid.

Let the equations to the generator and to the ellipsoid be

$$\frac{\xi-x}{\mathrm{L}} = \frac{\eta-y}{\mathrm{M}} = \frac{\zeta-z}{\mathrm{N}}, \tag{42}$$

and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1. \tag{43}$$

Now as (42) is to touch (43), the point (ξ, η, ζ) is in the tangent plane of (43); and therefore we have

$$\frac{\xi x}{a^2}+\frac{\eta y}{b^2}+\frac{\zeta z}{c^2} = 1;$$

whence, operating on the members of the equality (42),

$$\frac{\xi-x}{\text{L}} = \frac{\eta-y}{\text{M}} = \frac{\zeta-z}{\text{N}} = \frac{\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}-1}{\frac{\text{L}\xi}{a^2}+\frac{\text{M}\eta}{b^2}+\frac{\text{N}\zeta}{c^2}}$$

$$= \frac{\left\{\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}-1\right\}^{\frac{1}{2}}}{\left\{\frac{\text{L}^2}{a^2}+\frac{\text{M}^2}{b^2}+\frac{\text{N}^2}{c^2}\right\}^{\frac{1}{2}}};$$

$$\therefore \quad \left(\frac{\text{L}\xi}{a^2}+\frac{\text{M}\eta}{b^2}+\frac{\text{N}\zeta}{c^2}\right)^2 = \left(\frac{\text{L}^2}{a^2}+\frac{\text{M}^2}{b^2}+\frac{\text{N}^2}{c^2}\right)\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}-1\right).$$

From this equation it appears that if (ξ, η, ζ) is a point on the cylindrical surface which is also on the ellipsoid, then $\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}-1 = 0$, and therefore $\frac{\text{L}\xi}{a^2}+\frac{\text{M}\eta}{b^2}+\frac{\text{N}\zeta}{c^2} = 0$; which is the equation to a central plane section of the ellipsoid; and is the polar plane of a pole on the line (42) at an infinite distance.

Ex. 8. To determine the conditions that the general equation of the second degree of three variables may represent a cylinder.

Let the general equation be

$$\begin{aligned} \text{F}(x,y,z) = \text{A}x^2+\text{B}y^2+\text{C}z^2+2\text{A}_1yz+2\text{B}_1zx+2\text{C}_1xy \\ +2\text{A}_2x+2\text{B}_2y+2\text{C}_2z+\text{K} = 0; \quad (44) \end{aligned}$$

$$\left(\frac{d\text{F}}{dx}\right) = 2(\text{A}x+\text{C}_1y+\text{B}_1z+\text{A}_2), \quad \left(\frac{d\text{F}}{dy}\right) = \ldots\ldots, \quad \left(\frac{d\text{F}}{dz}\right) = \ldots\ldots;$$

and therefore equation (38) becomes

$$\begin{aligned} \text{L}(\text{A}x+\text{C}_1y+\text{B}_1z+\text{A}_2)+\text{M}(\text{C}_1x+\text{B}y+\text{A}_1z+\text{B}_2) \\ +\text{N}(\text{B}_1x+\text{A}_1y+\text{C}z+\text{C}_2) = 0; \end{aligned}$$

and as this condition is to be satisfied for all values of x, y, z, we have

$$\begin{aligned} \text{AL}+\text{C}_1\text{M}+\text{B}_1\text{N} &= 0, \\ \text{C}_1\text{L}+\text{BM}+\text{A}_1\text{N} &= 0, \\ \text{B}_1\text{L}+\text{A}_1\text{M}+\text{CN} &= 0, \\ \text{A}_2\text{L}+\text{B}_2\text{M}+\text{C}_2\text{N} &= 0; \end{aligned}$$

therefore from the first three,

$$\text{ABC}-\text{AA}_1^2-\text{BB}_1^2-\text{CC}_1^2+2\text{A}_1\text{B}_1\text{C}_1 = 0; \quad (45)$$

and from the first three combined with the fourth

$$(\text{BC}-\text{A}_1^2)\text{A}_2+(\text{CA}-\text{B}_1^2)\text{B}_2+(\text{AB}-\text{C}_1^2)\text{C}_2 = 0;$$

and taking the first three two and two together, we have

$$\frac{\mathrm{L}}{\mathrm{A}_1^2-\mathrm{BC}}=\frac{\mathrm{M}}{\mathrm{CC}_1-\mathrm{A}_1\mathrm{B}_1}=\frac{\mathrm{N}}{\mathrm{BB}_1-\mathrm{C}_1\mathrm{A}_1},$$

$$\frac{\mathrm{L}}{\mathrm{CC}_1-\mathrm{A}_1\mathrm{B}_1}=\frac{\mathrm{M}}{\mathrm{B}_1^2-\mathrm{CA}}=\frac{\mathrm{N}}{\mathrm{AA}_1-\mathrm{C}_1\mathrm{B}_1},$$

$$\frac{\mathrm{L}}{\mathrm{BB}_1-\mathrm{A}_1\mathrm{C}_1}=\frac{\mathrm{M}}{\mathrm{AA}_1-\mathrm{B}_1\mathrm{C}_1}=\frac{\mathrm{N}}{\mathrm{C}_1^2-\mathrm{AB}};$$

$$\therefore\quad \frac{\mathrm{L}}{(\mathrm{A}_1^2-\mathrm{BC})^{\frac{1}{2}}}=\frac{\mathrm{M}}{(\mathrm{B}_1^2-\mathrm{CA})^{\frac{1}{2}}}=\frac{\mathrm{N}}{(\mathrm{C}_1^2-\mathrm{AB})^{\frac{1}{2}}};$$

and substituting these values in the last, we have

$$(\mathrm{A}_1^2-\mathrm{BC})^{\frac{1}{2}}\mathrm{A}_2+(\mathrm{B}_1^2-\mathrm{CA})^{\frac{1}{2}}\mathrm{B}_2+(\mathrm{C}_1^2-\mathrm{AB})^{\frac{1}{2}}\mathrm{C}_2=0. \tag{46}$$

Now (45) is the well-known condition that the first six terms of (44) are resoluble into two linear homogeneous factors; and (46) is the additional condition requisite that (44) should be of the form

$$(ax+by+cz+d)(a_1x+b_1y+c_1z+d_1)-k=0; \tag{47}$$

and therefore (47) is the equation to a cylindrical surface of the second degree. Now (47) may be resolved into two factors of the form

$$ax+by+cz+d-\frac{k^{\frac{1}{2}}}{m}=0,$$

$$a_1x+b_1y+c_1z+d_1+mk^{\frac{1}{2}}=0,$$

where m is an undetermined constant: because if m is eliminated, we obtain the equation (47). Each of these equations is that of a plane, and the generating line is the line of intersection of them; and the surface is generated by the line whose varying position is due to the variation of m. From these two equations it is evident that the direction cosines of the generating line are proportional to b_1c-c_1b, c_1a-a_1c, a_1b-b_1a; and these are proportional to L, M, N. Although at first sight it may appear that (47) is not of the forms (31) or (33), yet it will be found that with these values of L, M and N it satisfies (38), and is therefore the equation of a cylindrical surface.

The surface in its most general form represents an elliptical or a hyperbolic cylinder. If however (46) is satisfied identically, so that $\mathrm{A}_1^2=\mathrm{BC}$, $\mathrm{B}_1^2=\mathrm{CA}$, $\mathrm{C}_1^2=\mathrm{AB}$, then (47) is the equation to a parabolic cylinder.

359.] On the equation of developable surfaces.

As any and every two consecutive generators of a developable surface intersect, and as these two, as shewn in Art. 352, lie in one plane, it is convenient to consider such a surface as formed by the continual intersection of planes drawn according to a given law. Now the general equation to a plane involves only three independent constants, and may be put in the form,

$$\text{A}x + \text{B}y + \text{C}z = 1.$$

Suppose each of these constants to be a function of a variable parameter a, viz. let $\text{A} = f(a)$, $\text{B} = \phi(a)$, $\text{C} = \psi(a)$; then, as a continuously varies, the plane will have different positions, any two consecutive ones of which will intersect in a straight line; and will, as its position varies, generate a developable surface, of which the straight line of intersection of two consecutive planes will be the generator; thus the equation to one of the planes will be

$$x f(a) + y\,\phi(a) + z\,\psi(a) = 1. \tag{48}$$

And if we differentiate it in reference to a, we have

$$x f'(a) + y\,\phi'(a) + z\,\psi'(a) = 0, \tag{49}$$

which is the equation to another plane; and the line of intersection of the two is the generator of the developable surface. From these two eliminating a, we shall obtain an equation in terms of x, y, z which will be that to the required surface: but this cannot be determined in the general case, that is, so long as the functions involved in (48) and (49) are undetermined. If however we eliminate the functions by differentiation, we shall get a differential equation which will give a property common to all developable surfaces.

360.] To find the differential equation to developable surfaces.

In (48) and (49), a is a function of x, y, z. Taking then the partial differentials of (48), we have

$$\{x f'(a) + y\phi'(a) + z\psi'(a)\} \left(\frac{da}{dx}\right) + f(a) + \psi(a) \left(\frac{dz}{dx}\right) = 0,$$

$$\{x f'(a) + y\phi'(a) + z\psi'(a)\} \left(\frac{da}{dy}\right) + \phi(a) + \psi(a) \left(\frac{dz}{dy}\right) = 0;$$

whence, by means of (49), we have

$$f(a) + \psi(a) \left(\frac{dz}{dx}\right) = 0, \qquad \phi(a) + \psi(a) \left(\frac{dz}{dy}\right) = 0. \tag{50}$$

From these last equations, eliminating a, we have a relation of the form,

$$\left(\frac{dz}{dx}\right) = \mathrm{F}\left\{\left(\frac{dz}{dy}\right)\right\};$$

whence, eliminating F, there results

$$\left(\frac{d^2z}{dx^2}\right)\left(\frac{d^2z}{dy^2}\right) - \left(\frac{d^2z}{dx\,dy}\right)^2 = 0; \qquad (51)$$

and this is the general differential equation to developable surfaces.

Another differential equation equivalent to (51) may be found as follows:

Suppose the equation to the surface to be

$$\mathrm{F}(x, y, z) = 0;$$

and U, V, W to be its partial derived-functions; then, as in Art. 332,

$$\mathrm{U}\,dx + \mathrm{V}\,dy + \mathrm{W}\,dz = 0; \qquad (52)$$

and suppose the equation to the plane by the consecutive intersection of which the surface is formed to be

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 1; \qquad (53)$$

and as this is a tangent plane, see Art. 352,

$$\mathrm{A}\,dx + \mathrm{B}\,dy + \mathrm{C}\,dz = 0; \qquad (54)$$

therefore by comparing (52) and (54),

$$\frac{\mathrm{U}}{\mathrm{A}} = \frac{\mathrm{V}}{\mathrm{B}} = \frac{\mathrm{W}}{\mathrm{C}} = \lambda. \qquad (55)$$

Differentiating again (52) and (54), since the tangent plane touches the surface along the generating line,

$$\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z + d\mathrm{U}\,dx + d\mathrm{V}\,dy + d\mathrm{W}\,dz = 0, \qquad (56)$$

$$\mathrm{A}\,d^2x + \mathrm{B}\,d^2y + \mathrm{C}\,d^2z = 0; \qquad (57)$$

whence replacing U, V, W from (55) in terms of A, B, C,

$$\lambda\,\{\mathrm{A}\,d^2x + \mathrm{B}\,d^2y + \mathrm{C}\,d^2z\} + d\mathrm{U}\,dx + d\mathrm{V}\,dy + d\mathrm{W}\,dz = 0;$$

$$\therefore\quad d\mathrm{U}\,dx + d\mathrm{V}\,dy + d\mathrm{W}\,dz = 0, \qquad (58)$$

which is in fact identical with equation (49).

Comparing this with (52), we have

$$\frac{\mathrm{U}}{d\mathrm{U}} = \frac{\mathrm{V}}{d\mathrm{V}} = \frac{\mathrm{W}}{d\mathrm{W}} = \frac{1}{\mu}. \qquad (59)$$

Let u, v, w, u', v', w' represent the several second partial derived-functions of $\mathrm{F}(x, y, z)$, viz.

$$\left.\begin{aligned} u &= \left(\frac{d^2\mathrm{F}}{dx^2}\right), & v &= \left(\frac{d^2\mathrm{F}}{dy^2}\right), & w &= \left(\frac{d^2\mathrm{F}}{dz^2}\right), \\ u' &= \left(\frac{d^2\mathrm{F}}{dy\,dz}\right), & v' &= \left(\frac{d^2\mathrm{F}}{dz\,dx}\right), & w' &= \left(\frac{d^2\mathrm{F}}{dx\,dy}\right); \end{aligned}\right\} \qquad (60)$$

so that

$$\left.\begin{aligned} \mu\mathrm{u} &= d\mathrm{u} = u\,dx + w'dy + v'dz, \\ \mu\mathrm{v} &= d\mathrm{v} = w'dx + v\,dy + u'dz, \\ \mu\mathrm{w} &= d\mathrm{w} = v'dx + u'dy + w\,dz; \end{aligned}\right\} \qquad (61)$$

from which, and from (52), eliminating μ, dx, dy, dz, we have

$$\mathrm{u}^2(vw-u'^2)+\mathrm{v}^2(wu-v'^2)+\mathrm{w}^2(uv-w'^2)$$
$$+2\mathrm{v}\mathrm{w}(v'w'-uu')+2\mathrm{w}\mathrm{u}(w'u'-vv')+2\mathrm{u}\mathrm{v}(u'v'-ww') = 0; \qquad (62)$$

which result may easily be shewn to be identical with (51) by writing the equations in the form

$$\mathrm{F}(x, y, z) = z - f(x, y) = 0.$$

and (51) takes the form (62), when $\left(\frac{d^2z}{dx^2}\right)$, $\left(\frac{d^2z}{dx\,dy}\right)$, $\left(\frac{d^2z}{dy^2}\right)$ are replaced by their values given in Art. 83. These equations are of course satisfied by the equations to the cylinder and to the cone.

361.] Since every two consecutive generating lines of a developable surface intersect, a curve is formed, after the manner of an envelope, see Chapter XIII, Section 2, by the continual intersection of all these; and this must be a curve of double curvature, otherwise all the lines would be in one plane, and the developable surface would be only a plane. This curve bears the name of Edge of Regression (*Arête de Rebroussement*), and the generator of the surface is plainly always a tangent to it. Its equations may be found as follows:

Equations (48) and (49), if a is considered constant, are, taken together, the equations to a generating line of the developable surface, and therefore, from what has just been said, to the line whose envelope has to be determined: and the equations to which may therefore be found by making a to vary. Differentiating therefore (49) with respect to a, we have

$$xf''(a)+y\phi''(a)+z\psi''(a) = 0; \qquad (63)$$

and eliminating (a) between this (48) and (49), we shall get two equations in terms of x, y, and z which are those to the edge of regression. This line, as is plain from its mode of generation, bounds the developable surface towards one side of space; and on the other side the surface is continued.

362.] Hence also we arrive at a new conception of a developable surface; it is generated by a tangent of a curve of double curvature which moves continuously along the curve. Also since the osculating plane is that which contains two consecutive tangents, it may be conceived of as formed by the continuous intersection of such osculating planes. Suppose then that the equations to a curve of double curvature are given in the forms, see Ex. 2, Art. 347,

$$x = f(a), \qquad y = \phi(a), \qquad z = \psi(a), \tag{64}$$

a being a variable parameter; and so that

$$dx = f'(a)\,da, \qquad dy = \phi'(a)\,da, \qquad dz = \psi'(a)\,da; \tag{65}$$

then, by equations (2), Art. 341, the equations to the generating line are

$$\frac{\xi - f(a)}{f'(a)} = \frac{\eta - \phi(a)}{\phi'(a)} = \frac{\zeta - \psi(a)}{\psi'(a)}; \tag{66}$$

from which the equation to the surface will be found by the elimination of a.

Similarly also will developable surfaces be formed by the intersection of normal planes of a curve of double curvature; for suppose the equations to the curve to be of the form (64), then the equation to the normal plane is,

$$\{\xi - f(a)\}f'(a) + \{\eta - \phi(a)\}\,\phi'(a) + \{\zeta - \psi(a)\}\,\psi'(a) = 0; \tag{67}$$

an equation of the form (48), and therefore manifestly that of a developable surface. Fig. 129 indicates the mode of generation of such surfaces and edges of regression.

363.] Ex. 1. To find the equation to the surface generated by tangents to the helix; or, which is the same thing, formed by the continuous intersection of osculating planes.

By Ex. 2, Art. 347, the equation to the osculating plane is

$$\eta \cos\phi - \xi \sin\phi = \frac{\zeta}{k} - a\phi; \tag{68}$$

therefore taking the ϕ-differential,

$$\eta \sin\phi + \xi \cos\phi = a; \tag{69}$$

whence, squaring and adding,

$$\phi = \frac{\zeta - k\,(\xi^2 + \eta^2 - a^2)^{\frac{1}{2}}}{ka}; \tag{70}$$

whereby (69) becomes

$$\eta \sin \frac{\zeta - k(\xi^2+\eta^2-a^2)^{\frac{1}{2}}}{ka} + \xi \cos \frac{\zeta - k(\xi^2+\eta^2-a^2)^{\frac{1}{2}}}{ka} = a; \quad (71)$$

which is the equation to the developable helicoid, or screw-surface; the edge of regression of which is the helix itself.

Similarly will the equation to the surface formed by the intersection of consecutive normal planes to the helix be found to be

$$\eta \sin \frac{k\zeta - (\eta^2+\xi^2-k^4a^2)^{\frac{1}{2}}}{k^2a} + \xi \cos \frac{k\zeta - (\eta^2+\xi^2-k^4a^2)^{\frac{1}{2}}}{k^2a} + k^2a = 0. \quad (72)$$

Ex. 2. If the equations to a curve of double curvature are

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= k^2, \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1; \end{aligned}\right\} \quad (73)$$

the student will without difficulty find the following equation to the surface formed by the intersection of normal planes;

$$\left(\frac{\xi}{\alpha}\right)^{\frac{2}{3}} \pm \left(\frac{\eta}{\beta}\right)^{\frac{2}{3}} - \left(\frac{\zeta}{\gamma}\right)^{\frac{2}{3}} = 0; \quad (74)$$

which is manifestly the equation to a cone; the ambiguity of sign in the second term depending on k^2 being greater or less than b^2.

A further inquiry into the properties of developable surfaces is beyond the scope of the present treatise, but I cannot refrain from recommending the reader to study the works of Monge and Dupin on these subjects: works as they are of such intrinsic merit that I cannot venture to characterize them, for it may be that my praise would be below their due, and thus only tend towards disparagement.

364.] On skew surfaces.

Skew surfaces, see Art. 351, are those ruled surfaces, any two consecutive generating lines of which do not intersect; in the complete equations of such generators, (9) and (10) Art. 351, three conditions are left undetermined, and these may be, that the generator shall meet three directors. It is also manifest geometrically that such conditions fix the generator; for take a point chosen arbitrarily on any one to be the vertex of a cone, from which conceive two conical surfaces to be described with the other two generators as their directors; then these cones will intersect in one or more straight lines, which will be the generators of the skew surface.

Now in developable surfaces we shewed that a tangent plane at any point of the surface not only contained a generator, but touched the surface through the whole length of the generator. In skew surfaces however it is not so; the tangent plane contains the generator, but cuts the surface at every other point along it save at that of contact. For suppose P and P′ to be any two points at a finite distance apart on the generator of a skew surface, and Q and Q′ to be two other points respectively near to them; then the tangent plane at P contains the line PQ, and that at P′ contains P′Q′; but these tangent planes cannot be identical in position, for were they so, the line joining Q and Q′ would be in the same plane with that joining P and P′, and two consecutive straight lines would intersect; and this is inconsistent with the fact of the surface being skew. Hence the tangent plane of a skew surface cuts that surface along the length of the generator, save at the point of contact.

The equations therefore to the generator of a skew surface are the following:

Let a be a variable parameter, and let the arbitrary functions introduced into the equations be so determined that the generator may pass through three directors; accordingly we have

$$\frac{x-f(a)}{F(a)} = \frac{y-\phi(a)}{\Phi(a)} = \frac{z-\psi(a)}{\Psi(a)}. \tag{75}$$

365.] The differential equation to skew surfaces may be found as follows:

Let $F(x, y, z)$ be the equation to the surface; and let U, V, W be its partial derived-functions; so that

$$U\,dx + V\,dy + W\,dz = 0. \tag{76}$$

Now take three consecutive points along the generating line of such a surface; then by the last Article, the tangent plane changes its position at those three points; U, V, W therefore, which are proportional to the direction-cosines of the normal, change as we pass along the line, but the ratios $dx : dy : dz$ remain constant, because they are the same for all points of the line. Hence we have

$$U\,dx + V\,dy + W\,dz = 0, \tag{77}$$

$$dU\,dx + dV\,dy + dW\,dz = 0, \tag{78}$$

$$d^2U\,dx + d^2V\,dy + d^2W\,dz = 0; \tag{79}$$

and replacing du, dv, dw by their values given in (61), and d^2u, d^2v, d^2w by their similar values, (78) and (79) become respectively a quadratic and a cubic in terms of dx, dy, dz; and from them and (77), dx, dy, dz are to be eliminated; and the resulting equation, in terms of partial differential coefficients, will be that to the skew surface.

Of this class of surfaces two kinds require special mention; viz. conoidal surfaces, and those surfaces which are generated by a straight line moving on two directors, and always parallel to a given plane.

366.] *Conoidal surfaces* are those skew ones, the generating line of which always passes through and is perpendicular to some straight line, say to the axis of z, and meets some director.

Let x, y, z, x', y', z' be severally the coordinates of the generator and of the director; and let the equations to the director be

$$\text{F}_1(x', y', z') = 0, \qquad \text{F}_2(x', y', z') = 0; \tag{80}$$

then the equations to the generator being

$$\frac{x}{x'} = \frac{y}{y'}, \quad z = z', \tag{81}$$

(80) become

$$\text{F}_1\left(x', \frac{y}{x}x', z\right) = 0, \qquad \text{F}_2\left(x', \frac{y}{x}x', z\right) = 0; \tag{82}$$

and if we eliminate x', we have a result of the form,

$$\text{F}\left(\frac{y}{x}, z\right) = 0; \tag{83}$$

which is the general equation to conoidal surfaces.

(83) may also be put into the explicit form,

$$z = f\left(\frac{x}{y}\right). \tag{84}$$

367.] The differential equation to such surfaces may be thus found; taking the partial differentials of (84), as in Art. 103, we have ultimately

$$x\left(\frac{dz}{dx}\right) + y\left(\frac{dz}{dy}\right) = 0. \tag{85}$$

Or taking the implicit form, viz. equation (83), and writing F_1 and F_2 for the derived functions of F, according as the first or second subject-variable varies, we have

$$\left(\frac{d\mathrm{F}}{dx}\right) = -\frac{y}{x^2}\,\mathrm{F}_1, \qquad \left(\frac{d\mathrm{F}}{dy}\right) = \frac{1}{x}\,\mathrm{F}_1, \qquad \left(\frac{d\mathrm{F}}{dz}\right) = \mathrm{F}_2,$$

$$\therefore \quad x\left(\frac{d\mathrm{F}}{dx}\right) + y\left(\frac{d\mathrm{F}}{dy}\right) + 0\left(\frac{d\mathrm{F}}{dz}\right) = 0. \tag{86}$$

The geometrical meaning of which is, that the normal to the surface is perpendicular to the line drawn from the point of intersection perpendicular to the axis of z.

Ex. 1. To find the equation to the conoid whose director is the helix.

$$x' = a\cos\phi, \quad y' = a\sin\phi, \quad z' = ka\phi; \tag{87}$$

$$\frac{y}{x} = \frac{y'}{x'}, \qquad z = z';$$

$$\therefore \quad x' = a\cos\frac{z}{ka}, \quad y' = a\sin\frac{z}{ka}; \qquad \frac{y}{x} = \tan\frac{z}{ka};$$

$$y\cos\frac{z}{ka} - x\sin\frac{z}{ka} = 0; \tag{88}$$

which surface is called the Skew Helicoid, and is that of the under surface of spiral staircases.

Ex. 2. Let the director be a circle whose plane is parallel to that of xz at a distance c from it, and whose centre is in the axis of y; see fig. 126.

Let the radius of the circle $= a$; and let the generator pass through and be perpendicular to the axis of x; $\mathrm{OM} = x$, $\mathrm{MN} = y$, $\mathrm{NP} = z$; $\mathrm{OM} = x'$, $\mathrm{ML} = y'$, $\mathrm{LQ} = z'$.

Let $\quad \mathrm{OA} = \mathrm{CB} = \mathrm{CE} = a; \qquad \mathrm{OC} = \mathrm{AB} = c;$

$$\therefore \quad \left.\begin{aligned} x'^2 + z'^2 &= a^2, \\ y' &= c; \end{aligned}\right\} \qquad \left.\begin{aligned} x' &= x, \\ \frac{y'}{z'} &= \frac{y}{z}; \end{aligned}\right\}$$

$$\therefore \quad \left.\begin{aligned} z'^2 &= a^2 - x^2, \\ y'^2 &= c^2; \end{aligned}\right\} \qquad \frac{y^2}{z^2} = \frac{c^2}{a^2 - x^2};$$

$$c^2z^2 = y^2(a^2 - x^2). \tag{89}$$

which surface is known by the name of the Cono-Cuneus of Wallis; the figure contains but one-fourth of it, the remainder being in three other octants.

368.] To find the equation to a surface generated by a straight line moving on two directors and always parallel to a given plane.

It is manifest from the mode of generation, that a section of the surface made by a plane parallel to the given plane is a straight line.

Let $\text{A}x+\text{B}y+cz = a$ be the equation to the plane cutting the surface and parallel to the given plane, and therefore having A, B, C constant, and a a variable parameter; and let the equation to another plane passing through the generating straight line of the surface be

$$\text{A}_1 x + \text{B}_1 y + \text{C}_1 z = 0; \tag{90}$$

that is, let us conceive it to pass through the origin; then A_1, B_1, and C_1 are variable and may be considered to be functions of a; so that

$$\text{A}_1 = f(a), \quad \text{B}_1 = \phi(a), \quad \text{C}_1 = \psi(a);$$

and therefore the general equation to the surface is

$$xf(\text{A}x+\text{B}y+\text{C}z)+y\phi(\text{A}x+\text{B}y+\text{C}z)+z\psi(\text{A}x+\text{B}y+\text{C}z) = 0. \tag{91}$$

To find its differential equation: to simplify the process, suppose the director plane to be parallel to that of xy: then $\text{A} = \text{B} = 0$, and $\text{C} = 1$; and the equation becomes

$$xf(z)+y\,\phi(z)+z\,\psi(z) = 0; \tag{92}$$

which may be put into the form

$$z = x\,\text{F}(z)+y\,\Phi(z); \tag{93}$$

whence

$$\left(\frac{d^2z}{dx^2}\right)\left(\frac{dz}{dy}\right)^2 - 2\left(\frac{d^2z}{dx\,dy}\right)\left(\frac{dz}{dx}\right)\left(\frac{dz}{dy}\right) + \left(\frac{d^2z}{dy^2}\right)\left(\frac{dz}{dx}\right)^2 = 0. \tag{94}$$

SECTION 2.—*On surfaces generated by the motion of circles.*

369.] Surfaces of revolution.

A surface of revolution is generated by a curve which revolves about a straight line called the axis, and every point of which describes a circle about the axis.

Hence, if such a surface is cut by a plane perpendicular to the axis of revolution, the section is the circumference of a circle whose centre is on the axis, and all points of which are consequently at equal distances from the axis.

Let the equations to the axis be

$$\frac{\xi-a}{l} = \frac{\eta-b}{m} = \frac{\zeta-c}{n}, \tag{95}$$

(a, b, c) being a given point through which it passes, viz. the point A in fig. 127, and l, m, n being its direction-cosines.

Let x, y, z be the current coordinates of the surface; then the equation to the plane passing through (x, y, z), and perpendicular to (95), is

$$lx + my + nz = p. \tag{96}$$

Let BAQ be the axis of revolution, OB the perpendicular from the origin on it, RP the generating curve; and suppose the equation of it to be given in the form

$$\text{AP}^2 = f(\text{BQ}). \tag{97}$$

then
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = f(lx+my+nz); \tag{98}$$

and this is the general equation to surfaces of revolution.

If the axis of revolution is that of z, then $a = b = c = 0$, $l = m = 0$, $n = 1$, and

$$x^2 + y^2 + z^2 = f(z);$$

or, which is equivalent,
$$z = f(x^2 + y^2). \tag{99}$$

370.] To find the differential equation to surfaces of revolution.

Eliminating f from (98) according to the method of Art. 104, we have

$$m(x-a) - l(y-b) + \{n(x-a) - l(z-c)\}\left(\frac{dz}{dy}\right)$$
$$+ \{m(z-c) - n(y-b)\}\left(\frac{dz}{dx}\right) = 0. \tag{100}$$

Or putting (98) in the form

$$\text{F}(x, y, z) = \text{F}\{(x-a)^2 + (y-b)^2 + (z-c)^2, lx + my + nz\} = 0, \tag{101}$$

let us suppose, as in Art. 53, F_1 and F_2 to be the derived-functions of F, according as the first or second subject-variable varies, then

$$\left.\begin{aligned} \left(\frac{d\text{F}}{dx}\right) &= 2(x-a)\,\text{F}_1 + l\,\text{F}_2 \\ \left(\frac{d\text{F}}{dy}\right) &= 2(y-b)\,\text{F}_1 + m\,\text{F}_2 \\ \left(\frac{d\text{F}}{dz}\right) &= 2(z-c)\,\text{F}_1 + n\,\text{F}_2 \end{aligned}\right\}; \tag{102}$$

$$\{m(z-c) - n(y-b)\}\left(\frac{d\text{F}}{dx}\right) + \{n(x-a) - l(z-c)\}\left(\frac{d\text{F}}{dy}\right)$$
$$+ \{l(y-b) - m(x-a)\}\left(\frac{d\text{F}}{dz}\right) = 0. \tag{103}$$

The geometrical meaning of this condition is, that the normal to the surface always meets the axis of revolution whose equations are (95).

A plane which passes through the axis of revolution is called a Meridian plane; and its line of intersection with the surface of revolution is called a Meridian curve.

371.] Ex. 1. To find the equation to a surface described by a straight line revolving about the axis of z, which it does not meet.

Let the equations to the revolving line in a given position of it be

$$\frac{x'-\alpha}{\text{L}} = \frac{y'-\beta}{\text{M}} = \frac{z'-\gamma}{\text{N}};$$

then, if x and y are the current coordinates of the required surface, when $x'^2+y'^2 = x^2+y^2$, $z' = z$; but

$$x' = \alpha + \frac{\text{L}}{\text{N}}(z-\gamma), \qquad y' = \beta + \frac{\text{M}}{\text{N}}(z-\gamma);$$

$$\therefore \quad \left\{\alpha + \frac{\text{L}}{\text{N}}(z-\gamma)\right\}^2 + \left\{\beta + \frac{\text{M}}{\text{N}}(z-\gamma)\right\}^2 = x^2+y^2. \quad (104)$$

which is the equation to a hyperboloid of revolution of one sheet, the centre of which is on the axis of z.

Ex. 2. To determine the conditions that the general equation of the second degree,

$$\text{A}x^2+\text{B}y^2+\text{C}z^2+2\text{A}_1 yz+2\text{B}_1 zx+2\text{C}_1 xy$$
$$+2\text{A}_2 x+2\text{B}_2 y+2\text{C}_2 z+\text{K} = 0, \quad (105)$$

should express a surface of revolution.

The most general form that (98) admits of, so as to be an expression of the second degree is

$$(x-a)^2+(y-b)^2+(z-c)^2 = k^2(lx+my+nz)^2; \quad (106)$$

expanding which and equating coefficients of the same powers of the variables with those of (105), we have

$$\left.\begin{aligned} \text{A} &= 1-k^2l^2, \\ \text{B} &= 1-k^2m^2, \\ \text{C} &= 1-k^2n^2, \end{aligned}\right\} \qquad \left.\begin{aligned} \text{A}_1 &= -k^2mn, \\ \text{B}_1 &= -k^2nl, \\ \text{C}_1 &= -k^2lm \end{aligned}\right\};$$

$$\therefore \quad -k^2l^2 = \frac{\text{B}_1\text{C}_1}{\text{A}_1}, \qquad -k^2m^2 = \frac{\text{C}_1\text{A}_1}{\text{B}_1}, \qquad -k^2n^2 = \frac{\text{A}_1\text{B}_1}{\text{C}_1},$$

$$\therefore \quad \text{A} - \frac{\text{B}_1\text{C}_1}{\text{A}_1} = \text{B} - \frac{\text{C}_1\text{A}_1}{\text{B}_1} = \text{C} - \frac{\text{A}_1\text{B}_1}{\text{C}_1}. \quad (107)$$

372.] On tubular surfaces.

Tubular surfaces are the envelopes of spheres of constant radii, whose centres are situated in a given curve, which is called the axis of the tube or canal.

The general theory of envelopes having been explained in Chapter XIII, it is unnecessary to enter on the subject at any great length, but one or two points require further elucidation.

Let $F(x, y\ z, a) = 0$ be the equation to the surface, involving x, y, z its current coordinates and a a variable parameter; and therefore representing a family of surfaces as a varies, and a particular individual of it for a particular value of a. Then the equation to the envelope is found by eliminating a between

$$F = 0, \quad \text{and} \quad \frac{dF}{da} = 0, \tag{108}$$

whence will generally arise an equation in terms of x, y, z.

Now although (108) thus give the equation to a surface, yet, if a is considered a constant in them, each when taken separately represents a surface, and when taken together they represent the line of intersection of two surfaces, and which is in general a curve of double curvature. To this line Monge has given the name of the *characteristic*. Thus if we conceive developable surfaces as formed by the intersection of consecutive planes, since two planes intersect in a straight line, a straight line is the characteristic, and is the generator of the developable surface.

Further, let us suppose, after the characteristic has been found, the variable a to vary again; hereby another characteristic will be formed, and will be determined by those two different equations, which will in general be different in form and position, and will cut the former one; thus an envelope will be formed of such characteristics, which will be an edge of regression, see Art. 361, and of course generally a curve of double curvature. Thus we shall have three equations,

$$F = 0, \qquad \frac{dF}{da} = 0, \qquad \frac{d^2F}{da^2} = 0; \tag{109}$$

from which eliminating a, we shall have two equations in terms of x, y, z, which will by their intersection give the edge of regression, and such as we have before met with in the case of developable surfaces. Fig. 129 will perhaps give a better notion of the formation of such a curve; but we shall return to the subject in Chapter XVIII, and discuss it in a particular case at a length which will remove many difficulties.

The complete investigation of tubular surfaces requires three equations to be found: (1) that of the envelope of all the spheres: (2) that of the characteristic: (3) that of the edge of regression.

Let a = the constant radius of the sphere. And let the equations to the axis be expressed in terms of a single variable parameter α, so that the equation to a sphere may be

$$\{x-f(\alpha)\}^2+\{y-\phi(\alpha)\}^2+\{z-\psi(\alpha)\}^2 = a^2. \qquad (110)$$

The α-differential of this is

$$\{x-f(\alpha)\}f'(\alpha)+\{y-\phi(\alpha)\}\phi'(\alpha)+\{z-\psi(\alpha)\}\psi'(\alpha)=0; \qquad (111)$$

which, taken in combination with (110) when α is constant, represents the characteristic; and as (111) represents a plane, the characteristic is manifestly a great circle of the sphere.

Differentiating (111) again, we have

$$\{x-f(\alpha)\}f''(\alpha)+\{y-\phi(\alpha)\}\phi''(\alpha)+\{z-\psi(\alpha)\}\psi''(\alpha) \\ -\{(f'(\alpha))^2+(\phi'(\alpha))^2+(\psi'(\alpha))^2\} = 0. \qquad (112)$$

By means of which and (110) and (111) if α is eliminated, there will be two equations in terms of x, y, z, which, taken in combination, are those to the edge of regression formed by the characteristics.

373.] Now all tubular surfaces have a common property, which may be expressed as a differential equation. This we proceed to find.

Let the equation to the surface be $F(x,y,z)=0$, of which U, V, W are the partial derived-functions; and let the equation to the generating sphere be

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2 = a^2. \qquad (113)$$

Now (α, β, γ) being the centre of the sphere, and the centre being on a given curve, these quantities are connected by two equations of the form

$$F_1(\alpha,\beta,\gamma)=0, \qquad F_2(\alpha,\beta,\gamma)=0; \qquad (114)$$

and for the envelope of the spheres

$$(x-\alpha)\,d\alpha+(y-\beta)\,d\beta+(z-\gamma)\,d\gamma = 0, \qquad (115)$$

$d\alpha$, $d\beta$, $d\gamma$ having two other relations given by (114), into which however it is of no use generally to inquire further. Now differentiating (113) we have

$$(x-\alpha)(dx-d\alpha)+(y-\beta)(dy-d\beta)+(z-\gamma)(dz-d\gamma)=0; \quad (116)$$

and therefore by (115),

$$(x-\alpha)\,dx+(y-\beta)\,dy+(z-\gamma)\,dz = 0; \qquad (117)$$

which shews that a tangent to the sphere at the points of it which are common to the sphere and the envelope is perpendicular to the line drawn from that point to the centre of the sphere. This latter line therefore is the normal to the tubular surface; and as the direction-cosines of the normal of a surface are proportional to U, V, and W, we have

$$\frac{x-\alpha}{\mathrm{U}} = \frac{y-\beta}{\mathrm{V}} = \frac{z-\gamma}{\mathrm{W}} = \frac{a}{\mathrm{Q}}. \qquad (118)$$

Now as the envelope touches each of the enveloped spheres, and as the envelope and enveloped sphere have one and the same tangent plane, so for a point common to the two surfaces x, y, z, U, V, W, are the same, whether we consider the point as belonging to the sphere or to the envelope. And therefore we may differentiate the equations (118) under this condition; whence we have

$$\left.\begin{aligned} \mathrm{Q}\,dx &= a\,d\mathrm{U} - a\frac{\mathrm{U}}{\mathrm{Q}}\,d\mathrm{Q} \\ \mathrm{Q}\,dy &= a\,d\mathrm{V} - a\frac{\mathrm{V}}{\mathrm{Q}}\,d\mathrm{Q} \\ \mathrm{Q}\,dz &= a\,d\mathrm{W} - a\frac{\mathrm{W}}{\mathrm{Q}}\,d\mathrm{Q} \end{aligned}\right\}; \qquad (119)$$

and using the notation of Art. 361, equations (61),

$$\left.\begin{aligned} d\mathrm{U} &= u\,dx + w'dy + v'dz \\ d\mathrm{V} &= w'dx + v\,dy + u'dz \\ d\mathrm{W} &= v'dx + u'dy + w\,dz \end{aligned}\right\};$$

therefore the group (119) becomes

$$\left.\begin{aligned} (au-\mathrm{Q})\,dx + aw'dy + av'dz &= a\frac{\mathrm{U}}{\mathrm{Q}}\,d\mathrm{Q} \\ aw'dx + (av-\mathrm{Q})dy + au'dz &= a\frac{\mathrm{V}}{\mathrm{Q}}\,d\mathrm{Q} \\ av'dx + au'dy + (aw-\mathrm{Q})dz &= a\frac{\mathrm{W}}{\mathrm{Q}}\,d\mathrm{Q} \end{aligned}\right\}; \qquad (120)$$

Whence, by cross-multiplication,

$$dx\,\{(au-\text{Q})(av-\text{Q})(aw-\text{Q})$$
$$-a^2u'^2(au-\text{Q})-a^2v'^2(av-\text{Q})-a^2w'^2(aw-\text{Q})+2a^3u'v'w'\}$$
$$= a\frac{d\text{Q}}{\text{Q}}\{\text{U}(av-\text{Q})(aw-\text{Q})-aw'\text{V}(aw-\text{Q})$$
$$-av'\text{W}(av-\text{Q})+a^2u'(u'\text{U}+v'\text{V}+w'\text{W})-2a^2u'^2\text{U}\}; \quad (121)$$

and similarly may the values of dy and dz be found. Multiplying through therefore by U, V, W, adding, and by means of (17), Art. 332,

$$\text{U}^2(av-\text{Q})(aw-\text{Q})+\text{V}^2(aw-\text{Q})(au-\text{Q})+\text{W}^2(au-\text{Q})(av-\text{Q})$$
$$-2a\{u'\text{VW}(au-\text{Q})+v'\text{WU}(av-\text{Q})+w'\text{UV}(aw-\text{Q})\}$$
$$+a^2\{(u'\text{U}+v'\text{V}+w'\text{W})^2-2(u'^2\text{U}^2+v'^2\text{V}^2+w'^2\text{W}^2)\} = 0. \quad (122)$$

If $a=\infty$, this condition becomes identical with that given in equation (62) for developable surfaces.

If the director curve of the centre of the sphere is a plane curve in the plane of xy; then the equation to the sphere is

$$(x-\xi)^2+(y-\eta)^2+z^2 = a^2. \quad (123)$$

As the tubular surface is an envelope we may take partial differentials of this equation on the supposition that the terms involving the differentials of ξ and η vanish; so that

$$x-\xi+z\left(\frac{dz}{dx}\right) = 0, \qquad y-\eta+z\left(\frac{dz}{dy}\right) = 0;$$

$$\therefore \quad z^2\left(\frac{dz}{dx}\right)^2+z^2\left(\frac{dz}{dy}\right)^2 = a^2-z^2;$$

$$\therefore \quad 1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2 = \frac{a^2}{z^2}. \quad (124)$$

which is the differential equation of such tubular surfaces: and if it is expressed in terms of U, V, W, Q, it becomes

$$\text{Q}z = a\text{W}. \quad (125)$$

The geometrical meaning of (124) and (125) is, the part of the normal between the surface and the plane of xy is equal to the radius of the generating sphere.

374.] **Examples of tubular surfaces.**

Ex. 1. Let the axis of the tube be a straight line whose equations are

$$\frac{\xi}{\mathrm{L}} = \frac{\eta}{\mathrm{M}} = \frac{\zeta}{\mathrm{N}};$$

then the equation to the sphere is

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2 = a^2;$$

and therefore

$$\frac{d\xi}{\mathrm{L}} = \frac{d\eta}{\mathrm{M}} = \frac{d\zeta}{\mathrm{N}};$$

$$(x-\xi)\,d\xi+(y-\eta)\,d\eta+(z-\zeta)\,d\zeta = 0;$$

$$\therefore \quad (x-\xi)\mathrm{L}+(y-\eta)\mathrm{M}+(z-\zeta)\mathrm{N} = 0;$$

$$\therefore \quad \frac{\xi}{\mathrm{L}} = \frac{\eta}{\mathrm{M}} = \frac{\zeta}{\mathrm{N}} = \frac{\mathrm{L}\xi+\mathrm{M}\eta+\mathrm{N}\zeta}{\mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2} = \frac{\mathrm{L}x+\mathrm{M}y+\mathrm{N}z}{\mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2};$$

whence ξ, η, ζ may be found in terms of x, y, z; and thence, substituting in the equation to the sphere,

$$\{\mathrm{L}(\mathrm{M}y+\mathrm{N}z)-x(\mathrm{M}^2+\mathrm{N}^2)\}^2+\{\mathrm{M}(\mathrm{N}z+\mathrm{L}x)-y(\mathrm{N}^2+\mathrm{L}^2)\}^2$$
$$+\{\mathrm{N}(\mathrm{L}x+\mathrm{M}y)-z(\mathrm{L}^2+\mathrm{M}^2)\}^2 = a^2(\mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2)^2.$$

Ex. 2. To find the equation to the surface of a circular ring.

Let the director curve of the centre of the generating sphere be

$$\xi^2+\eta^2 = c^2;$$

so that the equation of the sphere is

$$(x-\xi)^2+(y-\eta)^2+z^2 = a^2.$$

$$\xi\,d\xi+\eta\,d\eta = 0, \qquad (x-\xi)\,d\xi+(y-\eta)\,d\eta = 0;$$

$$\therefore \quad \frac{\xi}{x-\xi} = \frac{\eta}{y-\eta} = \frac{c}{(a^2-z^2)^{\frac{1}{2}}}:$$

$$\therefore \quad \xi\{c+(a^2-z^2)^{\frac{1}{2}}\} = cx, \qquad \eta\{c+(a^2-z^2)^{\frac{1}{2}}\} = cy;$$

$$\therefore \quad \{c+(a^2-z^2)^{\frac{1}{2}}\}^2 = x^2+y^2;$$

which is the equation to the required surface.

CHAPTER XVII.

ON CURVATURE OF CURVES IN SPACE, AND ON CERTAIN KINDRED AFFECTIONS.

375.]* CERTAIN principles, names, and modes of estimation which were discussed in Chapter XII, as to the curvature of curves, are stated with breadth sufficient to include kindred properties of curves in space; a difference however of great importance exists between the two classes, and which it is necessary at once to bring out into greater prominence. In the former case the whole of the curve lies in one plane, and the curve is therefore called a plane curve; in the present, although every two consecutive elements, or every three consecutive points, must be in one plane, viz. the osculating plane, yet the third element, or the fourth point, may be, and generally will be, in a different plane. For this reason such curves are called non-plane curves, and from this general property arise other affections of a more complex character, and which we proceed to inquire into.

Consider a portion of a curve in space, at no point of the part of which under investigation is there a point of abrupt termination, or of discontinuity, and at which the derived-functions of the equations to the curve are not indeterminate. Now as every three consecutive points must be in one plane, and as the mode of estimating curvature as explained in Art. 290 requires only three points in the curve's plane, the principles therein investigated are immediately applicable, and we propose to apply them by a similar process, viz. by drawing in the osculating plane, which is the plane containing three consecutive points, two consecutive normals, which will generally meet at a finite distance from the curve; the ratio of the infinitesimal angle contained between which to the element of the curve is curvature, as we heretofore defined it; see Art. 280; but which, for the sake of greater distinctness, we shall now call *absolute curvature*; and in accordance with the expression we shall use the following terms: the *radius of absolute curvature* is the distance from the curve of the intersection of two consecutive normals drawn in the osculating plane; the *centre of absolute curvature* is the point of intersection of two such normals; the *angle of curvature* is the angle contained between them; and since two such normals are perpendicular to two consecutive tangents, it is equal to the angle between them, and is accordingly called the *angle of contingence*; see Art. 284.

---

* For a most masterly exposition of the properties considered in this Chapter, and for geometrical proofs of them by the infinitesimal method, the reader is requested to consult a Memoir by M. de Saint Venant in "Trentième Cahier du Journal de l'École Royal Polytechnique."—Bachelier, Paris, 1845.

Suppose (x, y, z), $(x+dx, y+dy, z+dz)$ to be two consecutive points on a curve, and ds to be the distance between them, or to be the length-element of the curve; and suppose two consecutive normals to be drawn in the osculating plane, and to contain between them the angle $d\tau$; then, if $\rho =$ the radius of absolute curvature, and $d\tau =$ the angle of curvature,

$$ds = \pm\, \rho\, d\tau, \tag{1}$$

$$\rho = \pm\, \frac{ds}{d\tau}. \tag{2}$$

The radius of absolute curvature, which is mathematically defined by equation (2), is therefore the distance from the curve at which two consecutive normals drawn in the osculating plane intersect, or is the ratio of the length-element of the curve to the angle of curvature.

376.] In a curve however of the most general nature, as we pass continuously along it, the third element will be in a plane different to that of the two preceding ones; that is, two consecutive osculating planes will be inclined to each other; or, what is the same thing, two consecutive binormals are not parallel. This then is an affection different to any of those of plane curves, and which has been called by various names*, "second curvature," "torsion," "flexure," "cambrure;" we shall call it *torsion*, and curves which are affected with it we shall call non-plane curves. In plane curves it vanishes, and is of greater or less amount according to the deviation of the curve from a plane curve; we propose to measure it according to the principles of Art. 281. If therefore ds is a length-element of the curve, and $d\omega$ is the angle contained between two consecutive binormals, the torsion will vary directly as $d\omega$, and inversely as ds. Let us therefore define as follows:

---

* On the use of these terms, consult Note I. of M. de Saint Venant's Memoir in the Journal de l'École Polytechnique.

$$\text{torsion} = \frac{d\omega}{ds}. \tag{3}$$

Let us imagine a circle whose radius is R, of which let two radii be drawn parallel to two consecutive binormals, so that $d\omega$ is the angle between them, and ds is the intercepted arc; then

$$\mathrm{R} = \frac{ds}{d\omega}; \tag{4}$$

and the torsion $= \frac{1}{\mathrm{R}}$; calling then R the radius of torsion, and $d\omega$ the angle of torsion, we have the following definition of R:

The radius of torsion is the ratio of the length-element of a curve to the angle of inclination of two consecutive binormals.

It is of course manifest that two consecutive binormals do not of necessity intersect; but this will appear more distinctly hereafter.

It is also to be observed, that the torsion vanishes in the case of plane curves, but that the absolute curvature vanishes only in the case of straight lines; hence we shall derive analytical conditions of lines in space being plane and being straight.

On account of these two affections, curves in space have been called curves of double curvature.

377.] On the radius of absolute curvature.

Let (x, y, z) be the point on the curve at which the radius of absolute curvature is drawn; ρ its length; ds a length-element of the curve; (ξ, η, ζ) the centre of curvature which lies in the osculating plane; then

$$\rho^2 = (\xi-x)^2+(\eta-y)^2+(\zeta-z)^2; \tag{5}$$

To abbreviate the notation, let

$$\left.\begin{aligned} dy\,d^2z - dz\,d^2y &= \mathrm{X}, \\ dz\,d^2x - dx\,d^2z &= \mathrm{Y}, \\ dx\,d^2y - dy\,d^2x &= \mathrm{Z}, \end{aligned}\right\} \tag{6}$$

$$\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = \mathrm{P}^2; \tag{7}$$

so that we have by reason of equation (14), Art. 343,

$$(\xi-x)\,\mathrm{X}+(\eta-y)\,\mathrm{Y}+(\zeta-z)\,\mathrm{Z} = 0. \tag{8}$$

And as the centre of absolute curvature is at the point of intersection of two consecutive normals which are in the osculating plane, it is on the line of intersection of two consecutive normal planes; whence we have

$$(\xi-x)\,dx+(\eta-y)\,dy+(\zeta-z)\,dz = 0, \tag{9}$$

$$(\xi-x)\,d^2x+(\eta-y)\,d^2y+(\zeta-z)\,d^2z = ds^2; \tag{10}$$

and therefore by cross-multiplication from (8), (9) and (10),

$$\left.\begin{aligned} \xi-x &= \frac{ds^2(\mathrm{Y}\,dz-\mathrm{Z}\,dy)}{\mathrm{P}^2}, \\ \eta-y &= \frac{ds^2(\mathrm{Z}\,dx-\mathrm{X}\,dz)}{\mathrm{P}^2}, \\ \zeta-z &= \frac{ds^2(\mathrm{X}\,dy-\mathrm{Y}\,dx)}{\mathrm{P}^2}; \end{aligned}\right\} \tag{11}$$

$$\rho = \pm\frac{ds^2}{\mathrm{P}^2}\{(\mathrm{Y}\,dz-\mathrm{Z}\,dy)^2+(\mathrm{Z}\,dx-\mathrm{X}\,dz)^2+(\mathrm{X}\,dy-\mathrm{Y}\,dx)^2\}^{\frac{1}{2}}. \tag{12}$$

Which expressions may be simplified as follows:

$$\begin{aligned} \mathrm{Y}\,dz-\mathrm{Z}\,dy &= dz\,(dz\,d^2x-dx\,d^2z)-dy\,(dx\,d^2y-dy\,d^2x), \\ &= (dx^2+dy^2+dz^2)\,d^2x-dx\,(dx\,d^2x+dy\,d^2y+dz\,d^2z), \\ &= ds^2\,d^2x-dx\,ds\,d^2s, \qquad (13) \\ &= ds^3\,d.\frac{dx}{ds}. \qquad (14) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{Z}\,dx-\mathrm{X}\,dz &= ds^2\,d^2y-dy\,ds\,d^2s, \qquad (15) \\ &= ds^3\,d.\frac{dy}{ds}; \qquad (16) \end{aligned}$$

$$\begin{aligned} \mathrm{X}\,dy-\mathrm{Y}\,dx &= ds^2\,d^2z-dz\,ds\,d^2s, \qquad (17) \\ &= ds^3\,d.\frac{dz}{ds}; \qquad (18) \end{aligned}$$

$$\begin{aligned} \mathrm{P}^2 &= \mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2, \\ &= ds^2\,\{(d^2x)^2+(d^2y)^2+(d^2z)^2-(d^2s)^2\}. \qquad (19) \end{aligned}$$

Therefore from (13), (15), (17) and (19),

$$\frac{ds^2}{\rho} = \pm\{(d^2x)^2+(d^2y)^2+(d^2z)^2-(d^2s)^2\}^{\frac{1}{2}}; \quad 20)$$

and from (19), $$\rho^2 = \frac{ds^6}{\mathrm{P}^2}; \tag{21}$$

and therefore from (14), 16), (18), and (21),

$$\left.\begin{aligned} \xi - x &= \frac{\rho^2}{ds} d.\frac{dx}{ds}, \\ \eta - y &= \frac{\rho^2}{ds} d.\frac{dy}{ds}, \\ \zeta - z &= \frac{\rho^2}{ds} d.\frac{dz}{ds}; \end{aligned}\right\} \tag{22}$$

whence, squaring and adding, and by means of (5),

$$\frac{1}{\rho} = \pm \frac{1}{ds} \left\{ \left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2 + \left(d.\frac{dz}{ds}\right)^2 \right\}^{\frac{1}{2}}; \tag{23}$$

(20) and (23) being the same values of $\frac{1}{\rho}$, though in different forms.

If s is equicrescent, (23) may be put in the form

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right) + \left(\frac{d^2z}{ds^2}\right)^2. \tag{24}$$

378.] Let λ, μ, ν be the direction-angles of the radius of absolute curvature; then

$$\left.\begin{aligned} \cos\lambda &= \frac{\xi - x}{\rho} = \frac{\rho}{ds} d.\frac{dx}{ds}, \\ \cos\mu &= \frac{\eta - y}{\rho} = \frac{\rho}{ds} d.\frac{dy}{ds}, \\ \cos\nu &= \frac{\zeta - z}{\rho} = \frac{\rho}{ds} d.\frac{dz}{ds} \end{aligned}\right\}; \tag{25}$$

and if s is equicrescent,

$$\cos\lambda = \rho\frac{d^2x}{ds^2}, \qquad \cos\mu = \rho\frac{d^2y}{ds^2}, \qquad \cos\nu = \rho\frac{d^2z}{ds^2}; \tag{26}$$

on comparing which results with Art. 346, it appears that the radius of absolute curvature coincides in direction with the principal normal.

379.] To determine the angle of curvature.

Since by equation (2), Art. 375, $d\tau = \pm \frac{ds}{\rho}$; therefore from equation (23),

$$d\tau = \left\{ \left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2 + \left(d.\frac{dz}{ds}\right)^2 \right\}^{\frac{1}{2}}; \tag{27}$$

$$= ds\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right\}^{\frac{1}{2}}, \qquad (28)$$

if s is equicrescent.

Now equation (27) is remarkable and deserves attention; $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the direction-cosines of the tangent, and $d\tau$ is the infinitesimal angle between it and the consecutive tangent. This result therefore may be generalised. Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$, $\cos\alpha+d.\cos\alpha$, $\cos\beta+d.\cos\beta$, $\cos\gamma+d.\cos\gamma$ be the direction-cosines of two lines inclined to each other at an infinitesimal angle; then that infinitesimal angle is equal to

$$\{(d.\cos\alpha)^2+(d.\cos\beta)^2+(d.\cos\gamma)^2\}^{\frac{1}{2}}\,*. \qquad (29)$$

380.] We may also, as follows, immediately obtain the value of ρ, and as the process exhibits the power of the infinitesimal method in its greatest simplicity, we recommend the reader to study it attentively.

Let P, Q, R, fig. 128, be three consecutive points in the curve, all three lying in the plane of the paper which is the osculating plane. Then PQ $= ds$, QR $= ds+d^2s$; produce PQ to S, making QS $=$ PQ; complete the parallelograms PQRV, VQSR, so that QS $=$ VR $=$ PQ $= ds$, then the angle RQS $= d\tau$; and from Q as a centre, and with QS as a radius, describe a small arc ST, which is therefore perpendicular to QR; hence QT $=$ QS $= ds$; and therefore RT $= d^2s$, and ST $= ds\times d\tau$.

Now the projections on the coordinate axes of PQ are dx, dy, dz; and of QR, or of PV, $dx+d^2x$, $dy+d^2y$, $dz+d^2z$; therefore the projections of VQ are d^2x, d^2y, d^2z; and therefore

$$\text{VQ}^2 = (d^2x)^2+(d^2y)^2+(d^2z)^2;$$

but

$$\begin{aligned}\text{VQ}^2 &= \text{SR}^2,\\ &= \text{ST}^2+\text{TR}^2,\\ &= ds^2\,d\tau^2+(d^2s)^2;\end{aligned}$$

---

* Or thus: suppose $\cos\alpha$, &c. $\cos\alpha+d.\cos\alpha$, &c. to be the direction-cosines of two lines making an infinitesimal angle $d\tau$ with each other; take along the lines two distances each equal to unity, and commencing from the point of intersection of the lines. Then the projections of these distances on the coordinate axes are respectively $\cos\alpha\ldots$, $\cos\alpha+d.\cos\alpha\ldots$; and therefore the projections of the line joining their extremities are $d.\cos\alpha$, $d.\cos\beta$, $d.\cos\gamma$, but this line measures $d\tau$, since it subtends $d\tau$, at a distance unity; and therefore $d\tau^2 = (d.\cos\alpha)^2+(d.\cos\beta)^2+(d.\cos\gamma)^2$.

equating which values of VQ^2, we have

$$ds^2\, d\tau^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2. \qquad (30)$$

And therefore by (2),

$$\frac{1}{\rho} = \pm \frac{1}{ds^2}\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}^{\frac{1}{2}}. \qquad (31)$$

Hence also we have the following value of the angle of contingence,

$$d\tau = \frac{1}{ds}\{(d^2x)^2 + (d^2y)^2 + (d^2z) - (d^2s)^2\}^{\frac{1}{2}}. \qquad (32)$$

381.] Again consider fig. 128, and project the parallelogram PQRV on the plane of xy; the projections on the axes of x and y of the sides PQ and QR are severally dx, dy, and $dx + d^2x$, $dy + d^2y$; so that, as was shewn in Art. 295, the area of the projected parallelogram is equal to $dx\, d^2y - dy\, d^2x$, that is, to the quantity represented by Z in Art. 377. Similarly the projections of the parallelogram PQRV on the planes of yz and zx are X and Y; also, as in Art. 295,

$$\begin{aligned} \text{area of PQRV} &= \mathrm{PQ} \times \mathrm{QR} \times \sin \mathrm{RQS}, \\ &= ds\,(ds + d^2s) \sin d\tau, \\ &= ds^2 \times d\tau, \\ &= \frac{ds^3}{\rho}. \end{aligned} \qquad (33)$$

Hence by a property of projected areas,

$$\frac{ds^6}{\rho^2} = \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2;$$

the same as equation (21). Hence also we have the following values of the direction-cosines of the binormal, which is perpendicular to the plane in which the parallelogram PQRS lies, viz.

$$\frac{\rho \mathrm{X}}{ds^3}, \quad \frac{\rho \mathrm{Y}}{ds^3}, \quad \frac{\rho \mathrm{Z}}{ds^3}. \qquad (34)$$

382.] On torsion and the radius of torsion.

Since in Art. 376 $d\omega$ is the angle between two consecutive binormals, and since the direction-cosines of the first binormal are

$$\frac{\mathrm{X}}{\mathrm{P}}, \quad \frac{\mathrm{Y}}{\mathrm{P}}, \quad \frac{\mathrm{Z}}{\mathrm{P}},$$

or in terms of (34), $\frac{\rho \mathrm{X}}{ds^3}, \quad \frac{\rho \mathrm{Y}}{ds^3}, \quad \frac{\rho \mathrm{Z}}{ds^3};$

it follows from equation (29), Art. 379, that

$$d\omega = \left\{\left(d.\frac{X}{P}\right)^2 + \left(d.\frac{Y}{P}\right)^2 + \left(d.\frac{Z}{P}\right)^2\right\}^{\frac{1}{2}}; \tag{35}$$

$$\therefore \quad d\omega^2 = \left(\frac{P\,dX - X\,dP}{P^2}\right)^2 + \left(\frac{P\,dY - Y\,dP}{P^2}\right)^2 + \left(\frac{P\,dZ - Z\,dP}{P^2}\right)^2, \tag{36}$$

$$= \frac{P^2(dX^2 + dY^2 + dZ^2) - P^2\,dP^2}{P^4},$$

$$= \frac{(X^2+Y^2+Z^2)(dX^2+dY^2+dZ^2) - (X\,dX + Y\,dY + Z\,dZ)^2}{(X^2+Y^2+Z^2)^2},$$

$$= \frac{(Y\,dZ - Z\,dY)^2 + (Z\,dX - X\,dZ)^2 + (X\,dY - Y\,dX)^2}{(X^2+Y^2+Z^2)^2}. \tag{37}$$

But differentiating the several terms of (6), we have

$$\left.\begin{aligned} dX &= dy\,d^3z - dz\,d^3y, \\ dY &= dz\,d^3x - dx\,d^3z, \\ dZ &= dx\,d^3y - dy\,d^3x; \end{aligned}\right\} \tag{38}$$

$$\therefore \quad Y\,dZ - Z\,dY = Y(dx\,d^3y - dy\,d^3x) - Z(dz\,d^3x - dx\,d^3z),$$

$$= dx(X\,d^3x + Y\,d^3y + Z\,d^3z) - d^3x(X\,dx + Y\,dy + Z\,dz):$$

but $$X\,dx + Y\,dy + Z\,dz = 0;$$

$$\therefore \quad Y\,dZ - Z\,dY = dx\,(X\,d^3x + Y\,d^3y + Z\,d^3z); \tag{39}$$

and as similar values are true for the other terms of the numerator of (37),

$$d\omega^2 = \frac{ds^2}{R^2} = \frac{ds^2\,(X\,d^3x + Y\,d^3y + Z\,d^3z)^2}{(X^2+Y^2+Z^2)^2};$$

$$\therefore \quad \frac{1}{R} = \frac{X\,d^3x + Y\,d^3y + Z\,d^3z}{X^2+Y^2+Z^2}. \tag{40}$$

And since $X^2 + Y^2 + Z^2 = \dfrac{ds^6}{\rho^2}$,

$$\therefore \quad \frac{1}{R} = \frac{\rho^2}{ds^6}\,\{X\,d^3x + Y\,d^3y + Z\,d^3z\}; \tag{41}$$

and $$d\omega = \frac{\rho^2}{ds^5}\,\{X\,d^3x + Y\,d^3y + Z\,d^3z\}. \tag{42}$$

Also substituting the second set of the values of the direction-cosines, given in (34),

$$\frac{ds}{R} = \pm\left\{\left(d.\frac{\rho X}{ds^3}\right)^2 + \left(d.\frac{\rho Y}{ds^3}\right)^2 + \left(d.\frac{\rho Z}{ds^3}\right)^2\right\}^{\frac{1}{2}}. \tag{43}$$

383.] In reference to singular forms of the values of $d\tau$ and $d\omega$ which have been determined by equations (27) and (42), it is to be observed that

(α) If at any point of a curve $d\tau = 0$, or, which is the same thing,

$$d.\frac{dx}{ds} = 0, \quad d.\frac{dy}{ds} = 0, \quad d.\frac{dz}{ds} = 0, \tag{44}$$

then there is no change of direction of the tangent as we pass to a third point on the curve; that is, the curve becomes a straight line, or the curvature is suspended, and there is what is called a point of suspended curvature.

(β) And if $d\tau = 0$, and changes its sign, then not only does the curve at the point run into and coincide with its tangent, but it intersects it; and we have what corresponds to a point of inflexion of plane curves, and is called a point of Inflected Curvature.

(γ) And if at all points of a curve in space $d\tau = 0$, then the line is straight, as is apparent from the three conditions (44).

(δ) If at any point $d\omega = 0$, or what is equivalent,

$$\mathrm{X}\, d^3x + \mathrm{Y}\, d^3y + \mathrm{Z}\, d^3z = 0, \tag{45}$$

the osculating plane does not change position as we pass from a third to a fourth consecutive point, and the torsion at that point is suspended, or the curve is plane; such a point is conveniently called a point of Suspended Torsion.

(ϵ) And if $d\omega = 0$, and changes its sign, then the direction in which the torsion takes place is changed, and we have what is called a point of Inflected Torsion.

(ζ) And if at all points of a curve $d\omega = 0$, then the binormals are all parallel, and the curve is plane. This, it is to be observed, is the same condition as that before found in Art. 348.

384.] We proceed now to consider certain surfaces which are generated by planes and straight lines, the positions and directions of which depend on properties of curves of double curvature; and first let us consider the effects of the intersection of consecutive normal planes.

Two consecutive normal planes of course generally intersect; for did they not, the elements of the curve to which they are perpendicular would be in the same straight line, which is true of only particular parts of curves; and they intersect in a straight line, which, in accordance with the nomenclature of M. de Saint Venant, and taken from Monge and Lancret, we will call the *polar line*: the polar line manifestly passes through the centre of absolute curvature, and is perpendicular to the osculating plane. Consider again a third consecutive normal plane; this will intersect the second one in a straight line, which is a consecutive polar line; and so on; thus a developable surface will be formed by the intersection of such normal planes; and as it is also generated by the polar lines according to the properties of developable surfaces investigated in the last Chapter, it is called the Polar Surface.

Also a curve is formed by the intersection of the polar lines; for the lines of intersection of the first two and of the last two of three consecutive normal planes lie in the second plane, and, not being parallel, necessarily intersect, and must by their intersection form a curve of double curvature, which is the edge of regression of the polar surface. To this edge of regression therefore the polar lines are tangents; and as two consecutive polar lines are perpendicular to two consecutive osculating planes of the original curve, it follows that the angle of curvature of this edge of regression is equal to the corresponding angle of torsion of the primitive curve. Also it is manifest from the mode of generation that the normal planes of the primitive curve are osculating planes of the edge of regression; and therefore the angle of torsion of the edge of regression is equal to the angle of curvature of the primitive curve. We cannot however hence conclude, that the curvatures and torsions of the two curves are reciprocally equal, because the lengths of the corresponding elements of the two curves are not always equal.

385.] Investigation of mathematical expressions connected with polar surfaces and polar lines.

Let ξ, η, ζ be the current coordinates of the normal plane; (x, y, z) the point on the curve at which it is drawn; then the equation to the normal plane is

$$(\xi-x)\,dx+(\eta-y)\,dy+(\zeta-z)\,dz = 0; \qquad (46)$$

and the equation to the consecutive normal plane is

$$(\xi-x)\,d^2x+(\eta-y)\,d^2y+(\zeta-z)\,d^2z = ds^2; \qquad (47)$$

by means of which, and of the two equations to the curve, x, y, z

may be eliminated, and the resulting equation in terms of ξ, η, ζ is that of the polar surface.

386.] Again, to find the equations to the polar line.

Let ξ, η, ζ be the current coordinates of the line; then, as it passes through the centre of absolute curvature, and is perpendicular to the osculating plane, or is parallel to the binormal, its equations are, by means of (22),

$$\frac{\xi - x - \frac{\rho^2}{ds} d.\frac{dx}{ds}}{\text{X}} = \frac{\eta - y - \frac{\rho^2}{ds} d.\frac{dy}{ds}}{\text{Y}} = \frac{\zeta - z - \frac{\rho^2}{ds} d.\frac{dz}{ds}}{\text{Z}}. \qquad (48)$$

387.] Also to find the coordinates of the point of intersection of two consecutive polar lines: that is, the coordinates to a point on the edge of regression of the polar surface: we must again differentiate (47), whereby we have

$$(\xi - x) d^3x + (\eta - y) d^3y + (\zeta - z) d^3z = 3\, ds\, d^2s; \qquad (49)$$

and by cross-multiplication between (46), (47), and (49),

$$(\xi - x)(\text{X}\, d^3x + \text{Y}\, d^3y + \text{Z}\, d^3z) = 3\, ds\, d^2s\, \text{X} - ds^2 d\text{X}, \qquad (50)$$

$$= -ds^5 d.\frac{\text{X}}{ds^3}; \qquad (51)$$

and similar values are true for the symmetrical expressions. Therefore by means of (41),

$$\left.\begin{aligned} \xi - x &= -\frac{\rho^2 \text{R}}{ds} d.\frac{\text{X}}{ds^3}, \\ \eta - y &= -\frac{\rho^2 \text{R}}{ds} d.\frac{\text{Y}}{ds^3}, \\ \zeta - z &= -\frac{\rho^2 \text{R}}{ds} d.\frac{\text{Z}}{ds^3}. \end{aligned}\right\} \qquad (52)$$

By means of which and the two equations to the curve, x, y, z may be eliminated, and the two resulting equations in terms of ξ, η, ζ will be those to the edge of regression of the polar surface.

388.] Let us further consider these properties in relation to a sphere which has contact with the primitive curve.

As a circle is definite when it passes through three consecutive points on a curve, and cannot in general pass through more, so a sphere is definite when it passes through four points, provided that these four points have not such a position as to give a singular form to the sphere, as for instance to make it a plane; this is manifest geometrically, inasmuch as an infinite number of spheres may be made to pass through three points, but of these that which passes through the fourth point has a definite radius; and also algebraically, because the equation to a sphere involves four arbitrary constants, and these may be expressed in terms of the coordinates of four given points. Consider then a sphere to be so placed as to pass through a given point on a curve, and to touch the tangent to the curve at the point; the sphere passes through two consecutive points on the curve, and its centre may be at any point which is equidistant from these two; it may therefore be any where in the normal plane. Suppose also that the sphere passes through a third consecutive point in addition to the former two, then its centre must also be in the second consecutive normal plane, and must therefore be in the intersection of these two normal planes; that is, it must be in the polar line, but it may be at any point in that line; that line therefore may be considered as the locus of the centres of spheres which pass through the same three consecutive points on a curve. Suppose again that the sphere also passes through four consecutive points, then its centre must be in the point at which the polar line is intersected by its consecutive polar line, and is therefore at a definite point, and the radius of the sphere is of definite length. The point then at which the centre of such a sphere must be placed is a definite point on the edge of regression of the polar surface; such edge therefore may be defined to be, the locus of the centres of spheres which pass through four consecutive points on a curve of double curvature.

Therefore ξ, η, ζ of equations (52) are the coordinates to the centre of the osculating sphere, and $\xi - x$, $\eta - y$, $\zeta - z$ are the projections on the coordinate axes of its radius. Now after some long but not difficult reductions, equations (52) assume the forms

$$\left.\begin{aligned} \xi - x &= \rho \frac{\rho}{ds} d.\frac{dx}{ds} + \frac{\text{R}\, d\rho}{ds} \frac{\rho \text{X}}{ds^3}, \\ \eta - y &= \rho \frac{\rho}{ds} d.\frac{dy}{ds} + \frac{\text{R}\, d\rho}{ds} \frac{\rho \text{Y}}{ds^3}, \\ \zeta - z &= \rho \frac{\rho}{ds} d.\frac{dz}{ds} + \frac{\text{R}\, d\rho}{ds} \frac{\rho \text{Z}}{ds^3}; \end{aligned}\right\} \qquad (53)$$

and since $\frac{\rho}{ds} d.\frac{dx}{ds}, \ldots\ldots$ are the direction-cosines of the radius of curvature, and $\frac{\rho \mathrm{X}}{ds^3}$, ... are the direction-cosines of the polar line, it follows that the projections on the coordinate axes of the radius of the osculating sphere is equal to the sum of the projections on the same axes of the radius of absolute curvature, and of a line equal in length to $\frac{\mathrm{R}\,d\rho}{ds}$ measured from the centre of curvature along the polar line; and therefore as the polar line is perpendicular to the radius of absolute curvature,

$$\text{the radius of the osculating sphere} = \left\{\rho^2 + \left(\frac{\mathrm{R}\,d\rho}{ds}\right)^2\right\}^{\frac{1}{2}}. \qquad (54)$$

And if the radius of absolute curvature of a curve is constant for all points of the curve, so that $d\rho = 0$, then the centres of absolute curvature and of the osculating sphere are coincident; and if at a point on a curve ρ is a maximum or a minimum, the same result follows.

And by differentiating (52) or (53) we may find $d\xi$, $d\eta$, $d\zeta$, and thereby the length of an element of the edge of regression of the polar surface; and thence the curvature and torsion of the edge of regression in terms of the coordinates of the corresponding point of the primitive curve.

389.] The above investigations lead us immediately to an inquiry respecting those properties of curves of double curvature which are analogous to evolutes of plane curves.

Let a normal line be drawn at any point of a curve of double curvature; it will be in the normal plane at the point, and will therefore touch the polar surface.

Now conceive a second and consecutive normal plane to be drawn; it will meet the first normal line on the polar surface, and at the point of meeting let a normal line be drawn to the curve: and conceive again a third consecutive normal plane to be drawn, and to meet the second normal line: and another normal line to be drawn to the curve, and by a method similar to the former one: and so on; then will a curve be described on the polar surface, the elements of which are elements of these successive normal lines, and which curve is such that if a perfectly flexible and inextensible string is fixed at any point of it, and of such a length as when stretched will reach to the curve; then, if it be wrapped round the polar surface and along, and tangential to, the curve thereon described, the extremity of the string will describe the original curve of double curvature. On this account the curve described on the polar surface is called the Evolute, and the original curve is called the Involute with respect to it.

Thus, let (x, y, z) be the point on the original curve at which the normal line is drawn, and let the point on the polar surface at which the normal meets it be (ξ, η, ζ), and let $d\sigma$ be a length-element of the evolute; and let r be the distance between the two points; then, as r is to be a tangent to the evolute,

$$\frac{d\xi}{d\sigma} = \frac{x-\xi}{r}, \quad \frac{d\eta}{d\sigma} = \frac{y-\eta}{r}, \quad \frac{d\zeta}{d\sigma} = \frac{z-\zeta}{r};$$

Also
$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2,$$

$$-r\,dr = (x-\xi)\,d\xi + (y-\eta)\,d\eta + (z-\zeta)\,d\zeta,$$

$$-dr = \frac{d\xi^2 + d\eta^2 + d\zeta^2}{d\sigma} = \pm\, d\sigma;$$

$$\therefore \quad dr \pm d\sigma = 0;$$

and taking the negative sign $r - \sigma =$ a constant; so much of the string therefore is taken off from its length by the wrapping, as to leave the remainder equal to the distance of the point on the old curve from the point on the evolute where the wrapping ends.

Hence if from two points on an evolute tangents to the evolute are drawn to the involute, the difference of their lengths is equal to the length of the arc of the evolute between the points of contact.

As the basis of the construction of the evolute thus far has been an arbitrarily chosen normal line at a given point of the original curve, so may any other normal line be taken; and thus there may be any number of evolutes, all of which will be on the polar surface, and which may therefore be considered as the locus surface of such evolutes.

The locus of the centres of absolute curvature is not an evolute, although it is a curve described on the polar surface; and for this reason; suppose P, Q, R ...... to be consecutive points on the curve, and p, q, r ...... to be the centres of curvature corresponding to the points; then, if the line pqr ...... were an evolute of PQR ......, the arc pq should lie in the line Pp produced, the arc qr should lie in Qq produced, and so on: and such can be the case only when P*p*, Q*q*, R*r* ...... are two and two in the same plane; but all these lie severally in different planes, viz. in the osculating planes at P, Q, R ...... The radii of absolute curvature therefore cannot meet and by their intersections form an evolute, unless all are in the same plane, or, in other words, unless the curve is plane. A surface however is generated by such radii of curvature, which is ruled, but of the class termed Skew.

There is also another remarkable property of evolutes of curves of double curvature, viz. when the polar surface is developed into a plane, they become straight lines all diverging from the same point. For consider two consecutive normal planes P*p* and Q*q* drawn at P and Q, and imagine the second one to turn about the polar line in P*p*, until the two planes coincide; then, as is manifest from the construction, Q falls on and coincides with P, and the line P*p* coincides with Q*q*, so that *pq*, the element of the evolute, coincides in direction with P*p*; and as a similar result would follow from a similar operation being performed on the other normal planes, it follows that the evolute of which *pq* is an element becomes a straight line emanating from P, the point into which the whole original curve becomes absorbed; and as a similar result is true of all the other evolutes, it follows that when the polar surface is developed, the evolutes become a pencil of straight lines diverging from the point into which the curve falls.

And as the length of an element of an evolute is not altered by the development, it follows that the element of an evolute is the shortest distance between the two extremities of the element. The evolute therefore is the geodesic between any two points on the polar surface.

390.] To enable the reader to obtain an adequate conception of the results of the last few Articles, the geometrical figure 129* is given.

Let P_1, P_2, P_3, P_4 ...... be successive points in a curve of double curvature; and through the middle points M_1, M_2, M_3, ... of successive elements let the normal planes L_1, L_2, L_3, ... be drawn, intersecting each other consecutively in the straight lines

---

* The figure is the same as that given by Monge in his "Application d'Analyse," and thence has been copied into most of the ordinary text books.

A_1B_1, A_2B_2, ... which are therefore the polar lines; and the surface formed by the intersection of the normal planes is the developable polar surface; then, as all the elements of the curve are not in the same plane, the polar lines are not parallel, and therefore intersect consecutively, and thereby form an envelope, viz. the non-plane curve $Q_1Q_2Q_3$......, which is the edge of regression of the polar surface.

Also let the normal planes L_1, L_2 be cut by the osculating plane containing the elements P_1P_2 and P_2P_3, and which is therefore perpendicular to L_1 and L_2, and let the lines of intersection be M_1C_1, M_2C_1; then C_1 is the centre of absolute curvature of the curve at P_1, and P_1C_1 is the radius of absolute curvature.

Again, let the consecutive osculating plane be drawn containing the elements P_2P_3 and P_3P_4, and let its lines of intersection with L_2 and L_3 be M_2C_2 and M_3C_2; then C_2 is the centre of absolute curvature of the curve at P_2, and P_2C_2 is the radius of absolute curvature. It is manifest now that the line M_2C_2 does not coincide with M_2C_1, because they are the lines of intersection of the same plane L_2 by different planes; M_2C_2 therefore does not cut A_1B_1 in the point C_1; and therefore the consecutive radii M_1C_1 and M_2C_2 do not meet. The successive centres of curvature therefore do not arise from the intersection of consecutive radii of curvature, and consequently these radii are not tangents to the locus of the centres; and therefore it follows that the curve C_1C_2...... cannot be regarded as an evolute of the original curve. It is manifest however from the construction that such will be the case, if the original curve is plane.

The diagram gives us also a clear notion of the formation of evolutes. From M_1 let any line M_1D_1 be drawn in the normal plane, meeting at D_1 the polar surface, to which it is tangential; and from D_1 let the line D_1M_2 be drawn to M_2, the middle point of the next element; then this line lies in the consecutive normal plane, and is tangential to the polar surface, and has an element D_1D_2 in contact with it; and let a similar process be continued on other consecutive normal planes; then there will be described on the polar surface a curve D_1D_2......, such that each successive element on it being produced will pass through and be normal to the original curve, and such that the difference between two successive lines drawn from the new to the old curve is equal to an element of the new curve, as is manifest from the construction; therefore the curve $D_1 D_2 \dots\dots$ is an evolute to the original curve, which is called an involute relatively to it. Hereby also a developable surface will be formed of which the curve $D_1 D_2 \dots\dots$ is the edge of regression, the osculating planes to which are those containing the successive elements of the original curve.

And as the first normal $P_1 D_1$ was drawn to a point D_1, chosen arbitrarily, so might any other point have been taken on the polar line $B_1 A_1 Q_1$; and thus there may be any number of evolutes on the polar surface, and the polar surface may be considered as the locus-surface of such curves.

It is manifest from the geometry that if the polar surface is developed into a plane, $D_1 D_2 \dots\dots$ would become a straight line; therefore $D_1 D_2 \dots\dots$ is a geodesic line on the polar surface.

391.] From the figure also may be deduced results, some of which are identical with those above investigated by algebraical methods.

Through the points $M_1, M_2, \dots\dots$, the middle points of successive elements, let there be drawn the binormals $M_1 L_1$, $M_2 L_2$, $\dots\dots$ which are parallel to the polar lines $A_1 B_1$, $A_2 B_2 \dots\dots$; then the angle $A_1 Q_1 A_2$, being that between two successive polar lines, is equal to the angle of torsion. Also let the line $M_2 C_2$ cut the polar line $A_1 B_1$ in I; then we have the following values:

$$M_1 C_1 = \rho = \text{the radius of absolute curvature,}$$

$$M_2 I = d\rho = \text{the differential of ditto,}$$

$$M_1 Q_1 = \text{the radius of the osculating sphere,}$$

$$A_1 Q_1 A_2 = \text{the angle of torsion} = d\omega = \frac{ds}{R}.$$

Since then $C_2 I$ may be considered as an arc of a circle, subtending an angle $A_1 Q_1 A_2$ at Q_1, we have

$$C_2 I = Q_1 C_2 \times \text{angle } A_1 Q_1 A_2;$$

$$\therefore \quad d\rho = Q_1 C_2 \times d\omega,$$

$$\therefore \quad Q_1 C_2 = \frac{d\rho}{d\omega} = R\frac{d\rho}{ds}; \tag{55}$$

and since $M_1 C_1 Q_1$ is a right angle,

$$M_1 Q_1^2 = M_1 C_1^2 + C_1 Q_1^2, \quad .$$

$$\{\text{the radius of the osculating sphere}\}^2 = \rho^2 + \left(\mathrm{R}\frac{d\rho}{ds}\right)^2; \quad (56)$$

$$\tan \mathrm{Q}_1\mathrm{M}_1\mathrm{C}_1 = \frac{\mathrm{Q}_1\mathrm{C}_1}{\mathrm{P}_1\mathrm{C}_1} = \frac{\mathrm{R}\,d\rho}{\rho\, ds}. \quad (57)$$

Also as $\mathrm{M}_2\mathrm{C}_1$, $\mathrm{M}_2\mathrm{C}_2$ are the lines of intersection of the normal plane at P_2, by two consecutive osculating planes which are both perpendicular to it, the angle $\mathrm{C}_1\mathrm{M}_2\mathrm{C}_2$ is that between the osculating planes, and is therefore equal to the angle of torsion: that is,

$$\mathrm{C}_1\mathrm{M}_2\mathrm{C}_2 = d\omega;$$

and as $\mathrm{M}_2\mathrm{C}_1$ is perpendicular to $\mathrm{A}_1\mathrm{B}_1$, $\mathrm{C}_1\mathrm{I}$ may be considered the arc of a circle subtending at M_2 an angle $d\omega$; therefore

$$\begin{aligned} \mathrm{C}_1\mathrm{I} &= \mathrm{M}_2\mathrm{C}_1 \times \text{angle } \mathrm{C}_1\mathrm{M}_2\mathrm{C}_2, \\ &= \rho\, d\omega; \end{aligned} \quad (58)$$

and therefore if $\mathrm{C}_1\mathrm{C}_2$ is joined, $\mathrm{C}_1\mathrm{C}_2$ is an element of the curve-locus of the centres of curvature;

$$\begin{aligned} \therefore \quad \mathrm{C}_1\mathrm{C}_2^2 &= \mathrm{C}_1\mathrm{I}^2 + \mathrm{I}\mathrm{C}_2^2, \\ &= \rho^2\, d\omega^2 + d\rho^2. \end{aligned} \quad (59)$$

If the curve is plane $d\omega = 0$, and $\mathrm{C}_1\mathrm{C}_2 = d\rho$, in which case $\mathrm{C}_1\mathrm{C}_2$ is an element of the evolute of the original curve; but if the original curve is non-plane, the curve $\mathrm{C}_1\mathrm{C}_2, \ldots\ldots$ is not an evolute.

As $\mathrm{C}_1\mathrm{C}_2$ is an element of the curve-locus of the centres of absolute curvature, it is equal to $\{d\xi^2 + d\eta^2 + d\zeta^2\}^{\frac{1}{2}}$ of the expressions given in equations (11) or (22) of the present Chapter; that is,

$$d\rho^2 + \rho^2 d\omega^2 = \left\{dx + d.\left(\frac{\rho^2}{ds}\, d.\frac{dx}{ds}\right)\right\}^2 + \ldots\ldots + \ldots; \quad (60)$$

whence, after several reductions,

$$\frac{ds^2}{\rho^2} + \frac{ds^2}{\mathrm{R}^2} = \left\{d.\left(\frac{\rho}{ds}\, d.\frac{dx}{ds}\right)\right\}^2 + \left\{d.\left(\frac{\rho}{ds}\, d.\frac{dy}{ds}\right)\right\}^2 + \left\{d.\left(\frac{\rho}{ds}\, d.\frac{dz}{ds}\right)\right\}^2. \quad (61)$$

392.] Now this last equation deserves attention; for on referring to equations (25) it appears, that the direction-cosines of the radius of absolute curvature are

$$\frac{\rho}{ds}\,d.\frac{dx}{ds}, \quad \frac{\rho}{ds}\,d.\frac{dy}{ds}, \quad \frac{\rho}{ds}\,d.\frac{dz}{ds}; \tag{62}$$

and therefore on comparison of (61) with (29), it appears that the right-hand member of (61) is the square of the infinitesimal angle contained between two consecutive radii of absolute curvature. Therefore if $d\Omega$ is the angle between these, viz. the angle between M_1C_1 and M_2C_2 in fig. 129; then

$$d\Omega^2 = \Big\{d.\Big(\frac{\rho}{ds}\,d.\frac{dx}{ds}\Big)\Big\}^2 + \Big\{d.\Big(\frac{\rho}{ds}\,d.\frac{dy}{ds}\Big)\Big\}^2 + \Big\{d.\Big(\frac{\rho}{ds}\,d.\frac{dz}{ds}\Big)\Big\}^2, \tag{63}$$

$$= \frac{ds^2}{\rho^2} + \frac{ds^2}{R^2}; \tag{64}$$

and assuming $\mathfrak{R}$ to be such that $ds = \mathfrak{R}\,d\Omega$, we have

$$\frac{1}{\mathfrak{R}^2} = \frac{1}{\rho^2} + \frac{1}{R^2}; \tag{65}$$

and calling $\frac{1}{\mathfrak{R}}$ complex flexure, and $\mathfrak{R}$ the radius of complex flexure, we have

$$(\text{complex flexure})^2 = (\text{curvature})^2 + (\text{torsion})^2. \tag{66}$$

Fig. 129 affords an easy geometrical proof of (64).

393.] On the osculating surface of a non-plane curve.

The osculating planes of a non-plane curve by their consecutive intersections form a developable surface, whose edge of regression is the curve itself; or the surface may be considered to be generated by the tangent lines in their successive positions. The tangent line therefore is the characteristic of the surface, which is called the *osculating surface*.

Let the equation to an osculating plane be

$$(\xi-x)\,X + (\eta-y)\,Y + (\zeta-z)\,Z = 0; \tag{67}$$

then the equation to the consecutive plane is

$$(\xi-x)\,dX + (\eta-y)\,dY + (\zeta-z)\,dZ = 0, \tag{68}$$

$$\because \quad X\,dx + Y\,dy + Z\,dz = 0.$$

Eliminating therefore x, y, z from (67), (68), and the two equations to the curve, there will result an equation in terms of ξ, η, ζ which will be that to the osculating surface.

From the mode of generation of the surface, it is manifest that all involutes to the given curve lie in it; and that each involute cuts all the generating lines of the surface at right angles.

An example of a problem of this kind has been solved in Ex. 1, Art. 363.

394.] Again, suppose planes to be drawn at every point of a non-plane curve and perpendicular to the radius of absolute curvature at the point; such will by their intersections generate a developable surface, the equation to which is found as follows.

Let ξ, η, ζ be the current coordinates of the plane; then, since it is perpendicular to the radius of absolute curvature, or to the principal normal whose direction-cosines are given by equations (25), its equation is

$$(\xi-x)\, d.\frac{dx}{ds} + (\eta-y)\, d.\frac{dy}{ds} + (\zeta-z)\, d.\frac{dz}{ds} = 0; \qquad (69)$$

and the equation to the consecutive plane is

$$(\xi-x)\, d^2.\frac{dx}{ds} + (\eta-y)\, d^2.\frac{dy}{ds} + (\zeta-z)\, d^2.\frac{dz}{ds} = 0. \qquad (70)$$

The elimination therefore of x, y, z between (69), (70), and the two equations to the curve, will leave an equation involving ξ, η, ζ which will be that to the required surface.

The planes, by the consecutive intersection of which the surface is formed, are tangential to the original curve, and are by their position perpendicular to both the osculating and normal planes to the curve at the common point. This surface therefore always cuts at right angles the osculating surface of the curve; and since to the osculating surface the curve is the edge of regression, this surface is perpendicular to the osculating surface, through the whole length of its edge of regression. Now when the osculating surface is developed, the original curve becomes a straight line; but what effect has such a development of the osculating surface, or such a rectification of its edge of regression, on the surface which is perpendicular to it through its edge? it developes this orthogonal surface into a plane, and the line of contact of the original curve with it becomes a straight line. This surface is accordingly called *the rectifying developable surface*, as being that which touches the curve, and the curve of contact of which, with the given curve, becomes a straight line when the surface is developed into a plane. It is also for this reason that the plane which touches a curve, and is perpendicular to the radius of absolute curvature, is called *the rectifying plane* *.

395.] The line of intersection of two successive rectifying planes is called the *rectifying line;* hence it is the line of intersection of the two planes whose equations are (69) and (70); therefore its equations are

$$\frac{\xi-x}{d.\frac{dy}{ds}d^2.\frac{dz}{ds}-d.\frac{dz}{ds}d^2.\frac{dy}{ds}}=\frac{\eta-y}{d.\frac{dz}{ds}d^2.\frac{dx}{ds}-d.\frac{dx}{ds}d^2.\frac{dz}{ds}}$$
$$=\frac{\zeta-z}{d.\frac{dx}{ds}d^2.\frac{dy}{ds}-d.\frac{dy}{ds}d^2.\frac{dx}{ds}}; \quad (71)$$

also it is perpendicular to two consecutive principal normals.

Also by means of the denominators of the above fractions the infinitesimal angle between two consecutive rectifying lines may be determined.

And the edge of regression of the rectifying surface may be determined by differentiating (70), and then determining two equations in terms of ξ, η, ζ between (69), (70) and its differential, and the two equations to the curve.

396.] Let us apply to the helix the results which have been arrived at in the present Chapter. As in Ex. 2, Art. 347, let the equations to the curve be

$$\left.\begin{aligned} x &= a\cos\phi,\\ y &= a\sin\phi,\\ z &= ka\phi; \end{aligned}\right\} \quad \therefore \quad \left.\begin{aligned} dx &= -a\sin\phi\,d\phi &&= -y\,d\phi,\\ dy &= a\cos\phi\,d\phi &&= x\,d\phi,\\ dz &= ka\,d\phi &&= ka\,d\phi; \end{aligned}\right\} \quad (72)$$

$$ds^2 = (1+k^2)\,a^2\,d\phi^2;$$

and taking ϕ to be equicrescent,

$$\left.\begin{aligned} d^2x &= -x\,d\phi^2,\\ d^2y &= -y\,d\phi^2,\\ d^2z &= 0; \end{aligned}\right\} \qquad \left.\begin{aligned} d^3x &= y\,d\phi^3,\\ d^3y &= -x\,d\phi^3,\\ d^3z &= 0; \end{aligned}\right\} \quad (73)$$

---

* See a paper by Lancret in "Mémoires des Savans Etrangères," tome I^{ere}, Paris, An. XIV (1805).

and from the results of Ex. 2, Art. 347, and from (6) and (7) of Art. 377,

$$\left.\begin{aligned}\frac{d^2x}{ds^2} &= \frac{-x}{a^2(1+k^2)}, \\ \frac{d^2y}{ds^2} &= \frac{-y}{a^2(1+k^2)}, \\ \frac{d^2z}{ds^2} &= 0;\end{aligned}\right\} \qquad \left.\begin{aligned}\mathrm{X} &= kay\,d\phi^3, \\ \mathrm{Y} &= -kax\,d\phi^3, \\ \mathrm{Z} &= a^2 d\phi^3; \\ \mathrm{P}^2 &= a^4(1+k^2)\,d\phi^6;\end{aligned}\right\} \tag{74}$$

therefore by (24) and (25), $\rho = a(1+k^2)$;

$$\cos\lambda = -\cos\phi, \qquad \cos\mu = -\sin\phi, \qquad \cos\nu = 0;$$

$$d\tau = \frac{d\phi}{(1+k^2)^{\frac{1}{2}}}. \tag{75}$$

The radius of absolute curvature therefore coincides in direction with the radius of the base-cylinder of the helix; and as its value is constant, its extremity describes another helix of the same axis and the same thread, and which is traced on a base-cylinder whose radius is k^2a.

Again, from (40) and (42),

$$\left.\begin{aligned}\mathrm{R} &= a\frac{1+k^2}{k}, \qquad \therefore\ \mathrm{R} = \frac{\rho}{k}; \\ d\omega &= \frac{k}{(1+k^2)^{\frac{1}{2}}}\,d\phi.\end{aligned}\right\} \tag{76}$$

Also for the direction-cosines of the binormal, we have

$$\left.\begin{aligned}\cos l &= \frac{\mathrm{X}}{\mathrm{P}} = \frac{k}{(1+k^2)^{\frac{1}{2}}}\sin\phi, \\ \cos m &= \frac{\mathrm{Y}}{\mathrm{P}} = -\frac{k}{(1+k^2)^{\frac{1}{2}}}\cos\phi, \\ \cos n &= \frac{\mathrm{Z}}{\mathrm{P}} = \frac{1}{(1+k^2)^{\frac{1}{2}}}.\end{aligned}\right\} \tag{77}$$

The equation to the polar surface has been found in the last Chapter, see Ex. 1, equation (72), Art. 363, and therefore it is unnecessary to apply equations (46) and (47) to the same problem.

By means of (48) the equations to a polar line are

$$\frac{\xi+k^2x}{ky} = \frac{\eta+k^2y}{-kx} = \frac{\zeta-z}{a}. \tag{78}$$

Again, to find the equations to the edge of regression of the polar surface, we have from equations (52),

$$\xi = -k^2x, \qquad \eta = -k^2y, \qquad \zeta = z; \tag{79}$$

substituting which in the equations to the helix, viz.

$$x = a\cos\frac{z}{ka}, \qquad y = a\sin\frac{z}{ka}, \tag{80}$$

we have

$$\xi = -k^2a\cos\frac{\zeta}{ka}, \qquad \eta = -k^2a\sin\frac{\zeta}{ka}; \tag{81}$$

which represent a helix described on a cylinder, whose axis is the axis of the original base-cylinder, and radius $= k^2a$, and the inclination of whose thread to the plane of xy is the same as that of the original helix.

Also from (54),

$$\text{the radius of the osculating sphere} = a(1+k^2); \tag{82}$$

and is therefore equal to the radius of absolute curvature; and similarly the coordinates to the centre of the osculating sphere are those given in (79).

From (64), $$d\Omega = d\phi, \tag{83}$$

that is, two consecutive radii of absolute curvature are inclined to each other at the same angle as the corresponding radii of the base-cylinder.

Also from (65), $$\mathfrak{R} = (1+k^2)^{\frac{1}{2}}a. \tag{84}$$

The equation to the osculating surface has been investigated in Ex. 1, Art. 368.

By equation (69) the equation to the rectifying plane is

$$\xi x+\eta y = a^2; \tag{85}$$

that is, the plane is parallel to the axis of z, and perpendicular to the radius of the base-cylinder.

And by (71) the equations to the rectifying line are,

$$\xi = x, \qquad \eta = y, \qquad \zeta = \frac{0}{0}. \tag{86}$$

CHAPTER XVIII.

ON THE CURVATURE OF SECTIONS OF CURVED SURFACES.

397.] CONSIDER on a curved surface a point which is not one of discontinuity, and at which the derived-functions of the equation to the surface are neither singular nor indeterminate. Through this point let any number of lines be drawn on the surface; these will generally be curved, and the curvature of them will vary continuously as we pass from one to another; to the determination of their curvature we propose to apply the principles already explained; and by means of them to form a conception of the actual flexure or curvature of the surface at the point under consideration.

Now our principles require an investigation into the inclination of two consecutive elements, or into the relative position of three consecutive points, see Art. 281. The first point is that under consideration, and the next two are of necessity in the same plane with it; and according as the plane of the three consecutive points does or does not contain the normal to the surface at the first point, will the process of inquiry be different.

Firstly, let the three points be such that their plane may contain the normals at the first and second points; then the two consecutive normals lie in the same plane, and therefore intersect. Accordingly the principles on which we have estimated radius of curvature are directly applicable; this is the case of *principal normal section.*

Secondly, let the plane of the three points contain the first normal, but not the second; then the principles of estimating curvature which were adopted in the last Chapter are applicable; and we must investigate the distance from the surface of the point at which the normal plane of the second element meets the normal to the surface at the given point; for such a point will be identical with the point of intersection on the osculating plane of two consecutive normal planes: this is the case of *ordinary normal section.*

Thirdly, the three points may be such that their plane may not contain the normal line at the first point; in which case we must then apply the results of Art. 375 to determine the radius of absolute curvature. This is the case of *oblique section*, the osculating plane of the two elements being inclined at a given angle to the normal of the surface.

398.] I propose first to investigate the curvature of principal normal sections.

Let the equation to the surface be

$$\mathrm{F}(x, y, z) = 0; \tag{1}$$

and for the sake of abbreviating the notation let us employ the following symbols, as in Art. 332, and in Art. 360,

$$\left.\begin{aligned} &\left(\frac{d\mathrm{F}}{dx}\right) = \mathrm{U}, & &\left(\frac{d\mathrm{F}}{dy}\right) = \mathrm{V}, & &\left(\frac{d\mathrm{F}}{dz}\right) = \mathrm{W}; \\ &\left(\frac{d^2\mathrm{F}}{dx^2}\right) = u, & &\left(\frac{d^2\mathrm{F}}{dy^2}\right) = v, & &\left(\frac{d^2\mathrm{F}}{dz^2}\right) = w; \\ &\left(\frac{d^2\mathrm{F}}{dy\,dz}\right) = u', & &\left(\frac{d^2\mathrm{F}}{dz\,dx}\right) = v', & &\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) = w'; \end{aligned}\right\} \tag{2}$$

$$\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2 = \mathrm{Q}^2. \tag{3}$$

Then the equations to the normal at the point x, y, z are,

$$\frac{\xi - x}{\mathrm{U}} = \frac{\eta - y}{\mathrm{V}} = \frac{\zeta - z}{\mathrm{W}} = \frac{\rho}{\mathrm{Q}} = \Omega, \text{ (say)}, \tag{4}$$

where ρ is the distance between (x, y, z) and (ξ, η, ζ). Hence we have

$$\xi - x = \mathrm{U}\Omega, \qquad \eta - y = \mathrm{V}\Omega, \qquad \zeta - z = \mathrm{W}\Omega. \tag{5}$$

Now when two consecutive normals intersect, (5) must consist with their differentials, when x, y, z, and therefore when U, V, W, Ω vary; accordingly, differentiating, we have

$$\left.\begin{aligned} dx + \mathrm{U}\,d\Omega + \Omega\,d\mathrm{U} &= 0, \\ dy + \mathrm{V}\,d\Omega + \Omega\,d\mathrm{V} &= 0, \\ dz + \mathrm{W}\,d\Omega + \Omega\,d\mathrm{W} &= 0; \end{aligned}\right\} \tag{6}$$

whence, eliminating Ω and $d\Omega$, we have

$$(\mathrm{V}\,d\mathrm{W} - \mathrm{W}\,d\mathrm{V})\,dx + (\mathrm{W}\,d\mathrm{U} - \mathrm{U}\,d\mathrm{W})\,dy + (\mathrm{U}\,d\mathrm{V} - \mathrm{V}\,d\mathrm{U})\,dz = 0; \tag{7}$$

and as this is independent of ξ, η, ζ it is true for any point on the surface, and is therefore the differential equation of lines drawn on the surface, which are such that the consecutive normals to the surface along them intersect. These lines are called the *lines of curvature*, to the consideration of which we shall hereafter return. The geometrical meaning of (7) is, that the line of intersection of the two tangent planes drawn at the extremities of ds, the element of a line on the surface, is at right angles to the tangent line whose direction-cosines are proportional to dx, dy, dz.

The equation to the lines of curvature may also be expressed as follows: since

$$\begin{aligned} \mathrm{V}\,d\mathrm{W} - \mathrm{W}\,d\mathrm{V} &= \mathrm{V}(v'dx + u'dy + w\,dz) - \mathrm{W}(w'dx + v\,dy + u'dz), \\ &= (\mathrm{V}v' - \mathrm{W}w')dx + (\mathrm{V}u' - \mathrm{W}v)dy + (\mathrm{V}w - \mathrm{W}u')dz; \end{aligned}$$

and since, similarly,

$$\mathrm{W}\,d\mathrm{U} - \mathrm{U}\,d\mathrm{W} = (\mathrm{W}u - \mathrm{U}v')dx + (\mathrm{W}w' - \mathrm{U}u')dy + (\mathrm{W}v' - \mathrm{U}w)\,dz,$$

$$\mathrm{U}\,d\mathrm{V} - \mathrm{V}\,d\mathrm{U} = (\mathrm{U}w' - \mathrm{V}u)dx + (\mathrm{U}v - \mathrm{V}w')\,dy + (\mathrm{U}u' - \mathrm{V}v')\,dz,$$

the equation to the lines of curvature becomes

$$\begin{aligned} (\mathrm{V}v' - \mathrm{W}w')dx^2 + (\mathrm{W}w' - \mathrm{U}u')dy^2 + (\mathrm{U}u' - \mathrm{V}v')dz^2 \\ + (\mathrm{W}v' - \mathrm{U}w + \mathrm{U}v - \mathrm{V}w')dy\,dz + (\mathrm{U}w' - \mathrm{V}u + \mathrm{V}w - \mathrm{W}u')dz\,dx \\ + (\mathrm{V}u' - \mathrm{W}v + \mathrm{W}u - \mathrm{U}v')dx\,dy = 0. \quad (8) \end{aligned}$$

And as this is of two dimensions in terms of dx, dy, dz, there are in general two distinct lines of curvature passing through every point on a surface; these we shall shew to be perpendicular to each other at the point.

399.] Let therefore, as in (4), ρ be the length of the radius of curvature of a principal normal section; and let ξ, η, ζ be the coordinates to its extremity, so that

$$\rho^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2;$$

then as ξ, η, ζ, ρ are the same for the next consecutive normal, and therefore for the next *two* consecutive points, we have

$$(\xi - x)\,dx + (\eta - y)\,dy + (\zeta - z)\,dz = 0, \qquad (9)$$

$$(\xi - x)\,d^2x + (\eta - y)\,d^2y + (\zeta - z)\,d^2z = ds^2; \qquad (10)$$

and as the centre of curvature is on the normal,

$$\frac{\xi - x}{\mathrm{U}} = \frac{\eta - y}{\mathrm{V}} = \frac{\zeta - z}{\mathrm{W}} = \frac{\rho}{\mathrm{Q}} = \frac{ds^2}{\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z}; \qquad (11)$$

$$\therefore \quad \rho = \frac{\mathrm{Q}\,ds^2}{\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z}. \qquad (12)$$

Now differentiating the equation to the surface twice, we have

$$\mathrm{U}\,d^2x+\mathrm{V}\,d^2y+\mathrm{W}\,d^2z+u\,dx^2+v\,dy^2+w\,dz^2$$
$$+2\,\{u'dy\,dz+v'dz\,dx+w'dx\,dy\} = 0, \qquad (13)$$

and employing l, m, n, to represent the direction-cosines of the element of the curve whose curvature we are investigating, so that

$$\frac{dx}{ds} = l, \qquad \frac{dy}{ds} = m, \qquad \frac{dz}{ds} = n, \qquad (14)$$

we have, neglecting the ambiguity of sign,

$$\frac{\mathrm{Q}}{\rho} = ul^2+vm^2+wn^2+2u'mn+2v'nl+2w'lm. \qquad (15)$$

Suppose that at the point on the surface neither U, V, nor W vanishes, then this expression admits of the following modification; since

$$\mathrm{U}dx+\mathrm{V}dy+\mathrm{W}dz = 0; \qquad (16)$$
$$\therefore \quad \mathrm{U}l+\mathrm{V}m+\mathrm{W}n = 0; \qquad (17)$$
$$\therefore \quad \mathrm{V}m+\mathrm{W}n = -\mathrm{U}l;$$

$$\left.\begin{aligned} \therefore \quad 2mn &= \frac{\mathrm{U}^2l^2-\mathrm{V}^2m^2-\mathrm{W}^2n^2}{\mathrm{VW}}; \\ \text{similarly} \qquad 2nl &= \frac{\mathrm{V}^2m^2-\mathrm{W}^2n^2-\mathrm{U}^2l^2}{\mathrm{WU}}, \\ 2lm &= \frac{\mathrm{W}^2n^2-\mathrm{U}^2l^2-\mathrm{V}^2m^2}{\mathrm{UV}}; \end{aligned}\right\} \qquad (18)$$

and substituting as follows,

$$\left.\begin{aligned} \mathrm{H} &= u+\frac{\mathrm{U}}{\mathrm{VW}}(\mathrm{U}u'-\mathrm{V}v'-\mathrm{W}w'), \\ \mathrm{K} &= v+\frac{\mathrm{V}}{\mathrm{WU}}(\mathrm{V}v'-\mathrm{W}w'-\mathrm{U}u'), \\ \mathrm{L} &= w+\frac{\mathrm{W}}{\mathrm{UV}}(\mathrm{W}w'-\mathrm{U}u'-\mathrm{V}v'); \end{aligned}\right\} \qquad (19)$$

we have

$$\frac{\mathrm{Q}}{\rho} = \mathrm{H}l^2+\mathrm{K}m^2+\mathrm{L}n^2. \qquad (20)$$

But as the section whose curvature we are considering has an element coincident with that of a line of curvature, we may introduce this condition, and modify (20) accordingly; returning to (7) and (8),

$$\mathrm{V}d\mathrm{W}-\mathrm{W}d\mathrm{V} = (\mathrm{V}v'-\mathrm{W}w')\,dx+(\mathrm{V}u'-\mathrm{W}v)\,dy+(\mathrm{V}w-\mathrm{W}u')\,dz;$$

whence, by means of (16) and (19),

$$V\,dW - W\,dV = VL\,dz - WK\,dy. \qquad (21)$$

Similarly
$$W\,dU - U\,dW = WH\,dx - UL\,dz, \qquad (22)$$
$$U\,dV - V\,dU = UK\,dy - VH\,dx; \qquad (23)$$

whereby the differential equation to the lines of curvature becomes,

$$U(K-L)\,dy\,dz + V(L-H)\,dz\,dx + W(H-K)\,dx\,dy = 0, \qquad (24)$$

or
$$U(K-L)mn + V(L-H)nl + W(H-K)lm = 0. \qquad (25)$$

Also from (17),
$$Ul + Vm + Wn = 0;$$

whence, eliminating U, V, W in order, and simplifying the result by means of (20), we have

$$\frac{U}{l\left(H-\frac{Q}{\rho}\right)} = \frac{V}{m\left(K-\frac{Q}{\rho}\right)} = \frac{W}{n\left(L-\frac{Q}{\rho}\right)}; \qquad (26)$$

whence
$$\frac{U^2}{H-\frac{Q}{\rho}} + \frac{V^2}{K-\frac{Q}{\rho}} + \frac{W^2}{L-\frac{Q}{\rho}} = 0; \qquad (27)$$

a quadratic equation, which gives the two values of ρ corresponding to the two radii of curvature of the principal normal sections.

If however at the point of the surface under investigation either U, V or W vanishes, the process, by which H, K, L have been formed, must be modified accordingly; and neither can the equation of the principal radii of curvature be expressed in the forms (20) and (27), nor the equation of the lines of curvature in the form (25). In this case then we are obliged to recur to the original forms, which are also most general, viz. (8) and (15).

400.] Equations* (26) give us also values for l, m, n which are the direction-cosines of the lines of curvature, and therefore of the principal normal sections; whereby it may be shewn, that the principal normal sections are at right angles to each other.

For let $l_1 m_1 n_1$, $l_2 m_2 n_2$ be the direction-cosines of the normal sections corresponding to which the radii of curvature are

---

* For this Article, as well as for a great part of the preceding one, and for Art. 402, I am indebted to Gregory's Solid Geometry, Cambridge, 1845.

ρ_1, ρ_2; then we have, writing in (27) each value of ρ, and subtracting one from the other,

$$\left(\frac{Q}{\rho_1}-\frac{Q}{\rho_2}\right)\left\{\frac{U^2}{\left(H-\frac{Q}{\rho_1}\right)\left(H-\frac{Q}{\rho_2}\right)} + \frac{V^2}{\left(K-\frac{Q}{\rho_1}\right)\left(K-\frac{Q}{\rho_2}\right)} + \frac{W^2}{\left(L-\frac{Q}{\rho_1}\right)\left(L-\frac{Q}{\rho_2}\right)}\right\} = 0. \quad (28)$$

Hence, if ρ_1 and ρ_2 are unequal, as they will usually be, the second factor must be equal to 0; and therefore by reason of (26),

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0; \quad (29)$$

and therefore the principal normal sections intersect each other at right angles.

Hence also the lines of curvature at any point of a surface cut each other at right angles.

The radii of curvature of the principal normal sections are called the *principal radii of curvature.*

401.] To determine the radius of curvature of any normal section.

Observing the mode of determining the radius of curvature of such a section which was mentioned in Art. 397, it appears that we have to find the point of intersection of the normal whose equations are

$$\frac{\xi-x}{U} = \frac{\eta-y}{V} = \frac{\zeta-z}{W},$$

and the plane which is perpendicular to the second element, and whose equation is given in (10); whence, using the same notation and the same method of investigation as that of Art. 399, we have

$$\frac{Q}{\rho} = ul^2+vm^2+wn^2+2u'mn+2v'nl+2w'lm; \quad (30)$$

and if neither U, V, nor W vanishes, then

$$\frac{Q}{\rho} = Hl^2+Km^2+Ln^2, \quad (31)$$

H, K, L being given in equations (19), and l, m, n being the direction-cosines of the element of the curve at the point on the surface, and the curvature of which we are examining.

402.] Of all radii of curvature of normal sections, to determine whether any, and what, are maxima and minima.

For all normal sections passing through a given point Q, U, V, W, H, K, L being, or involving, the partial derived-functions of the equation to the surface at the point, are constant, and l, m, n vary. Hence we have to determine the maximum and minimum of (31), having given the conditions

$$Ul + Vm + Wn = 0, \tag{32}$$

$$l^2 + m^2 + n^2 = 1; \tag{33}$$

$$\left.\begin{aligned} \therefore\quad D\left(\frac{Q}{\rho}\right) = 2\{Hl\,dl + Km\,dm + Ln\,dn\} &= 0, \\ U\,dl + V\,dm + W\,dn &= 0, \\ l\,dl + m\,dm + n\,dn &= 0; \end{aligned}\right\} \tag{34}$$

and using indeterminate multipliers $-\lambda$ and $-\mu$, according to Art. 167,

$$\left.\begin{aligned} Hl - \lambda U - \mu l &= 0, \\ Km - \lambda V - \mu m &= 0, \\ Ln - \lambda W - \mu n &= 0; \end{aligned}\right\} \tag{35}$$

whence multiplying severally through by l, m, n, and adding,

$$\frac{Q}{\rho} - \mu = 0;$$

$$\therefore\quad l = \frac{\lambda U}{H - \frac{Q}{\rho}}, \qquad m = \frac{\lambda V}{K - \frac{Q}{\rho}}, \qquad n = \frac{\lambda W}{L - \frac{Q}{\rho}}; \tag{36}$$

and therefore
$$\frac{U^2}{H - \frac{Q}{\rho}} + \frac{V^2}{K - \frac{Q}{\rho}} + \frac{W^2}{L - \frac{Q}{\rho}} = 0; \tag{37}$$

which last results are identical with (26) and (27); whence it follows, that of all normal sections the principal ones are those whose curvature is respectively a maximum and a minimum. Hence the normal sections of greatest and least curvature are at right angles to each other.

403.] The radius of curvature of any normal section may be expressed as follows, in terms of the radii of curvature of the principal normal sections.

The proposition having been discovered by Euler, is known by the name of Euler's Theorem of Normal Sections.

Let ρ_1 and ρ_2 be the principal radii of curvature, and $l_1 m_1 n_1$,

$l_2 m_2 n_2$ be the direction-cosines of the elements of the lines of curvature passing through the given point on the surface, and therefore of the directions of the principal normal sections at the point; let ρ be the radius of curvature of any other normal section, and lmn the direction-cosines of its element; and let a be the angle between its plane and that of the maximum principal radius of curvature; then we have

$$\left.\begin{aligned} \mathrm{U}l + \mathrm{V}m + \mathrm{W}n &= 0, \\ \mathrm{U}l_1 + \mathrm{V}m_1 + \mathrm{W}n_1 &= 0, \\ \mathrm{U}l_2 + \mathrm{V}m_2 + \mathrm{W}n_2 &= 0; \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} ll_1 + mm_1 + nn_1 &= \cos a, \\ ll_2 + mm_2 + nn_2 &= \sin a; \end{aligned}\right\} \tag{39}$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0. \tag{40}$$

Also if we suppose the lines of curvature, and the normal of the surface, to constitute a system of rectangular axes at the point under consideration as the origin, we have

$$\left.\begin{aligned} l &= l_1 \cos a + l_2 \sin a, \\ m &= m_1 \cos a + m_2 \sin a, \\ n &= n_1 \cos a + n_2 \sin a; \end{aligned}\right\} \tag{41}$$

the last term of the general formula for such cosines vanishing, because the element of the normal section is perpendicular to the normal to the surface.

Also from (36),

$$\left.\begin{aligned} \left(\mathrm{H} - \frac{\mathrm{Q}}{\rho_1}\right) l_1 &= \lambda \mathrm{U}, \\ \left(\mathrm{K} - \frac{\mathrm{Q}}{\rho_1}\right) m_1 &= \lambda \mathrm{V}, \\ \left(\mathrm{L} - \frac{\mathrm{Q}}{\rho_1}\right) n_1 &= \lambda \mathrm{W}; \end{aligned}\right\} \qquad \left.\begin{aligned} \left(\mathrm{H} - \frac{\mathrm{Q}}{\rho_2}\right) l_2 &= \lambda \mathrm{U}, \\ \left(\mathrm{K} - \frac{\mathrm{Q}}{\rho_2}\right) m_2 &= \lambda \mathrm{V}, \\ \left(\mathrm{L} - \frac{\mathrm{Q}}{\rho_2}\right) n_2 &= \lambda \mathrm{W}; \end{aligned}\right\} \tag{42}$$

whence multiplying severally by l, m, n, and adding, and by reason of (38),

$$\left.\begin{aligned} \mathrm{H}\, ll_1 + \mathrm{K}\, mm_1 + \mathrm{L}\, nn_1 &= \frac{\mathrm{Q}}{\rho} \cos a, \\ \mathrm{H}\, ll_2 + \mathrm{K}\, mm_2 + \mathrm{L}\, nn_2 &= \frac{\mathrm{Q}}{\rho} \sin a; \end{aligned}\right\} \tag{43}$$

and multiplying the first of (43) by $\cos a$, and the second by $\sin a$, and adding and reducing by means of (41), we have

$$\text{H}\,l^2 + \text{K}\,m^2 + \text{L}\,n^2 = \frac{\text{Q}}{\rho_1}(\cos a)^2 + \frac{\text{Q}}{\rho_2}(\sin a)^2; \tag{44}$$

and therefore by (31),

$$\frac{1}{\rho} = \frac{(\cos a)^2}{\rho_1} + \frac{(\sin a)^2}{\rho_2}; \tag{45}$$

which is Euler's Theorem; and is of importance, as by it the radius of curvature of any normal section is expressed in terms of the principal radii of curvature at the point in question.

404.] Hence we have the following proposition as to the radii of curvature of any two normal sections which are perpendicular to each other.

Let ρ and ρ' be the radii of curvature of two normal sections perpendicular to each other; then

$$\frac{1}{\rho} = \frac{(\cos a)^2}{\rho_1} + \frac{(\sin a)^2}{\rho_2}, \qquad \frac{1}{\rho'} = \frac{(\sin a)^2}{\rho_1} + \frac{(\cos a)^2}{\rho_2},$$

$$\therefore \quad \frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{\rho_1} + \frac{1}{\rho_2}; \tag{46}$$

that is, the sum of the curvatures of any two normal sections perpendicular to each other is constant.

405.] As an example of the preceding formulæ, let us take the ellipsoid whose equation is

$$\text{F}(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \tag{47}$$

$$\therefore \quad \text{U} = \frac{2x}{a^2}, \qquad \text{V} = \frac{2y}{b^2}, \qquad \text{W} = \frac{2z}{c^2};$$

$$u = \frac{2}{a^2}, \qquad v = \frac{2}{b^2}, \qquad w = \frac{2}{c^2};$$

$$u' = 0, \qquad v' = 0, \qquad w' = 0;$$

$$\text{Q} = 2\left\{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right\}^{\frac{1}{2}} = \frac{2}{p},$$

if p is the perpendicular from the centre on the tangent plane; see equation (31), Art. 338. Hence equation (27) becomes

$$\frac{x^2}{a^2(p\rho - a^2)} + \frac{y^2}{b^2(p\rho - b^2)} + \frac{z^2}{c^2(p\rho - c^2)} = 0, \tag{48}$$

the roots of which quadratic equation are the greatest and least principal radii of curvature at any point on the ellipsoid.

And as the last term of the quadratic when written in an integral form is $\frac{a^2 b^2 c^2}{p^4}$, it follows that the product of the greatest and least radii of curvature is invariable for all points for which p is constant.

Again, putting (48) in the form

$$\frac{x^2}{a^2\left(1-\frac{a^2}{p\rho}\right)}+\frac{y^2}{b^2\left(1-\frac{b^2}{p\rho}\right)}+\frac{z^2}{c^2\left(1-\frac{c^2}{p\rho}\right)}=0;$$

and subtracting it from the equation to the ellipsoid, we have

$$\frac{x^2}{a^2-p\rho}+\frac{y^2}{b^2-p\rho}+\frac{z^2}{c^2-p\rho}=1;$$

which is the equation to a concentric and confocal surface of the second order.

Also by means of (26), the directions of the principal normal sections may be determined at any point of the ellipsoid.

406.] We proceed now to consider certain singular values of ρ_1 and ρ_2, and the nature of a point on a surface whereat the singular values exist.

In equations (11) and (12) an ambiguity of sign exists, which is introduced in extracting the root of $\text{U}^2+\text{V}^2+\text{W}^2$, and therefore ρ may be affected with a + or a − sign; and the same ambiguity of sign continues in (27).

As U, V, W are the same for different normal sections at the same point, and as ρ is an absolute length of line, it appears from (11) that the change of sign arises from $\xi-x$, $\eta-y$, $\zeta-z$; and therefore the change of sign implies, that the centres of curvature are for different normal sections situated on different sides of the surface.

With respect to (45) it is to be borne in mind, that ρ_1 and ρ_2 are both taken with the positive sign, and that Q has the same sign, viz. +, in both. From (45) therefore it follows, that if ρ_1 and ρ_2 have the same sign, ρ has always the same sign as either of them; and that therefore all normal sections have their curvature in the neighbourhood of the point turned in the same direction. The analytical condition derived from equation (27) that this should be the case, is, that

$$\text{U}^2\text{KL}+\text{V}^2\text{LH}+\text{W}^2\text{HK} \text{ must be positive.} \qquad (49)$$

Also it is manifest that, as ρ_1 and ρ_2 are a maximum and a minimum value of the radii of curvature, ρ always lies between them.

Again, if the signs of ρ_1 and ρ_2 are different, that is, if

$$\mathrm{U}^2\mathrm{KL} + \mathrm{V}^2\mathrm{LH} + \mathrm{W}^2\mathrm{HK} \text{ is negative,} \tag{50}$$

the radii of curvature of some normal sections are turned in a direction contrary to that of others, and (45) becomes

$$\frac{1}{\rho} = \frac{(\cos a)^2}{\rho_1} - \frac{(\sin a)^2}{\rho_2}; \tag{51}$$

let a' be such that $\qquad \tan a' = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}},$ (52)

then for all values of a, from $-a'$ to $+a'$, and from $\pi - a'$ to $\pi + a'$, the radii of curvature of normal sections are turned in the same direction; and when $a = \pm a'$, and $= \pi \pm a'$, $\rho = \infty$; then the normal section becomes a straight line in its consecutive elements which abut at the point, or the curvature is suspended; see Art. 383; and for all values of a outside those limits, the curvature of the normal sections is turned in a contrary direction.

According to our hypothesis ρ_1 is the maximum and ρ_2 is the minimum radius of curvature.

In the case in which one of the principal radii of curvature is infinite, say $\rho_1 = \infty$,

$$\rho = \frac{(\sin a)^2}{\rho_2}, \tag{53}$$

the analytical condition of which derived from (27) is

$$\mathrm{U}^2\mathrm{KL} + \mathrm{V}^2\mathrm{LH} + \mathrm{W}^2\mathrm{HK} = 0, \tag{54}$$

and which, when expanded, becomes identical with that determined in Art. 360, equation (62), as the differential equation of developable surfaces. Hence we have the geometrical meaning of this equation. One of the principal normal sections of the surface lies along the straight generating line, and therefore the curvature of this section vanishes.

Again, suppose the two values of ρ_1 and ρ_2 to be equal and of opposite signs; then the coefficient of the second term of the quadratic (27) must be equal to zero, whereby we have

$$\mathrm{U}^2(\mathrm{K}+\mathrm{L}) + \mathrm{V}^2(\mathrm{L}+\mathrm{H}) + \mathrm{W}^2(\mathrm{H}+\mathrm{K}) = 0. \tag{55}$$

In this case, a' of (52) $= 45°$; and therefore of the surface about the point, the curvature of one-half is turned in one direction, and that of the other half in the opposite direction, and the dividing lines of these districts of opposite curvature are two straight lines passing through the point, and perpendicular to each other.

407.] Lastly, let us consider the case when the principal radii of curvature are equal and have the same sign. Here $\rho_1 = \rho_2$, and equation (45) becomes

$$\rho = \rho_1 = \rho_2; \tag{56}$$

that is, the radii of curvature of all the normal sections at such a point are equal. The point is called an ***umbilic.***

At it (27) when arranged in powers of ρ is a complete square; whence the condition might be found; but the following process is easier. As all the principal radii of curvature must be equal, the direction-cosines which determine their directions must be indeterminate; and as these are the same as those which determine the lines of curvature, equations (25) or (26) must be satisfied identically and independently of any particular values of l, m, n; but this is effected if

$$\text{H} = \text{K} = \text{L}; \tag{57}$$

these two equation therefore, together with that to the surface, determine the position of an umbilic. In these however the simultaneous values of xyz must not be such as to make to vanish either U, V or W, for, if so, the process according to which H, K, L were determined in Art. 399 fails. And if the two equations (57) are equivalent to only one, then this, together with the equation to the surface, will determine a line on the surface which is the locus of such umbilical points, and is called the ***line of spherical curvature.***

Also if $\text{H} = \text{K} = \text{L}$, we have from equation (20)

$$\rho = \frac{\text{Q}}{\text{H}},$$

and all the radii of curvature are equal. And as in this case (49) is satisfied, so all the curvatures of the normal section are in the same direction.

In the case however in which either U, V, or W vanishes, and thereby H, K and L are rendered indeterminate, we may proceed as follows:

Suppose U to vanish; then returning to equation (8) we have

$$(\mathrm{V}v' - \mathrm{W}w')\,dx^2 + \mathrm{W}w'dy^2 - \mathrm{V}v'dz^2 + (\mathrm{W}v' - \mathrm{V}w')\,dy\,dz$$
$$+ (\mathrm{V}w - \mathrm{V}u - \mathrm{W}u')\,dz\,dx + (\mathrm{V}u' + \mathrm{W}u - \mathrm{W}v)\,dx\,dy = 0. \qquad (58)$$

And since $\mathrm{U} = 0$, equation (16) becomes

$$\mathrm{V}\,dy + \mathrm{W}\,dz = 0;$$
$$\therefore \quad dz = -\frac{\mathrm{V}}{\mathrm{W}}\,dy; \qquad (59)$$

substituting which in (58) we have

$$(\mathrm{V}v' - \mathrm{W}w')\left\{dx^2 - \frac{\mathrm{V}^2 + \mathrm{W}^2}{\mathrm{W}^2}\,dy^2\right\}$$
$$+ \frac{2\mathrm{V}\mathrm{W}u' - \mathrm{W}^2 v - \mathrm{V}^2 w + (\mathrm{V}^2 + \mathrm{W}^2)\,u}{\mathrm{W}}\,dy\,dx = 0, \qquad (60)$$

which, for an umbilic, must be satisfied independently of $\frac{dy}{dx}$; hence the conditions are

$$\text{when } \mathrm{U} = 0, \quad \left.\begin{aligned} \mathrm{V}v' - \mathrm{W}w' &= 0, \\ u &= \frac{\mathrm{V}^2 w - 2\,\mathrm{V}\mathrm{W}u' + \mathrm{W}^2 v}{\mathrm{V}^2 + \mathrm{W}^2} \end{aligned}\right\}. \qquad (61)$$

Similarly the conditions are

$$\text{when } \mathrm{V} = 0, \quad \left.\begin{aligned} \mathrm{W}w' - \mathrm{U}u' &= 0, \\ v &= \frac{\mathrm{W}^2 u - 2\,\mathrm{W}\mathrm{U}v' + \mathrm{U}^2 w}{\mathrm{W}^2 + \mathrm{U}^2} \end{aligned}\right\}. \qquad (62)$$

$$\text{And when } \mathrm{W} = 0, \quad \left.\begin{aligned} \mathrm{U}u' - \mathrm{V}v' &= 0, \\ w &= \frac{\mathrm{U}^2 v - 2\,\mathrm{U}\mathrm{V}w' + \mathrm{V}^2 u}{\mathrm{U}^2 + \mathrm{V}^2} \end{aligned}\right\}. \qquad (63)$$

Hence to find all the umbilics on a surface, we must first seek the number of points which satisfy the general conditions (57); and also inquire when any and what points satisfy either of the three systems (61), (62), and (63).

408.] Examples of umbilics.

Ex. 1. The ellipsoid.

$$\mathrm{F}(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0;$$

whence
$$\mathrm{U} = \frac{2x}{a^2}, \quad \mathrm{V} = \frac{2y}{b^2}, \quad \mathrm{W} = \frac{2z}{c^2};$$
$$u = \frac{2}{a^2}, \quad v = \frac{2}{b^2}, \quad w = \frac{2}{c^2};$$

$$u' = v' = w' = 0;$$

$$\therefore \quad \text{H} = \frac{2}{a^2}, \quad \text{K} = \frac{2}{b^2}, \quad \text{L} = \frac{2}{c^2}.$$

It is evident therefore that H, K, L can never be equal. Also of conditions (61), (62) and (63), the first and last lead to impossible results; and as to (62), let V = 0, therefore $y = 0$, and

$$\frac{1}{b^2}\left\{\frac{z^2}{c^4} + \frac{x^2}{a^4}\right\} = \frac{x^2}{a^4c^2} + \frac{z^2}{c^4a^2};$$

$$\therefore \quad x^2 = a^2\frac{a^2-b^2}{a^2-c^2}, \qquad z^2 = c^2\frac{b^2-c^2}{a^2-c^2}.$$

The ellipsoid therefore has four umbilics situated symmetrically in the plane of the greatest and least principal axes; and the tangent planes at these points are parallel to the circular plane sections of the surface. The radius of curvature at the umbilic $= \dfrac{b^3}{ac}$.

If $a = b$, the surface becomes an oblate spheroid; $x = 0$, $z = \pm c$, and the umbilics are the points where the axis of revolution meets the surface.

Every point of a sphere is an umbilic; and a sphere is the only surface possessing this property.

Ex. 2. Find the umbilic of the surface $xyz = k^3$.

$$\text{U} = yz, \qquad \text{V} = zx, \qquad \text{W} = xy,$$
$$u = 0, \qquad v = 0, \qquad w = 0,$$
$$u' = x, \qquad v' = y, \qquad w' = z;$$

$$\therefore \quad \text{H} = -\frac{yz}{x}, \quad \text{K} = -\frac{zx}{y}, \quad \text{L} = -\frac{xy}{z},$$

$$\therefore \quad \text{H} = \text{K} = \text{L}, \text{ if } x = y = z = k.$$

and the umbilic is at the point (k, k, k).

If the position of an umbilic is determined by the condition (57), then (24) or (25) is identically satisfied independently of the values of l, m, n; and therefore through such an umbilic the number of lines of curvature may be infinite. But if it is determined by either of the conditions (61), (62), or (63), one only, or at all events only a determinate number of lines of curvature will pass through the umbilical point. Thus through the umbilic on the ellipsoid found in Ex. 1, only one line of curvature passes, and corresponding to it we have

$$\frac{dx}{dz} = -\frac{a}{c}\left(\frac{b^2-c^2}{a^2-b^2}\right)^{\frac{1}{2}};$$

and therefore the line of curvature is in the plane of the a and c semi-axes.

409.] To return to the consideration of lines of curvature, and of the locus of the centres of principal curvature which is closely allied to them.

A normal at a point of a curved surface is always intersected by two other adjacent normals, drawn in the two principal normal planes which are perpendicular to each other. Now imagine a point to pass from a given point on a surface to the adjacent point in one of its principal normal sections; and on again in the same direction from this point to the infinitesimally adjacent point in its principal normal section; and so on; it is evident that the point passes along a curve described on the surface; and if the same process is carried on for all the points of the surface, it is evident that the surface will be divided into a series of zones or bands of varying width.

Imagine again the moving point to set out from the first point on the surface along its principal normal section in the direction perpendicular to the former path; and that it passes along this path to a second point; and thence to a third and so on; it is evident that it traverses a path which cuts each of the other curves at right angles; and if the same process is performed in reference to other points on the surface, the surface will be divided into a series of bands of varying width the lines of which are perpendicular to those by which the bands of the former series were formed: and thus the surface will be divided into a series of small curvilinear rectangles, all the angles of which are right, the sides of them being portions of lines of curvature of the surface, and these always intersecting at right-angles.

Let us illustrate these propositions by means of a surface of revolution; say, to fix the ideas, of a paraboloidal form. The lines of intersection of the surface by its meridianal planes evidently form one series of lines of curvature; because all normals along any of these meridianal curves pass through the axis of the figure. And all "parallels" intersect the surface in lines of curvature of the other series, because all the normals drawn at points on the same parallel meet in the axis. The surface-elements into which the whole surface is divided by these lines of curvature are rectangular. If these lines of curvature are projected on a plane perpendicular to the axis of revolution of the surface, the parallels give concentric circles, and the meridians give right lines radiating from the common centre, and therefore cut all the concentric circles at right-angles.

Let us for an instant consider the lines of curvature of the ellipsoid, whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

From (7), Art. 398, we have

$$(b^2-c^2)\frac{x}{dx}+(c^2-a^2)\frac{y}{dy}+(a^2-b^2)\frac{z}{dz}=0. \qquad (64)$$

and this is the differential equation to the lines of curvature; from which however we have at present no means of deducing the integral equation: and therefore we must defer the discussion to the succeeding volume*.

The preceding properties of the lines of curvature, and of the developable surfaces formed by the consecutive normal planes, are of great importance in architecture. If an area is to be covered with a vaulted dome of stone or similar material, the plane joints of the voussoirs ought to be at right angles to the exposed surface of the stone; and therefore the lines of division of the stones in the vault ought to be lines of curvature of the vaulted dome; and then these side-surfaces of the voussoirs will be perpendicular to the exposed surfaces, and the pressures will be at right-angles to the surfaces which bear them. To the observations made by Monge as to the lines of joints of the voussoirs in a vault, I may add, that not only in vaulted domes, but in all kinds of curved masonry and woodwork the lines of joint ought to be the lines of curvature, and the surface joints of the voussoirs ought to be the developable surfaces which correspond to these lines. It is important therefore that we should be able to construct these lines of curvature; and with this object M. Ch. Dupin discovered the following general theorem:

---

* For a graphic description of the lines of curvature on an ellipsoid I must refer the reader to "Application de l'Analyse a Géométrie" of Monge.

410.] If there are three systems of surfaces which mutually intersect each other at right-angles, every two of them trace on the third its lines of curvature.

Let the three systems of surfaces be represented by the equations

$$\mathrm{F}_1(x,y,z) = f_1(a), \quad \mathrm{F}_2(x,y,z) = f_2(a), \quad \mathrm{F}_3(x,y,z) = f_3(a); \tag{65}$$

where a is a variable parameter by the variation of which each member of the several systems is formed; and where f_1, f_2, f_3 are symbols of functions and are such that the several systems intersect at right angles.

Let $\mathrm{U}_1, \mathrm{V}_1, \mathrm{W}_1$, u_1, v_1, w_1, u_1', v_1', w_1', and similar symbols, with the subscripts 2 and 3, represent the partial derived functions of (65) as heretofore; then, as the three surfaces intersect at right angles,

$$\left.\begin{array}{r}\mathrm{U}_2\mathrm{U}_3 + \mathrm{V}_2\mathrm{V}_3 + \mathrm{W}_2\mathrm{W}_3 = 0 \\ \mathrm{U}_3\mathrm{U}_1 + \mathrm{V}_3\mathrm{V}_1 + \mathrm{W}_3\mathrm{W}_1 = 0 \\ \mathrm{U}_1\mathrm{U}_2 + \mathrm{V}_1\mathrm{V}_2 + \mathrm{W}_1\mathrm{W}_2 = 0\end{array}\right\}. \tag{66}$$

Let dx, dy, dz be the projections on the axes of a length-element of the intersection of two members of F_2 and F_3; this is therefore perpendicular to a member of F_1; so that

$$\frac{dx}{\mathrm{U}_1} = \frac{dy}{\mathrm{V}_1} = \frac{dz}{\mathrm{W}_1}; \tag{67}$$

and the x, y, z are subject to the relation given in the third equation of (66); so that differentiating we have

$$\mathrm{U}_2\, d\mathrm{U}_1 + \mathrm{U}_1\, d\mathrm{U}_2 + \mathrm{V}_2\, d\mathrm{V}_1 + \mathrm{V}_1\, d\mathrm{V}_2 + \mathrm{W}_2\, d\mathrm{W}_1 + \mathrm{W}_1\, d\mathrm{W}_2 = 0; \tag{68}$$

and replacing $d\mathrm{U}_1, \ldots\ldots$ by their values, which are given in (61), Art. 360, we have

$$\begin{aligned}&\mathrm{U}_1\{\mathrm{U}_2 u_3 + \mathrm{U}_3 u_2 + \mathrm{V}_2 w_3' + \mathrm{V}_3 w_2' + \mathrm{W}_2 v_3' + \mathrm{W}_3 v_2'\} \\ &+ \mathrm{V}_1\{\mathrm{U}_2 w_3' + \mathrm{U}_3 w_2' + \mathrm{V}_2 v_3 + \mathrm{V}_3 v_2 + \mathrm{W}_2 u_3' + \mathrm{W}_3 u_2'\} \\ &+ \mathrm{W}_1\{\mathrm{U}_2 v_3' + \mathrm{U}_3 v_2' + \mathrm{V}_2 u_3' + \mathrm{V}_3 u_2' + \mathrm{W}_2 w_3 + \mathrm{W}_3 w_2\} = 0;\end{aligned} \tag{69}$$

and this is in terms of second partial derived functions the condition that the line of intersection of two members of F_2 and F_3 should at the common point of intersection with the member of F_1 be perpendicular to F_1. And two other similar expressions can be found, because the lines of intersection of members of F_3 and F_1, and of F_1 and F_2 respectively, are perpendicular to the members of F_2 and of F_3. If

$$\mathrm{U}_2\mathrm{U}_3 u_1 + \mathrm{V}_2\mathrm{V}_3 v_1 + \mathrm{W}_2\mathrm{W}_3 w_1$$
$$+ (\mathrm{V}_2\mathrm{W}_3 + \mathrm{V}_3\mathrm{W}_2) u_1' + (\mathrm{W}_2\mathrm{U}_3 + \mathrm{W}_3\mathrm{U}_2) v_1' + (\mathrm{U}_2\mathrm{V}_3 + \mathrm{U}_3\mathrm{V}_2) w_1' = \mathrm{A}_1; \quad (70)$$

and if similar values are formed by the interchange of subscripts for A_2 and A_3; then (69) and the other two similar expressions become

$$\mathrm{A}_2 + \mathrm{A}_3 = 0, \qquad \mathrm{A}_3 + \mathrm{A}_1 = 0, \qquad \mathrm{A}_1 + \mathrm{A}_2 = 0; \quad (71)$$

$$\therefore \quad \mathrm{A}_1 = \mathrm{A}_2 = \mathrm{A}_3 = 0; \quad (72)$$

so that the conditions requisite that the three systems of surfaces should intersect at right angles are

$$\mathrm{U}_2\mathrm{U}_3 u_1 + \mathrm{V}_2\mathrm{V}_3 v_1 + \mathrm{W}_2\mathrm{W}_3 w_1$$
$$+ (\mathrm{V}_2\mathrm{W}_3 + \mathrm{V}_3\mathrm{W}_2) u_1' + (\mathrm{W}_2\mathrm{U}_3 + \mathrm{W}_3\mathrm{U}_2) v_1' + (\mathrm{U}_2\mathrm{V}_3 + \mathrm{U}_3\mathrm{V}_2) w_1' = 0; \quad (73)$$

$$\mathrm{U}_3\mathrm{U}_1 u_2 + \mathrm{V}_3\mathrm{V}_1 v_2 + \mathrm{W}_3\mathrm{W}_1 w_2$$
$$+ (\mathrm{V}_3\mathrm{W}_1 + \mathrm{V}_1\mathrm{W}_3) u_2' + (\mathrm{W}_3\mathrm{U}_1 + \mathrm{W}_1\mathrm{U}_3) v_2' + (\mathrm{U}_3\mathrm{V}_1 + \mathrm{U}_1\mathrm{V}_3) w_2' = 0; \quad (74)$$

$$\mathrm{U}_1\mathrm{U}_2 u_3 + \mathrm{V}_1\mathrm{V}_2 v_3 + \mathrm{W}_1\mathrm{W}_2 w_3$$
$$+ (\mathrm{V}_1\mathrm{W}_2 + \mathrm{V}_2\mathrm{W}_1) u_3' + (\mathrm{W}_1\mathrm{U}_2 + \mathrm{W}_2\mathrm{U}_1) v_3' + (\mathrm{U}_1\mathrm{V}_2 + \mathrm{U}_2\mathrm{V}_1) w_3' = 0; \quad (75)$$

now from the first of these I propose to shew that an element of the line of intersection of two members of F_1 and F_2, at the point where they are intersected by the corresponding member of F_3, is an element of a line of curvature of the member of F_1. From (66) we have

$$-\mathrm{U}_2\mathrm{U}_3 = \mathrm{V}_2\mathrm{V}_3 + \mathrm{W}_2\mathrm{W}_3,$$

$$\mathrm{U}_2\mathrm{U}_3 = \frac{(\mathrm{V}_1\mathrm{V}_2 + \mathrm{W}_1\mathrm{W}_2)(\mathrm{V}_1\mathrm{V}_3 + \mathrm{W}_1\mathrm{W}_3)}{\mathrm{U}_1^2};$$

$$\therefore \quad (\mathrm{V}_2\mathrm{V}_3 + \mathrm{W}_2\mathrm{W}_3)\mathrm{U}_1^2 + (\mathrm{V}_1\mathrm{V}_2 + \mathrm{W}_1\mathrm{W}_2)(\mathrm{V}_1\mathrm{V}_3 + \mathrm{W}_1\mathrm{W}_3) = 0;$$

$$\therefore \quad \mathrm{V}_2\mathrm{W}_3 + \mathrm{V}_3\mathrm{W}_2 = \frac{\mathrm{U}_1^2\mathrm{U}_2\mathrm{U}_3 - \mathrm{V}_1^2\mathrm{V}_2\mathrm{V}_3 - \mathrm{W}_1^2\mathrm{W}_2\mathrm{W}_3}{\mathrm{V}_1\mathrm{W}_1}; \quad (76)$$

similarly,
$$\mathrm{W}_2\mathrm{U}_3 + \mathrm{W}_3\mathrm{U}_2 = \frac{\mathrm{V}_1^2\mathrm{V}_2\mathrm{V}_3 - \mathrm{W}_1^2\mathrm{W}_2\mathrm{W}_3 - \mathrm{U}_1^2\mathrm{U}_2\mathrm{U}_3}{\mathrm{W}_1\mathrm{U}_1}, \quad (77)$$

$$\mathrm{U}_2\mathrm{V}_3 + \mathrm{U}_3\mathrm{V}_2 = \frac{\mathrm{W}_1^2\mathrm{W}_2\mathrm{W}_3 - \mathrm{U}_1^2\mathrm{U}_2\mathrm{U}_3 - \mathrm{V}_1^2\mathrm{V}_2\mathrm{V}_3}{\mathrm{U}_1\mathrm{V}_1}; \quad (78)$$

and substituting these in (73), we have

$$\mathrm{U}_2\mathrm{U}_3 u_1 + \mathrm{V}_2\mathrm{V}_3 v_1 + \mathrm{W}_2\mathrm{W}_3 w_1 + \frac{u_1'}{\mathrm{V}_1\mathrm{W}_1}\{\mathrm{U}_1^2\mathrm{U}_2\mathrm{U}_3 - \mathrm{V}_1^2\mathrm{V}_2\mathrm{V}_3 - \mathrm{W}_1^2\mathrm{W}_2\mathrm{W}_3\}$$
$$+ \frac{v_1'}{\mathrm{W}_1\mathrm{U}_1}\{\mathrm{V}_1^2\mathrm{V}_2\mathrm{V}_3 - \mathrm{W}_1^2\mathrm{W}_2\mathrm{W}_3 - \mathrm{U}_1^2\mathrm{U}_2\mathrm{U}_3\}$$
$$+ \frac{w_1'}{\mathrm{U}_1\mathrm{V}_1}\{\mathrm{W}_1^2\mathrm{W}_2\mathrm{W}_3 - \mathrm{U}_1^2\mathrm{U}_2\mathrm{U}_3 - \mathrm{V}_1^2\mathrm{V}_2\mathrm{V}_3\} = 0. \quad (79)$$

Let ds be the length-element on F_1, which is the intersection of F_1 and F_2, and is perpendicular to F_3; then

$$\mathrm{U}_2\,dx+\mathrm{V}_2\,dy+\mathrm{W}_2\,dz = 0, \qquad \mathrm{U}_2\mathrm{U}_1+\mathrm{V}_2\mathrm{V}_1+\mathrm{W}_2\mathrm{W}_1 = 0;$$

$$\therefore \quad \frac{\mathrm{U}_2}{\mathrm{W}_1\,dy-\mathrm{V}_1\,dz} = \frac{\mathrm{V}_2}{\mathrm{U}_1\,dz-\mathrm{W}_1\,dx} = \frac{\mathrm{W}_2}{\mathrm{V}_1\,dx-\mathrm{U}_1\,dy}; \tag{80}$$

also

$$\frac{\mathrm{U}_3}{dx} = \frac{\mathrm{V}_3}{dy} = \frac{\mathrm{W}_3}{dz}. \tag{81}$$

Therefore in (79) replacing U_2, V_2, W_2, U_3, V_3, W_3, by these proportionals, we have, after all reductions,

$$dy\,dz\,\mathrm{U}_1\,\Big\{v_1-w_1+\frac{\mathrm{V}_1}{\mathrm{W}_1\mathrm{U}_1}(\mathrm{V}_1v_1'-\mathrm{W}_1w_1'-\mathrm{U}_1u_1')$$
$$-\frac{\mathrm{W}_1}{\mathrm{U}_1\mathrm{V}_1}(\mathrm{W}_1w_1-\mathrm{U}_1u_1'-\mathrm{V}_1v_1')\Big\}+\ldots+\ldots = 0; \tag{82}$$

and if we substitute in this expression the values of H, K, L given in (19), Art. 399, we have

$$\mathrm{U}_1(\mathrm{K}_1-\mathrm{L}_1)\,dy\,dz+\mathrm{V}_1(\mathrm{L}_1-\mathrm{H}_1)\,dz\,dx+\mathrm{W}_1(\mathrm{H}_1-\mathrm{K}_1)\,dx\,dy = 0, \tag{83}$$

which is the same as (24), Art. 399, and is the differential equation to a line of curvature on F_1. It is also evident that we should have obtained the same equation if by means of (73) we had investigated the condition that the line of intersection of F_1 and F_3 should be perpendicular to F_2 at the common point of intersection: and as (83) is a quadratic equation, in terms of the proportions of the length-element, its two roots refer to the two lines of intersection of F_2 and F_3 with F_1; and therefore we conclude that the lines of intersection of F_2 and F_3 with F_1 at the common point are the lines of curvature of F_1.

If processes in all respects similar are performed on (74) and (75), two equations with the subscripts 2 and 3, and similar in form to (83), will result: and therefore the members of F_3 and F_1 will intersect all the members of F_2, and the members of F_1 and F_2 will intersect all the members of F_3, along their respective lines of curvature: and this is Dupin's Theorem.

If therefore a system of surfaces of the form F_1 is given, and if we can find any other two systems such as F_2 and F_3, the several members of which intersect all the other members of F_1 and of each other at right-angles, then the lines of intersection are lines of curvature.

411.] Of this proposition we have one remarkable example.

Let us call surfaces of the second order confocal, when their principal sections are confocal. Then three systems of confocal surfaces of the second order, which are severally an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets, intersect at right angles; as we may thus prove. As the surfaces are confocal we may take their equations to be

$$\left.\begin{aligned}
F_1(x,y,z) &= \frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2-b^2}+\frac{z^2}{\lambda^2-c^2} = 1,\\
F_2(x,y,z) &= \frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}+\frac{z^2}{\mu^2-c^2} = 1,\\
F_3(x,y,z) &= \frac{x^2}{\nu^2}+\frac{y^2}{\nu^2-b^2}+\frac{z^2}{\nu^2-c^2} = 1;
\end{aligned}\right\} \qquad (84)$$

where $\lambda^2 > b^2 > c^2$; $\mu^2 > b^2 < c^2$; $\nu^2 < b^2 < c^2$; so that the three equations represent severally an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. Now

$$\begin{aligned}
U_2U_3+V_2V_3+W_2W_3 &= \frac{4x^2}{\mu^2\nu^2}+\frac{4y^2}{(\mu^2-b^2)(\nu^2-b^2)}+\frac{4z^2}{(\mu^2-c^2)(\nu^2-c^2)}\\
&= \frac{4}{\nu^2-\mu^2}\left\{\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}+\frac{z^2}{\mu^2-c^2}-\frac{x^2}{\nu^2}-\frac{y^2}{\nu^2-b^2}-\frac{z^2}{\nu^2-c^2}\right\}\\
&= \frac{4}{\nu^2-\mu^2}\{1-1\}\\
&= 0;
\end{aligned}$$

and therefore all the members of F_2 and F_3 intersect each other at right-angles. Similarly it may be shewn that all the members of F_1 intersect at right-angles all the members of F_2 and F_3.

Hence by Dupin's Theorem we infer that the lines of curvature on an ellipsoid are the lines of intersection of it with two confocal hyperboloids, which are respectively of one and two sheets; and thus the construction of the lines of curvature is reduced to the much more simple problem of the intersection of surfaces.

412] On the locus-surface of the centres of principal curvature.

Imagine at a given point on a surface the two lines of curvature to be drawn; and at every point along one of these lines of curvature imagine normal planes to be drawn touching it; these will by their intersection generate a developable surface, cutting the given surface at right angles; the normal line will be the characteristic of this developable surface; and the curve formed by their intersection will be its edge of regression, which will also be the locus curve of the centres of principal curvature whose section touches the line of curvature.

Similarly, if normal planes are drawn touching the other line of curvature which passes through the given point, another developable surface will be generated which cuts the former at right angles; and there will also be another edge of regression, which is the locus-curve of these second centres of principal curvature.

And as a similar process will hold true for all points of a surface, so will the series of developable surfaces arising from the first line of curvature cut at right angles each of the series arising from the second line of curvature, and thus will space be filled with developable surfaces intersecting each other at right angles; and as the edges of regression belonging to the first line of curvature continuously vary, so will they generate a surface-locus of all the corresponding centres of curvature. And similarly will another surface be generated by the other centres of curvature. We shall hereby obtain a surface of two sheets, to each of which the normal of the surface is a tangent; and any two planes drawn through the normal and touching the two sheets are at right angles to each other.

Hence it appears that the surfaces which are the locus-surfaces of the two centres of principal curvature must have particular properties. They will be of two sheets; their algebraical equations therefore must be of even dimensions; these sheets must intersect at right angles. A surface therefore which does not fulfil these two conditions cannot be the surface of the centres of curvature; it may be the locus-surface of the centres of one principal curvature, but it will require another surface to be its conjugate, and this will be the locus-surface of the other centre of principal curvature.

If at any point the two sheets of the locus-surface of the centres of principal curvature intersect each other, so that their edges of regression have a common point, the principal radii of curvature of the original surface corresponding to that point are equal, and there is an umbilic; and if there is a continuous locus of such points of intersection, the corresponding line on the original surface is *a line of spherical curvature*. See Art. 407.

The analytical process for finding the equation of this surface-locus of two sheets is obvious enough. The equations to the surface, to its lines of curvature through a given point, and to the points of intersection of consecutive normals, or the coordinates to the centre of curvature given by equations (11), are sufficient for eliminating $x\,y\,z$, and for giving an equation in terms of $\xi\ \eta$ and ζ.

As at an umbilic the equations (8) or (24) for determining the directions of the lines of curvature give indeterminate results, we must evaluate them by differentiation according to the method of Art. 139. On examining which it will be seen that each differentiation increases by unity the power of dx, dy, dz; and therefore as we begin with a quadratic, after one differentiation we shall have a cubic; after two differentiations a biquadratic; and so on. Suppose then that the directions are determinate after one differentiation, if all the three roots are real there will be three lines of curvature passing through the point; if two roots are impossible, there will be but one line of curvature; and such is the case at the umbilics on the ellipsoid. Similarly may there be four or more lines of curvature at an umbilic; nay, an infinite number, as is the case at the pole of a surface of revolution.

The lines of curvature are generally non-plane curves, and have a contact of only the first order with the principal normal section, which they touch at the point of contact: this is manifest from the fact that the equation of the lines of curvature involves differentials of the first order only.

413.] If the equation to the surface is given in the explicit form

$$z = f(x, y), \tag{85}$$

we must replace as follows; and as the results assume forms from which most of the properties of the curvature of surfaces have been derived in former text books, we give them in order to exhibit the identity of the conclusions:

$$\mathrm{F}(x, y, z) = f(x, y) - z = 0. \tag{86}$$

Let $$\left(\frac{dz}{dx}\right) = p, \qquad \left(\frac{dz}{dy}\right) = q,$$

$$\left(\frac{d^2 z}{dx^2}\right) = r, \qquad \left(\frac{d^2 z}{dx\,dy}\right) = s, \qquad \left(\frac{d^2 z}{dy^2}\right) = t;$$

$$\therefore\quad \left.\begin{array}{lll} \text{U} = p, & \text{V} = q, & \text{W} = -1, \\ u = r, & v = t, & w = 0, \\ u' = 0, & v' = 0, & w' = s; \end{array}\right\} \tag{87}$$

also since
$$dz = p\,dx + q\,dy, \tag{88}$$
therefore equation (8) of the lines of curvature becomes

$$dx^2\{s(1+p^2)-pqr\}+dy\,dx\,\{t(1+p^2)-r(1+q^2)\} \\ -dy^2\{s(1+q^2)-pqt\} = 0. \tag{89}$$

This is a quadratic equation in terms of $dy : dx$, and is the differential equation to the projections of the lines of curvature on the plane of xy. Suppose the coordinate planes to be so chosen that the tangent plane at the point under consideration is parallel to the plane of xy, then $p = 0$, $q = 0$, and equation (89) becomes

$$\left(\frac{dy}{dx}\right)^2+\frac{r-t}{s}\frac{dy}{dx}-1 = 0; \tag{90}$$

and as the product of the two roots $= -1$, it follows that the lines of curvature are perpendicular to each other.

Hence it follows, that there is only one line of curvature through an umbilic of the ellipsoid, and that it is in the section of the greatest and least axes.

The equation (15) of the principal radii of curvature becomes

$$\rho = \frac{(1+p^2+q^2)^{\frac{1}{2}}}{rl^2+2slm+tm^2}; \tag{91}$$

the conditions (49), (50), (54) become

$$rt-s^2 >, <, = 0. \tag{92}$$

Hence the condition of a developable surface is, see Art. 360,

$$rt-s^2 = 0. \tag{93}$$

And the condition (55), that the two principal radii of curvature should be equal and affected with opposite signs, is

$$(1+q^2)r-2pqs+(1+p^2)t = 0. \tag{94}$$

Also the general conditions (57) for an umbilic become

$$\frac{r}{1+p^2} = \frac{s}{pq} = \frac{t}{1+q^2}, \tag{95}$$

and of the three sets of special conditions, (61) and (62) become

$$p = 0, \qquad s = 0, \qquad r(1+q^2) = t, \tag{96}$$

$$q = 0, \qquad s = 0, \qquad t(1+p^2) = r; \tag{97}$$

(63) cannot be satisfied, since W cannot vanish.

Hence also we have a geometrical interpretation of Lagrange's condition that a function of two variables, say $z = f(x, y)$, should admit of a maximum or of a minimum value.

The condition is that, see Art. 158 and 159,

$$\left(\frac{d^2 z}{dx^2}\right)\left(\frac{d^2 z}{dy^2}\right) - \left(\frac{d^2 z}{dx\,dy}\right)^2,$$

or $rt - s^2$, should be positive; whence it follows, that the principal radii of curvature must be measured in the same direction. Now at a point where $\left(\frac{dz}{dx}\right) = \left(\frac{dz}{dy}\right) = 0$, the tangent plane is parallel to that of xy; if then all the radii of curvature of the normal sections at that point are measured in the same direction, z is a maximum or a minimum: but if some are turned in one direction and others in the opposite direction, that is, if $rt - s^2$ is negative, there will be a partial maximum and a partial minimum, but no total maximum or minimum; and if $rt - s^2 = 0$, then the surface is developable, and the generating line will give a series of partial maxima or minima.

414.] Meunier's Theorem on the curvature of oblique sections of a surface.

The two cases of principal and of ordinary normal sections having thus been investigated, it remains for us to consider the third case of Art. 397; that in which a curve is described on a surface, but the osculating plane to which at the given point is not normal to the surface.

Let ρ' be the radius of absolute curvature of such a curve, and let λ, μ, ν be the direction-angles of its direction; then, taking s to be equicrescent, by (26), Art. 378,

$$\cos\lambda = \rho'\frac{d^2x}{ds^2}, \qquad \cos\mu = \rho'\frac{d^2y}{ds^2}, \qquad \cos\nu = \rho'\frac{d^2z}{ds^2}; \quad (98)$$

multiplying these severally by $\frac{\mathrm{U}}{\mathrm{Q}}, \frac{\mathrm{V}}{\mathrm{Q}}, \frac{\mathrm{W}}{\mathrm{Q}}$, and adding, we have

$$\rho'\frac{\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z}{\mathrm{Q}\,ds^2} = \frac{\mathrm{U}\cos\lambda + \mathrm{V}\cos\mu + \mathrm{W}\cos\nu}{\mathrm{Q}}; \quad (99)$$

but the right-hand member of the equation is the cosine of the angle between the radius of curvature of the oblique section and the normal to the surface, $= \cos\psi$, say; and by equation (12), if ρ is the radius of curvature of the normal section at the point, and which has the same tangent, the latter factor of the left-hand member $= \frac{1}{\rho}$,

$$\therefore \quad \rho' = \rho \cos \theta. \tag{100}$$

Hence the radius of curvature of an oblique section is equal to the projection on the osculating plane at the point, of the radius of curvature of the normal section of the surface which has the same tangent line with the oblique section.

Hence if a sphere is described having for centre and radius the centre and radius of curvature of any normal section, all the oblique sections which touch the normal section at the point on the surface have, for osculating circles at the common point, the small circles of the sphere made by their respective planes.

415.] As whatever tends to elucidate the difficulties of an obscure subject deserves attention, I do not hesitate to introduce the following process, although it proves theorems which have been discussed in the previous Articles; and it exhibits the relations existing between the curvatures of normal sections in a remarkable light, and hereby indicates the nature of a point of a surface at which the partial derived-functions are not indeterminate.

Let the point of the surface under consideration be taken as the origin, and let the tangent plane be that of xy; and therefore the normal is the axis of z. Let the equation to the surface be

$$z = f(x, y). \tag{101}$$

At an infinitesimal distance dz from the origin let a plane be drawn parallel to that of xy, and cutting the surface; the curve of section we will, after M. Ch. Dupin,* call the *indicatrix*, as the form of it indicates the nature of the surface at the origin; let dx, dy, dz be the coordinates to a point on this curve; and through that point and the axis of z let a normal section be drawn, making an angle a with the plane of xz, so that $\tan a = \frac{dy}{dx}$; and let ds be the arc of the normal section of the surface between the origin and the point (dx, dy, dz); then

$$ds^2 = dx^2 + dy^2 + dz^2; \tag{102}$$

and if ρ is the radius of curvature at the origin of this normal section, and which lies along the axis of z, from the geometry of the circle, we have

---

* Développements de Géométrie; par Ch. Dupin, Paris 1813, page 48.

$$\rho = \frac{ds^2}{2dz}; \tag{103}$$

that is, the radius of curvature of a normal section varies as the square of the distance between the point and the intersection of the normal plane with the indicatrix.

Using the notation of Art. 413, and expanding according to Art. 140, we have

$$z+dz = z+p\,dx+q\,dy+\frac{1}{1.2}\{r\,dx^2+2s\,dx\,dy+t\,dy^2\}+\dots, \tag{104}$$

and neglecting higher powers of the infinitesimals dx and dy, and observing that $p = q = 0$, because the normal at the origin is perpendicular to the axes of x and y, we have

$$dz = \frac{1}{2}\{r\,dx^2+2s\,dx\,dy+t\,dy^2\}; \tag{105}$$

and therefore

$$\rho = \frac{ds^2}{r\,dx^2+2s\,dx\,dy+t\,dy^2}; \tag{106}$$

which equation is equivalent to (15), and gives the value of the radius of curvature of the normal section.

As ρ is generally finite, it appears from equation (103) that dz is an infinitesimal of the same order as ds^2; therefore in equation (102) dz^2 must be neglected, and we have

$$ds^2 = dx^2+dy^2. \tag{107}$$

Hence

$$\rho = \frac{dx^2+dy^2}{r\,dx^2+2s\,dx\,dy+t\,dy^2}, \tag{108}$$

$$= \frac{1}{r(\cos a)^2+2s\sin a\cos a+t\,(\sin a)^2}; \tag{109}$$

which result is the same as equation (91), and from which therefore the properties of maxima and minima radii of curvature might be deduced.

Suppose the coordinate axes of x and y to be turned about the axis of z through an angle θ, such that

$$\tan 2\theta = \frac{2s}{r-t}; \tag{110}$$

then (105) becomes

$$2\,dz = r\,dx^2+t\,dy^2; \tag{111}$$

which equation, if we consider dx, dy, dz to be the coordinates to a point on the surface near to the origin, is that of a paraboloid; of which the principal sections are those made by the planes of xz and yz; and if ρ_x and ρ_y are the radii of curvature of these sections respectively, $\rho_x = \frac{1}{r}$, $\rho_y = \frac{1}{t}$. If the axes of x and y are turned about that of z as above, (109) becomes

$$\begin{aligned} \frac{1}{\rho} &= r(\cos a)^2 + t(\sin a)^2 \\ &= \frac{(\cos a)^2}{\rho_x} + \frac{(\sin a)^2}{\rho_y}, \end{aligned} \tag{112}$$

which is Euler's Theorem; see Art. 403.

To return to the equation of the indicatrix, viz. (105); dz, being the distance between the parallel planes of xy and of that of the curve, is constant; and dx and dy are the rectangular coordinates to the indicatrix, the origin being at the point where the axis of z cuts its plane, and ds is the radius vector; hence, replacing dx and dy by ξ and η, and dz by $\frac{c}{2}$ we have

$$r\xi^2 + 2s\xi\eta + t\eta^2 = c, \tag{113}$$

which is an equation of the second degree, referred to its centre as origin, and represents an ellipse or hyperbola according as $rt - s^2$ is positive or negative; and represents a circle if $r = t$, and $s = 0$; and two parallel straight lines if $rt - s^2 = 0$. Hence we conclude, that if a surface is cut by a plane parallel and infinitesimally near to a tangent plane, the curve of section is either an ellipse, a hyperbola, or two parallel straight lines; the ellipse of course admitting of the variety of a circle, and the hyperbola in certain cases being rectangular, and in other two intersecting straight lines.

If the indicatrix is an ellipse the surface is wholly concave towards it, such as is the case at all points of an ellipsoid; and, if it is a hyperbola, some part of the surface about the point has its curvature turned in one direction and some part in the opposite; and if the indicatrix is two parallel straight lines, the surface is concave towards them in a direction perpendicular to them, but is in a straight line in a direction parallel to them.

Also since ds is the central radius vector of the indicatrix, and since the radius of curvature of the normal section varies as ds^2, the latter quantity partakes of *singular* values analogous to those which the former admits of.

In the ellipse all the radii vectores are real; therefore if the indicatrix is an ellipse, that is, if $rt-s^2$ is positive, all the radii of curvature of normal sections are turned in the same direction: the radii vectores of the ellipse have two maxima and two minima values, which are at right angles to each other; therefore the radii of curvature of normal sections have values respectively a maximum and a minimum, which are perpendicular to each other. In the circle all the radii vectores are equal; therefore, if $r = t$ and $s = 0$, all the radii of curvature of normal sections are equal, and there is an umbilic; thus the tangent plane at an umbilic of an ellipsoid is parallel to a plane of circular sections.

In the hyperbola some of the radii vectores are real and some are impossible; therefore if $rt-s^2$ is negative, the radii of curvature of normal sections are turned in one direction for all real radii vectores of the hyperbola, and in the opposite direction for the impossible ones; the asymptotes being the lines bounding the parts which have their curvatures turned in opposite directions; and if the hyperbola is rectangular, equal portions of the surface at the point have their curvatures turned in opposite directions. Hence also, as the principal axes of the hyperbola are at right angles, one being real and the other being impossible, so will the sections of greatest and least curvature be at right angles to each other, and the radii will be turned in opposite directions.

If $rt = s^2$, that is if the indicatrix is two parallel straight lines, the origin being at a middle point between them, the radii vectores which are perpendicular to the lines are the least, and the normal section coincident with them is that of greatest curvature; but as the line, which is parallel to and bisects them, never meets them, the corresponding radius of curvature is infinite, and the curvature of the coincident normal section vanishes. This is manifestly the case with developable surfaces.

416.] Hence also it is plain, that if the condition of *osculation* of two surfaces is made to depend on the second derived functions as well as the first being the same in both, or in other words, on the two surfaces having the same indicatrix, a surface of the second order can always be found to osculate to a given surface at a given point; and that, in the case of an umbilic, the surface may be a sphere, and in a developable surface it becomes a cylinder.

417.] **On the measure of curvature of a surface at a given point.**

Gauss in his celebrated memoir "*Disquisitiones Generales circa Superficies Curvas,*" has introduced a definition of curvature of a surface which is derived analogously from the means of measuring the curvature of a plane curve; and from his definition has deduced a mathematical estimate of curvature.

Suppose Δs to be the finite arc of a plane curve commencing at a point P; and at the extremities of Δs let two normals to the curve be drawn. In the same plane take a circle whose radius is unity, and through its centre let two radii be drawn parallel to the two normals at the extremities of Δs, and let the radii include an angle or arc $\Delta\psi$; then the limit $\frac{d\psi}{ds}$ towards which $\frac{\Delta\psi}{\Delta s}$ converges, when the arc of the original curve is infinitesimal, is, according to the definition of Art. 281, the curvature of the curve at the point P.

Imagine now upon a curved surface a finite area enclosed by a contour, within which is a given point P; and also imagine a sphere whose radius is unity; and suppose normals to the surface to be drawn at every point of the enclosing contour, and radii of the sphere to be drawn parallel to these normals; by this process a spherical area will be enclosed on the surface of the sphere.

Let $\Delta \mathrm{s}$ be the area enclosed by the contour, and $\Delta \mathrm{z}$ the area of the enclosed figure on the surface of the sphere; then the limit $\frac{d\mathrm{z}}{d\mathrm{s}}$ towards which $\frac{\Delta\mathrm{z}}{\Delta\mathrm{s}}$ converges, when the contour becomes infinitesimal but still encloses the point P, is the curvature of the surface at the point P.

Let the area $\Delta\mathrm{z}$ on the given surface be a rectangle contained by four lines of curvature; and let $\Delta\phi_1$, $\Delta\phi_2$ be the angles subtended at the centres of principal curvature by two adjacent sides of the rectangle;

$$\therefore \quad \Delta\mathrm{s} = \rho_1\rho_2\,\Delta\phi_1\,\Delta\phi_2; \tag{114}$$

and similarly,

$$\Delta\mathrm{z} = \Delta\phi_1\,\Delta\phi_2; \tag{115}$$

$$\therefore \quad \text{the curvature} = \frac{d\mathrm{z}}{d\mathrm{s}} = \frac{1}{\rho_1\rho_2}; \tag{116}$$

and the curvature of the surface at any point is equal to the product of the curvatures of the principal normal sections at the same point.

The truth of the result is manifestly independent of the form of the small area; for whatever its form be, it can always be divided into a number of infinitesimal rectangles, for every one of which the result of equation (116) will be true; and therefore by simple addition the aggregate of all, which is expressed by equation (116), will be true also. The curvature then will be affected with a positive or a negative sign, according as the radii of the principal normal sections have the same or different signs.

In the case of developable surfaces, one of the principal normal sections has an infinite radius of curvature; it would therefore follow from (116), that the curvature of a developable surface is zero: but such is the case only with a plane. We must therefore retrace our steps and modify the process in the following manner, by operating on a right circular cylinder whose radius is unity instead of on a sphere:

Let the two containing sides of the rectangular area on the given surface, and which are coincident with the lines of curvature, be Δs_1 and Δs_2; of which let Δs_2 lie along a generating line of the developable surface, and Δs_1 be at right angles to it; then in the limit, if $d\phi_1$ be the angle subtended by ds_1 at the centre of principal curvature, $ds_1 = \rho_1 \, d\phi_1$. Let the axis of the cylinder be parallel to the generating line of the developable at the given point, and from a point in the axis of the cylinder let normals be drawn to the cylinder parallel to normals drawn to the developable surface along Δs_1, and let $\Delta \sigma_1$ be the intercepted arc of the circle on the surface of the cylinder; then $\Delta \sigma_1 = \Delta \phi_1$; also let a line equal to Δs_2 be taken on the surface of the cylinder and perpendicular to $\Delta \sigma_1$ through one of its extremities; by this process therefore

$$\begin{aligned} d\mathrm{S} &= \rho_1 \, d\phi_1 \, ds_2, \\ d\Sigma &= d\phi_1 \, ds_2, \end{aligned} \qquad \therefore \quad \frac{d\Sigma}{d\mathrm{S}} = \frac{1}{\rho_1}; \tag{117}$$

and therefore the curvature of a developable surface at a given point is equal to the curvature of the principal normal section which is perpendicular to a generating line.

I must in conclusion observe that a different definition of curvature of a surface has been given by Madame Sophie Germain; for which I must refer the reader to her own memoir on the Curvature of Surfaces; see Crelle, VII. 1; and Récherches sur la Théorie des Surfaces Élastiques, Paris, 1821 and 1826.

CHAPTER XIX.

THE CALCULUS OF OPERATIONS.

418.] MANY theorems, which occur in common Algebra and elsewhere, are true, not because the subjects of them have any special nature as quantities, but because they conform to certain laws and combine in certain manners; and these theorems therefore are equally true of all symbols which are subject to these laws. For the sake of an example let us take the binomial theorem, and its ordinary proof: n simple factors of the form $x-a$, $x-b$, ... are multiplied together; and the law of their combination having been detected, we are thence led to a general form; and if we assume $a=b=c= \dots$, we obtain the general expansion of $(x-a)^n$. Now this theorem is true, not because x, a, b, ... are symbols of quantity, but because being symbols of quantity they satisfy certain laws, in conformity with which the said theorem is. All other symbols therefore which satisfy these laws are subject to the same theorem. Similarly other algebraical theorems of expansion, and expansions such as those which have been given in the preceding pages for e^x, $\sin x$, $\tan^{-1} x$, and, in short, Maclaurin's and Taylor's Series, are true of all symbols which satisfy certain laws.

A distinction, and a division consequent upon that distinction, has been made of symbols; some are called symbols of operation, others symbols of quantity: those which indicate processes to be performed on subjects, such as symbols of differentiation; those of trigonometrical operations, as sin, tan, $\sec^{-1}$; those of logarithmic affection, as $\log_a$, $e^{(\ )}$, &c. are called symbols of operation: whereas the subjects on which these operations are performed are called symbols of quantity. Thus if a, b, x, y are symbols of the ordinary quantities of Algebra, they are called symbols of quantities. The distinction however is scarcely accurate; for may not x be the same as $x \times 1$? in which case x symbolizes a process, that of multiplication, performed on unity, and is a symbol of operation; but it is useful enough for our purpose.

419.] Let us investigate and define two or three laws to which symbols whether of operation or of quantity may be subject.

Let ϕ, ψ be symbols of operation; and let u and v be symbols of quantity, and of subjects on which ϕ and ψ are performed. And let us suppose ϕ, ψ, u to be such that

$$\phi\psi(u) = \psi\phi(u); \tag{1}$$

that is, if the two operations indicated by ϕ and ψ are performed successively on u, one on the back of the other, let us assume the result to be the same whatever is the order in which they are performed: two such symbols of operation are said to be commutative, and to satisfy the law of commutation.

Similarly, again, if ϕ, ψ, χ are such that

$$\phi\psi\chi(u) = \psi\chi\phi(u) = \chi\phi\psi(u), \tag{2}$$

these symbols are commutative. Quantities multiplied into each other, satisfy this commutative law: thus if a, b, c are constants

$$a \times b \times c \times u = b \times c \times a \times u = c \times b \times a \times u. \tag{3}$$

Symbols of differentiation are also subject to the same law: thus, if u is a function of x and y,

$$\frac{d}{dy}\frac{d}{dx}u = \frac{d}{dx}\frac{d}{dy}u; \tag{4}$$

and a similar theorem is true if u is a function of three or more variables. The form in which the law of commutation has been heretofore expressed as to differentiation is, "the order of differentiation is indifferent." See Art. 79. Trigonometrical functions are not generally subject to this law; thus, $\sin 2x$ is not equivalent to $2\sin x$; neither is $\sin\tan^{-1}x$ equivalent to $\tan^{-1}\sin x$.

Again, let ϕ be a symbol of operation, and u and v two symbols of quantity; and let ϕ be such that

$$\phi(u+v) = \phi(u) + \phi(v); \tag{5}$$

then the operation expressed by ϕ is said to be distributive, and ϕ is said to satisfy the distributive law. Similarly the number of subjects of the function may be n; and we may have

$$\phi(u_1 + u_2 + \dots\dots + u_n) = \phi(u_1) + \phi(u_2) + \dots\dots + \phi(u_n). \tag{6}$$

Let us at present take only two subjects, as in (5); and let us operate on them again with the symbol ϕ; then, if we symbolize two such operations, performed successively on a subject by ϕ^2, we have

$$\begin{aligned}\phi^2(u+v) &= \phi\{\phi(u+v)\} \\ &= \phi\{\phi(u)+\phi(v)\} \\ &= \phi^2(u)+\phi^2(v);\end{aligned}$$

and if the operation is performed n times successively

$$\phi^n(u+v) = \phi^n(u)+\phi^n(v). \tag{7}$$

Symbols of multiplication are in conformity with this law; thus, if a is a constant,

$$a(u+v) = au+av, \qquad a^n(u+v) = a^n u + a^n v.$$

Symbols of differentiation are also subject to it; thus, if u and v are functions of x,

$$\frac{d}{dx}(u+v) = \frac{du}{dx}+\frac{dv}{dx}; \tag{8}$$

$$\frac{d^n}{dx^n}(u+v) = \frac{d^n u}{dx^n}+\frac{d^n v}{dx^n}; \tag{9}$$

trigonometrical operations are not subject to it; thus, sin $(u+v)$ is not equivalent to $\sin u + \sin v$.

Constants and other symbols of multiplication are also subject to a law of notation; or rather a law of notation has been framed with respect to them to which other operative symbols may be subject, when they are repeated successively on the back of each other. The origin of this law which is analogically extended is the following; let a be a constant multiplied into u; then

$$a^m a^n u = a^{m+n} u;$$

that is, if u is multiplied by a first n times, and then m times, the result is the same as if u had been multiplied $m+n$ times by a. So if ϕ is a symbol of operation which is performed on u first n times and then m times, and if the result of these successive operations is the same as if the operation of which ϕ is the symbol had been performed $m+n$ times, then

$$\phi^m \phi^n(u) = \phi^{m+n}(u). \tag{10}$$

this law of notation is called the iterative or the repetitive law. Evidently differentiation is a process subject to it; because it has been shewn in the preceding pages that

$$\frac{d^m}{dx^m}\frac{d^n}{dx^n}u = \frac{d^{m+n}}{dx^{m+n}}u. \tag{11}$$

Now the calculus wherein these and other similar laws of operation, as distinct from symbols of quantity, are developed, is called the Calculus of Operations; and also is sometimes called the method of separation of symbols of operation from those of quantity: its laws and results I proceed to develope, so far as they are applicable to differential calculus. Our operations thus far have been those of direct differentiation, that is, m and n in (11) are taken to be positive; for, although with certain limitations, the results will be true when the operations are inverse, that is, when m and n are negative, yet at present it is unnecessary to fix their limitations, and therefore I shall confine myself for the most part to direct processes.

420.] First I must shew that any algebraic function of a distributive symbol is also itself distributive; and therefore that an algebraic function of the symbol of differentiation is also itself distributive.

Let ϕ be a distributive function of which $u+v$ is the subject: and let it satisfy (5) and (7); let

$$a_0 + a_1\phi + a_2\phi^2 + \dots\dots + a_n\phi^n$$

be an algebraical function of it; then, from (5) and (7), it is manifest that

$$\begin{aligned}\{a_0 + a_1\phi + a_2\phi^2 + \dots\dots + a_n\phi^n\}(u+v) &= \\ \{a_0 + a_1\phi + a_2\phi^2 + \dots\dots + a_n\phi^n\}\,u& \\ +\{a_0 + a_1\phi + a_2\phi^2 + \dots\dots + a_n\phi^n\}\,v;& \qquad (12)\end{aligned}$$

and if F symbolizes an algebraic function, then, also

$$\mathrm{F}\phi(u+v) = \mathrm{F}\phi(u) + \mathrm{F}\phi(v); \qquad (13)$$

and therefore as x-differentiation is a distributive operation,

$$\mathrm{F}\frac{d}{dx}(u+v) = \mathrm{F}\left(\frac{du}{dx}\right) + \mathrm{F}\left(\frac{dv}{dx}\right). \qquad (14)$$

Again, x-differentiation is subject to the commutative law: now suppose

$$\frac{d^nu}{dx^n} - p_1\frac{d^{n-1}u}{dx^{n-1}} + p_2\frac{d^{n-2}u}{dx^{n-2}} - \dots\dots \pm p_{n-1}\frac{du}{dx} \mp p_nu = 0 \quad (15)$$

to be a differential expression; then, omitting the symbol of quantity, we have

$$\frac{d^n}{dx^n} - p_1\frac{d^{n-1}}{dx^{n-1}} + \dots\dots \pm p^{n-1}\frac{d}{dx} \mp p_n = 0; \qquad (16)$$

let the roots of this equation be $a_1, a_2, \dots a_n$; and let us resolve (16) into its factors; so that it is equivalent to

$$\left(\frac{d}{dx}-a_1\right)\left(\frac{d}{dx}-a_2\right)\ldots\ldots\left(\frac{d}{dx}-a_n\right) = 0; \tag{17}$$

and if the operations indicated by these factorial symbols are successively, and one on the back of another, applied to the subject u, the result will be by the commutative law the same as if u had been operated on by the whole symbol (16).

Thus $$\left(\frac{d}{dx}-1\right)\left(\frac{d}{dx}-2\right)u = \left(\frac{d}{dx}-1\right)\left(\frac{du}{dx}-2u\right)$$

$$= \frac{d^2u}{dx^2}-3\frac{du}{dx}+2u$$

$$= \left\{\frac{d^2}{dx^2}-3\frac{d}{dx}+2\right\}u.$$

It is not my intention to enter on the processes of general differentiation, when the indices are fractional, because the theory is too imperfect for an elementary treatise; but I shall extend some of the theorems to a few cases wherein the index of differentiation is negative; in which it will be convenient to use a distinctive symbol for this negative differentiation; and I shall assume

$$d^{-1} = \int; \tag{18}$$

so that $$\left(\frac{d}{dx}\right)^{-1} = d^{-1}\times dx = \int dx, \tag{19}$$

and $$\left(\frac{d}{dx}\right)^{-n} = d^{-n}dx^n = \int^n dx^n; \tag{20}$$

so that $$\frac{d^m}{dx^m}\left(\frac{d}{dx}\right)^{-n} = \frac{d^{m-n}u}{dx^{m-n}} = \int^n dx^n \left(\frac{d}{dx}\right)^m. \tag{21}$$

421.] First let us investigate certain forms which are taken by results proved in the preceding pages; by Art. 47,

$$\mathrm{D}u = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy;$$

so that if we omit the subject of the operating symbol,

$$\mathrm{D} = \left(\frac{d}{dx}\right)dx + \left(\frac{d}{dy}\right)dy; \tag{22}$$

that is, the operation expressed by D is equivalent to the two operations expressed in the right-hand member of the equation.

As these operations are commutative, and as their symbols are subject to the index law, we may raise both sides of (22) to the nth power; whereby we have

$$\mathrm{D}^n = \left\{\left(\frac{d}{dx}\right) dx + \left(\frac{d}{dy}\right) dy\right\}^n; \tag{23}$$

$$= \left(\frac{d}{dx}\right)^n dx^n + \frac{n}{1}\left(\frac{d}{dx}\right)^{n-1}\left(\frac{d}{dy}\right) dx^{n-1} dy + \ldots + \left(\frac{d}{dy}\right)^n dy^n. \tag{24}$$

and adding the subject, we have

$$\mathrm{D}^n u = \left(\frac{d^n u}{dx^n}\right) dx^n + n\left(\frac{d^n u}{dx^{n-1}\,dy}\right) dx^{n-1} dy + \ldots + \left(\frac{d^n u}{dy^n}\right) dy^n; \tag{25}$$

which result is the same as (104), Art. 80.

If we use the notation of Art. 47, then

$$\mathrm{D}u = d_x u + d_y u;$$

$$\therefore \quad \mathrm{D} = d_x + d_y,$$

$$\text{and} \quad \mathrm{D}^n = d_x{}^n + n\,d_x{}^{n-1} d_y + \frac{n(n-1)}{1.2} d_x{}^{n-2} d_y{}^2 + \ldots + d_y{}^n. \tag{26}$$

And if u is a function of many variables, then because

$$\mathrm{D} = d_x + d_y + d_z + \ldots\ldots, \tag{27}$$

$$\mathrm{D}^n = (d_x + d_y + d_z + \ldots\ldots)^n; \tag{28}$$

the right-hand member of which must be expanded by the multinomial theorem, according to the process of Ex. 3, Art. 95; whereby the equivalent of D^n will be obtained in terms of partial differentials and derived functions.

422.] We proceed now to the investigation of the results consequent on other symbols of operation beside those of simple differentiation. Since

$$\frac{d}{d\theta} e^{m\theta} = m e^{m\theta}, \qquad \therefore \quad \frac{d^n}{d\theta^n} e^{m\theta} = m^n e^{m\theta}; \tag{29}$$

and therefore if F represents an algebraical function

$$\mathrm{F}\left(\frac{d}{d\theta}\right) e^{m\theta} = e^{m\theta}\,\mathrm{F}(m): \tag{30}$$

and I may observe that this result is true when n is negative. Next let us take the operating symbol $\frac{d}{d\theta} e^{m\theta}$; that is, let us suppose a subject u to be a function of θ, and to be multiplied by $e^{m\theta}$, and subsequently to be operated upon by the operation of which the symbol is $\frac{d}{d\theta}$; then

$$\frac{d}{d\theta} e^{m\theta} u = e^{m\theta} \frac{du}{d\theta} + m e^{m\theta} u$$

$$= e^{m\theta} \left\{ \frac{d}{d\theta} + m \right\} u; \tag{31}$$

so that omitting the subject u, we have

$$\frac{d}{d\theta} e^{m\theta} = e^{m\theta} \left(\frac{d}{d\theta} + m \right); \tag{32}$$

and therefore
$$\left(\frac{d}{d\theta}\right)^n e^{m\theta} = e^{m\theta} \left(\frac{d}{d\theta} + m \right)^n; \tag{33}$$

$$\therefore \quad \left(\frac{d}{d\theta} + m \right)^n u = e^{-m\theta} \left(\frac{d}{d\theta}\right)^n e^{m\theta} u; \tag{34}$$

and these equalities are true when n is negative; in which case, using the notation of Art. 420, we have

$$\int^n e^{m\theta} u \, d\theta^n = e^{m\theta} \left(\frac{d}{d\theta} + m \right)^{-n} u, \tag{35}$$

$$\left(\frac{d}{d\theta} + m \right)^{-n} u = e^{-m\theta} \int^n e^{m\theta} u \, d\theta^n; \tag{36}$$

and if F expresses an algebraical function,

$$\mathrm{F}\left(\frac{d}{d\theta}\right) e^{m\theta} = e^{m\theta} \mathrm{F} \left(\frac{d}{d\theta} + m \right). \tag{37}$$

From these expressions another theorem may be deduced. Since from (32) we have

$$\left(\frac{d}{d\theta} + m \right) u = e^{-m\theta} \frac{d}{d\theta} e^{m\theta} u; \tag{38}$$

in it let us substitute for m successively $0, -1, -2, \dots -(n-1)$; and let all these processes be performed successively on u; then we have

$$\left(\frac{d}{d\theta} - n + 1 \right) \left(\frac{d}{d\theta} - n + 2 \right) \dots\dots \left(\frac{d}{d\theta} - 1 \right) \frac{d}{d\theta} u$$

$$= e^{(n-1)\theta} \frac{d}{d\theta} e^{-\theta} \frac{d}{d\theta} \dots\dots e^{-\theta} \frac{d}{d\theta} u$$

$$= e^{n\theta} \left(e^{-\theta} \frac{d}{d\theta} \right)^n u; \tag{39}$$

which and (30) and (37) are three fundamental theorems of this kind of operating symbol.

423.] And they may by the following substitution take forms which are useful in the solution of many differential equations.

Let $e^{\theta} = x$; then $\frac{d}{d\theta} = x\frac{d}{dx}$; (40)

and from (29) and (30) we have

$$\left(x\frac{d}{dx}\right)^n x^m = m^n x^m, \tag{41}$$

$$\text{F}\left(x\frac{d}{dx}\right) x^m = x^m\,\text{F}(m). \tag{42}$$

Again, let the substitution (40) be made in (34) and (37); and we have

$$\left(x\frac{d}{dx}+m\right)^n u = x^{-m}\left(x\frac{d}{dx}\right)^n x^m u, \tag{43}$$

and
$$\text{F}\left(x\frac{d}{dx}\right) x^m u = x^m\,\text{F}\left(x\frac{d}{dx}+m\right) u. \tag{44}$$

Also let the substitution (40) be made in (39); and we have

$$\left(x\frac{d}{dx}-n+1\right)\left(x\frac{d}{dx}-n+2\right)\ldots\ldots\left(x\frac{d}{dx}-1\right)x\frac{d}{dx}u$$
$$= x^n\left(\frac{d}{dx}\right)^n u. \tag{45}$$

424.] By this last formula, differential equations, of which the differential terms are of the form $\left(x\frac{d}{dx}\right)^n u$, that is, where x is equicrescent, and the power of x in the coefficient and the order of the derived-function are the same, may be transformed into others of which the coefficients shall be constant. Because, if $x = e^{\theta}$,

$$\left(x\frac{d}{dx}\right)^n u = \left(\frac{d}{d\theta}-n+1\right)\left(\frac{d}{d\theta}-n+2\right)\ldots\left(\frac{d}{d\theta}-1\right)\frac{d}{d\theta}u. \tag{46}$$

Of this transformation some examples are subjoined.

Ex. 1. $x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + u = \log x.$

By (40) this becomes, if $x = e^{\theta}$,

$$\left(\frac{d}{d\theta}-1\right)\frac{d}{d\theta}u + \frac{du}{d\theta} + u = \theta,$$

$$\frac{d^2u}{d\theta^2} + u = \theta.$$

Ex. 2. $x^3\frac{d^3u}{dx^3} + 3x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + aux^n = 0.$

By reason of (40), if $x = e^{\theta}$, we have

$$\left(\frac{d}{d\theta} - 2\right)\left(\frac{d}{d\theta} - 1\right)\frac{du}{d\theta} + 3\left(\frac{d}{d\theta} - 1\right)\frac{du}{d\theta} + \frac{du}{d\theta} + aue^{n\theta} = 0,$$

$$\frac{d^3u}{d\theta^3} + aue^{n\theta} = 0.$$

425.] Let u and v be two functions of x: then

$$\frac{d}{dx}uv = v\frac{du}{dx} + u\frac{dv}{dx}. \tag{47}$$

In the right-hand member of this equation, let δ be the symbol of differentiation referring to u only and δ' that referring to v only: then

$$\frac{d}{dx}uv = \frac{\delta}{dx}vu + \frac{\delta'}{dx}vu; \tag{48}$$

$$\therefore \quad \frac{d}{dx} = \frac{\delta}{dx} + \frac{\delta'}{dx}. \tag{49}$$

Let both members be raised to the nth power; then

$$\frac{d^n}{dx^n} = \frac{\delta^n}{dx^n} + n\frac{\delta^{n-1}\delta'}{dx^n} + \frac{n(n-1)}{1.2}\frac{\delta^{n-2}\delta'^2}{dx^n} + \dots + \frac{\delta'^n}{dx^n}; \tag{50}$$

and adding the subject, we have

$$\frac{d^n.uv}{dx^n} = \frac{d^nu}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \frac{n(n-1)}{1.2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2} + \dots\dots$$
$$\dots\dots + u\frac{d^nv}{dx^n}; \tag{51}$$

which is Leibnitz's Theorem given in Art. 55.

Of this theorem the following are particular examples.

Let $v = e^{ax}$,

$$\therefore \quad \frac{d^n.ue^{ax}}{dx^n} = e^{ax}\left\{\frac{d^nu}{dx^n} + n\frac{d^{n-1}u}{dx^{n-1}}a + \frac{n(n-1)}{1.2}\frac{d^{n-2}u}{dx^{n-2}}a^2 + \dots + ua^n\right\} \tag{52}$$
$$= e^{ax}\left\{\frac{d}{dx} + a\right\}^n u;$$

which is the same as (34), proved above.

Let $n = -1$, in (52); then

$$\int ue^{ax}dx = e^{ax}\left\{\frac{d}{dx} + a\right\}^{-1}u;$$

$$\therefore \quad \left\{\frac{d}{dx} + a\right\}^{-1}u = e^{-ax}\int ue^{ax}dx. \tag{53}$$

426.] Also since

$$u\frac{dv}{dx} = \frac{d}{dx}uv - v\frac{du}{dx}; \qquad (54)$$

let us take $\text{D}, \delta, \delta'$ to be symbols of x-differentiation; and let us assume D to apply to both u and v, and δ and δ' to u and to v separately: so that taking symbols of operation only

$$\delta' = \text{D} - \delta. \qquad (55)$$

Let both members of this equation be raised to the nth power; then

$$\delta'^n = \text{D}^n - n\text{D}^{n-1}\delta + \frac{n(n-1)}{1.2}\text{D}^{n-2}\delta^2 + \ldots\ldots (-)^n\delta^n; \qquad (56)$$

and appending the subject uv,

$$u\frac{d^n v}{dx^n} = \frac{d^n.uv}{dx^n} - \frac{n}{1}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{du}{dx}\right) + \frac{n(n-1)}{1.2}\frac{d^{n-2}}{dx^{n-2}}\left(v\frac{d^2u}{dx^2}\right) - \ldots$$
$$(-)^n v\frac{d^n u}{dx^n}. \qquad (57)$$

In a paper by Mr. Hargreave in the Philosophical Transactions for 1848, (51) and (57) are extended to algebraical functions of $\frac{d}{dx}$.

427.] Taylor's series may be expressed in the following concise form, if the symbol of quantity is separated from that of operation.

By (76), Art. 71, if we replace derived functions by their equivalent ratios of differentials, we have

$$f(x+h) = f(x) + \frac{df(x)}{dx}\frac{h}{1} + \frac{d^2f(x)}{dx^2}\frac{h^2}{1.2} + \ldots\ldots \qquad (58)$$

and therefore

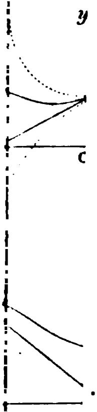
$$f(x+h) = \left\{1 + \frac{d}{dx}\frac{h}{1} + \left(\frac{d}{dx}\right)^2\frac{h^2}{1.2} + \left(\frac{d}{dx}\right)^3\frac{h^3}{1.2.3} + \ldots\right\}f(x)$$
$$= e^{h\frac{d}{dx}}f(x); \qquad (59)$$

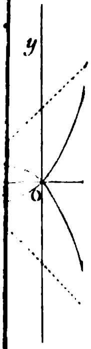
so that if $f(x)$ is operated upon by the symbol $e^{h\frac{d}{dx}}$, it is changed into $f(x+h)$.

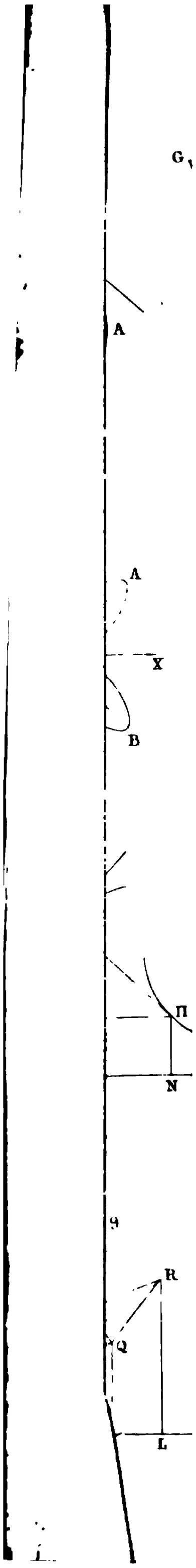
Similarly, if $f(y)$ is operated upon by the symbol $e^{k\frac{d}{dy}}$, it is changed into $f(y+k)$. And therefore if $\text{F}(x,y)$ is a function of two independent variables x and y,

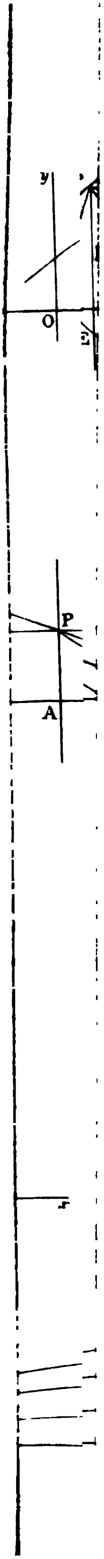
$$e^{h\frac{d}{dx}+k\frac{d}{dy}}\text{F}(x,y) = \text{F}(x+h, y+k); \qquad (60)$$

68






G
A
A
X
B
Π
N
ϑ
R
Q
L


y
O
P
A
x

Errata in

D. [illegible]

RSITY LIBRARY

p. 29 Art. 15. As there corrected.

p. 114 " 67. " " "

p. 150 " 90. Notation inconsistent.

p. 184 " 108, line −6; $-\frac{\sin I\cos I}{r}\left(\frac{dV}{dr}\right)$ omitted.

p. 177 " 104 Ex. 2. u should be $=0$.

p. 230 lines −11, 13. All Wrong. see M.S.

p. 283 Art. 173. " " " "

p. 347 last paragraph. ?

p. 365. Eq. (98) If x, U_{n-1} should disappear, then we could not conclude etc.

p. 366 line 12 up Signs wrong. Corrd on margin.

STORED AT NRLF

Misprints.

"Preface IX line −3 for "stop" read "step"

page 116. line −9 " "is good" " "is well".

" 167 " −17 " added to " "with"

" 230 " −17 rectify the "$\overset{=}{i}$".

" 222 " −10 $\frac{1}{1.2\ldots n}$ omitted before brackets!

" 219 " −7 } omitted

" 232 & 233 two eqs. numbered (56) [referred to p. 268 line +10

[illegible]

" 457 line −11 for y read Y